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SOME INTEGRAL INEQUALITIES RELATED TO HILBERT'S

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ABSTRACT. We prove some integral inequalities involving the Laplace transform. These are sharper than some known generalizations of the Hilbert integral inequality including a recent result of Yang.

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1. INTRODUCTION

By the Hilbert integral inequality we mean the following well-known result.

Theorem 1.1. *Suppose $p > 1$, $q = p/(p-1)$, functions f and g to be non-negative and Lebesgue measurable on $(0,\infty)$ then*

$$\int_0^\infty\int_0^\infty \frac{f(u)g(v)}{u+v}dudv \leq B\left(\frac{1}{p},\frac{1}{q}\right)\left[\int_0^\infty f(u)^p du\right]^{\frac{1}{p}}\left[\int_0^\infty g(v)^q dv\right]^{\frac{1}{q}}, \tag{1.1}$$

where B is the beta function. The inequality is strict unless f or g are null.

The constant $B\left(\frac{1}{p},\frac{1}{q}\right) = \pi \operatorname{cosec}\left(\frac{\pi}{p}\right)$ is known to be the best possible. See [3, Chapter 9], for the history of this result.

Many authors on integral inequalities follow the practice of Hardy, Littlewood and Polya [3] of implicitly assuming all functions mentioned are measurable and non-negative. We intend to make such conditions explicit. Further, if one of the integrals on the right of (1.1) is infinite then the strict inequality gives the impression that the left side is finite. Recent authors such as Yang [7] avoid this by adding the conditions $0 < \int_0^\infty f^p(u)du < \infty$, $0 < \int_0^\infty g^q(v)dv < \infty$ to give the strict inequality. Hence it becomes convenient to introduce a class of functions that satisfy all the required conditions. *We define $P(E)$ to be the class of functions $f : E \to \mathbb{R}$ such that on E:*

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(1) *f is measurable,*
(2) *f is non-negative,*
(3) *f is not null (hence positive on a set of positive measure so $\int_E f > 0$) and*
(4) *f is integrable (so $\int_E f < \infty$).*

Yang [7], [8] has recently found various inequalities related to that above. One of these (in our notation) is

Theorem 1.2. *If $p > 1$, $q = p/(p-1)$, $\lambda > 2 - \min(p,q)$ and $(u-a)^{1-\lambda} f^p(u)$ and $(v-a)^{1-\lambda} g^q(v)$ are in $P(a,\infty)$ then*

$$\int_a^\infty \int_a^\infty \frac{f(u)g(v)}{(u+v-2a)^\lambda} dudv < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \left[\int_a^\infty (u-a)^{1-\lambda} f(u)^p \, du\right]^{\frac{1}{p}} \left[\int_a^\infty (v-a)^{1-\lambda} g(v)^q dv\right]^{\frac{1}{q}}.$$

We will specifically write out the case $a = 0$ since this case is equivalent to the complete theorem.

Theorem 1.3. *If $p > 1$, $q = p/(p-1)$, $\lambda > 2 - \min(p,q)$, $u^{1-\lambda} f^p(u)$ and $v^{1-\lambda} g^q(v)$ are in $P(0,\infty)$ then*

$$\int_0^\infty \int_0^\infty \frac{f(u)g(v)}{(u+v)^\lambda} \, dudv < B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right) \left[\int_0^\infty u^{1-\lambda} f(u)^p du\right]^{\frac{1}{p}} \left[\int_0^\infty v^{1-\lambda} g(v)^q dv\right]^{\frac{1}{q}}.$$

To show that Theorem 1.3 implies Theorem 1.2, change the variables u and v in Theorem 1.3 to $s = u + a$ and $t = v + a$ and then replace $f(s-a)$, $g(t-a)$ by $\phi(s)$ and $\psi(t)$ respectively. Thus Theorem 1.3 is equivalent to Theorem 1.2. Note also that for the case $\lambda = 1$, Theorem 1.3 reduces to Theorem 1.1.

This paper will consider a further generalization of this inequality.

Theorem 1.4. *If $p > 1$, $q = p/(p-1)$, $b > -\frac{1}{p}$, $c > -\frac{1}{q}$ and $u^{p-pb-2} f^p(u)$ and $v^{q-qc-2} g^q(v)$ are in $P(0,\infty)$ then*

$$\int_0^\infty \int_0^\infty \frac{f(u)g(v)}{(u+v)^{b+c+1} dudv} < B\left(b + \frac{1}{p}, c + \frac{1}{q}\right) \left\|u^{1-b-2/p} f(u)\right\|_p \left\|v^{1-c-2/q} g(v)\right\|_q. \tag{1.2}$$

This reduces to Theorem 1.3 for the case $b = (\lambda + p - 3)/p$, $c = (\lambda + q - 3)/q$.

Theorem 1.4 is itself a special case of the Hardy-Littlewood-Polya inequality, Proposition 319 of [3]. Recall that a function $K(u,v)$ is homogeneous of degree -1 if $K(\lambda u, \lambda v) = \lambda^{-1} K(u,v)$ for allowed u, v, λ. The inequality is

Theorem 1.5. *If $p > 1$, $q = p/(p-1)$, ϕ^p and ψ^q are in $P(0,\infty)$ and $K(u,v)$ is positive on $(0,\infty) \times (0,\infty)$, homogeneous of degree -1 with*

$$\int_0^\infty K(u,1) u^{-\frac{1}{p}} du = k$$

then

$$\int_0^\infty \int_0^\infty K(u,v)\phi(u)\psi(v) \, dudv < k \|\phi\|_p \|\psi\|_q \tag{1.3}$$

and

$$\left\| \int_0^\infty K(u,v)\phi(u)\,du \right\|_p < k\,\|\phi\|_p\,. \tag{1.4}$$

The constant k is the best possible.

To obtain Theorem 1.4 substitute

$$K(u,v) = \frac{u^{b+\frac{2}{p}-1}v^{c+\frac{2}{q}-1}}{(u+v)^{b+c+1}}$$

and $\phi(u) = u^{1-b-\frac{2}{p}}f(u)$, $\psi(v) = v^{1-c-\frac{2}{q}}g(v)$.

In summary, we have a chain of generalizations:

Th.1.1 (Hilbert) $\Leftarrow$ Th.1.2 (Yang) $\Leftrightarrow$ Th.1.3 $\Leftarrow$ Th.1.4 $\Leftarrow$ Th.1.5 (HLP).

This paper presents a new inequality Theorem 2.1, sharper than Theorem 1.4, involving the Laplace transform. Logically the implications are

$$\left.\begin{array}{r}\text{Th.2.2}\\ \text{Th.1.5 (HLP)} \Rightarrow \text{Th.2.3}\end{array}\right\} \Rightarrow \text{Th.2.1} \Rightarrow \text{Th.1.4}.$$

In Section 2 we state and prove this result. Section 3 considers the extension of our theory to the case of non-conjugate p and q. In that section, Theorem 1.1 generalizes to Theorem 3.1, Theorem 1.5 generalizes to Theorem 3.2 and Theorem 2.1 to Theorem 3.4. The structure of the section is

$$\left.\begin{array}{r}\text{Th.2.2}\\ \text{Th.3.2 (Bonsall)} \Rightarrow \text{Th.3.3}\end{array}\right\} \Rightarrow \text{Th.3.4}.$$

2. A SHARPER INEQUALITY

Theorem 2.1. *Suppose $p > 1$, $q = p/(p-1)$, $b > -\frac{1}{p}$, $c > -\frac{1}{q}$, suppose $u^{p-pb-2}f^p(u)$ and $v^{q-qc-2}g^q(v)$ are $P(0,\infty)$ and F, G are the respective Laplace transforms of f and g, then*

$$\int_0^\infty\!\!\int_0^\infty \frac{f(u)g(v)}{(u+v)^{b+c+1}}\,dudv \le \frac{1}{\Gamma(b+c+1)}\left\|s^bF(s)\right\|_p \|s^cG(s)\|_q \tag{2.1}$$

$$< B\left(b+\frac{1}{p}, c+\frac{1}{q}\right)\left\|u^{1-b-2/p}f(u)\right\|_p \left\|v^{1-c-2/q}g(v)\right\|_q. \tag{2.2}$$

To show that (2.1) gives a considerably sharper bound than (1.2) consider the case $p = q = 2$, $b = c = 0$ and $f\,(x) = g\,(x) = e^{-x}$. Then $F\,(s) = G\,(s) = 1/(s+1)$ and

$$\left\|s^bF\,(s)\right\|_p = \|s^cG\,(s)\|_q = 1$$

while

$$\|f\|_p = \|g\|_q = \frac{1}{\sqrt{2}}.$$

The right side of (2.1) is thus equal to 1 while that of (1.2) is $\frac{\pi}{2}$.

The case $b = 1 - \frac{2}{p}$, $c = 1 - \frac{2}{q}$ for Theorem 2.1 has been considered by the author in [6, Theorem 15].

The proof uses the following identity

Theorem 2.2. *If $a > -1$, f and g are non-negative and measurable on $(0,\infty)$ with respective Laplace transforms F and G then*

$$\int_0^\infty\!\!\int_0^\infty \frac{f(u)g(v)}{(u+v)^{a+1}}dudv = \frac{1}{\Gamma(a+1)}\int_0^\infty s^aF\,(s)\,G\,(s)\;ds. \tag{2.3}$$

The proof is a simple application of Fubini's theorem,

$$\begin{aligned}\int_0^\infty s^a F(s)G(s)ds &= \int_0^\infty s^a\,ds \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-sv} g(v)\,dv \\ &= \int_0^\infty f(u)\,du \int_0^\infty g(v)\,dv \int_0^\infty s^a e^{-s(u+v)}\,ds \\ &= \int_0^\infty\int_0^\infty \frac{\Gamma(a+1)f(u)g(v)}{(u+v)^{a+1}}\,dudv\end{aligned}$$

which is equation (2.3).

Despite the simplicity of the proof of (2.3) and the obvious relation with Hilbert's inequality, we can find no explicit reference to this identity in the literature before the case $a = 0$ in [6]. That case appears implicitly in Hardy's proof of Widder's inequality, [2]. Mulholland [5] used the discrete analogue, namely

$$\sum_{m=0}^\infty \sum_{n=0}^\infty \frac{a_m b_n}{m+n+1} = \int_0^1 \sum_{m=0}^\infty a_m x^m \sum_{n=0}^\infty a_n x^n dx,$$

to prove the series case of Hilbert's inequality.

Theorem 2.1 also depends on the following bound on Laplace transforms.

Theorem 2.3. *Suppose* $p > 1$, $\alpha + \frac{1}{p} > 0$, $x^{p-p\alpha-2} f^p(x)$ *is* $P(0,\infty)$ *and* F *is the Laplace transform of* f, *then*

$$\|s^\alpha F(s)\|_p < \Gamma\left(\alpha + \frac{1}{p}\right) \left\| x^{1-\alpha-\frac{2}{p}} f(x)\right\|_p .$$

This is also a corollary of Theorem 1.5. For if we make $K(u,v) = e^{-\frac{u}{v}} u^{\alpha+\frac{2}{p}-1} v^{-\alpha-\frac{2}{p}}$ and $\phi(u) = u^{-\alpha-\frac{2}{p}+1} f(u)$ then we obtain from (1.4)

$$\left\| v^{-\alpha-\frac{2}{p}} F\left(\frac{1}{v}\right)\right\|_p < \Gamma\left(\alpha+\frac{1}{p}\right) \left\| x^{1-\alpha-\frac{2}{p}} f\left(x\right)\right\|_p$$

and a substitution $s = 1/v$ in the left side gives the theorem. We can find no direct mention of this result although the last two formulae in Propositions 350 of [3] are special cases.

We may now prove Theorem 2.1. Since $b + c > -1$, Theorem 2.2 gives

$$\int_0^\infty\int_0^\infty \frac{f(u)g(v)}{(u+v)^{b+c+1}} dudv = \frac{1}{\Gamma(b+c+1)} \int_0^\infty s^{b+c} F\left(s\right) G\left(s\right) ds.$$

Then by Hölder's inequality

$$\int_0^\infty\int_0^\infty \frac{f(u)g(v)}{(u+v)^{b+c+1}} dudv \le \frac{1}{\Gamma(b+c+1)} \left\| s^b F(s)\right\|_p \|s^c G(s)\|_q$$

which is equation (2.1). Applying Theorem 2.3 to each norm, the right side of this is less than

$$\frac{\Gamma(b+\frac{1}{p})\Gamma(c+\frac{1}{q})}{\Gamma(b+c+1)} \left\| u^{1-b-\frac{2}{p}} f(u)\right\|_p \left\| v^{1-c-\frac{2}{q}} g(v)\right\|_q$$

and this is (2.2).

3. NON-CONJUGATE PARAMETERS

Here we consider inequalities of the type considered in Section 2 but where $q \neq p/(p-1)$. (Henceforth we will use p' and q' for the respective conjugates of p and q.) The earliest investigation of this type seems to be Theorem 340 of Hardy, Littlewood and Polya, [3]. In our notation this is

Theorem 3.1. *Suppose $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} \geq 1$ and $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$ (so $0 < \lambda \leq 1$); suppose f^p and g^q are in $P(0,\infty)$ then*

$$\int_0^\infty\int_0^\infty \frac{f(u)g(v)}{(u+v)^\lambda}\,dudv < C\,\|f\|_p\,\|g\|_q\,,$$

where C depends on p and q only.

Hardy, Littlewood and Polya did not give a specific value for the constant C. An alternative proof by Levin, [4] established that $C = B^\lambda\left(\frac{1}{\lambda p'}, \frac{1}{\lambda q'}\right)$ suffices but the paper did not decide whether this was the best possible constant. This question remains open.

Just as Theorem 1.5 generalized Theorem 1.1 to a general kernel, Bonsall [1] has generalized the above result.

Theorem 3.2. *Suppose $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} \geq 1$, $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$; ϕ^p and ψ^q are in $P(0,\infty)$ and $K(u,v)$ is positive on $(0,\infty)\times(0,\infty)$, homogeneous of degree -1 with*

$$\int_0^\infty K(x,1)x^{-1/(\lambda q')}dx = k$$

then

$$\int_0^\infty\int_0^\infty K^\lambda(u,v)\phi(u)\psi(v)\,dudv < k^\lambda\,\|\phi\|_p\,\|\psi\|_q \tag{3.1}$$

and

$$\left\|\int_0^\infty K^\lambda(u,v)\phi(u)\,du\right\|_{q'} < k^\lambda\,\|\phi\|_p\,. \tag{3.2}$$

Since this theorem is more than that claimed by Bonsall we will repeat and extend his proof, using the standard methods of [3] Section 9.3.

Since $p'\lambda$ and $q'\lambda$ are conjugate,

$$\begin{aligned} &K^\lambda(u,v)\phi(u)\psi(v)\\ &\quad= \left[K(u,v)^{\frac{1}{p'}}\psi^{\frac{q}{p'}}(v)\left(\frac{u}{v}\right)^{\frac{-1}{p'q'\lambda}}\right]\left[K(u,v)^{\frac{1}{q'}}\phi^{\frac{p}{q'}}(u)\left(\frac{v}{u}\right)^{\frac{-1}{p'q'\lambda}}\right]\\ &\qquad\quad\times\left[\phi^{p(1-\lambda)}(u)\psi^{q(1-\lambda)}(v)\right]\\ &\quad= F_1\,F_2\,F_3 \quad \text{say.} \end{aligned}$$

Then since $\frac{1}{p'} + \frac{1}{q'} + (1-\lambda) = 1$, Holder's inequality gives

$$\begin{aligned} \int_0^\infty\int_0^\infty K(u,v)^\lambda\phi(u)\psi(v)\,dudv &\leq \left[\int_0^\infty\int_0^\infty F_1^{p'}\,dudv\right]^{\frac{1}{p'}}\left[\int_0^\infty\int_0^\infty F_2^{q'}\,dudv\right]^{\frac{1}{q'}}\\ &\qquad\times\left[\int_0^\infty\int_0^\infty F_3^{1/(1-\lambda)}\,dudv\right]^{(1-\lambda)}\\ &= I_1^{\frac{1}{p'}}\,I_2^{\frac{1}{q'}}\,I_3^{1-\lambda}. \end{aligned}$$

Here

$$I_1 = \int_0^\infty\int_0^\infty K(u,v)\psi^q(v)\left(\frac{u}{v}\right)^{-\frac{1}{q'\lambda}}\,dudv$$
$$= \int_0^\infty\int_0^\infty K(x,1)\psi^q(v)x^{-\frac{1}{q'\lambda}}\,dxdv = k\,\|\psi\|_q^q\,.$$

Similarly

$$I_2 = \int_0^\infty\int_0^\infty K(1,x)\phi^p(v)x^{-\frac{1}{p'\lambda}}dxdu = k\,\|\phi\|_p^p$$

and

$$I_3 = \int_0^\infty\int_0^\infty \phi(u)^p\psi(v)^q\,dudv = \|\phi\|_p^p\|\psi\|_q^q.$$

Substituting these I_i, gives

$$\int_0^\infty\int_0^\infty K^\lambda(u,v)\phi(u)\psi(v)dudv \le k^\lambda\,\|\phi\|_p\,\|\psi\|_q\,. \tag{3.3}$$

Note that equality in (3.3) can only occur if one of the F_i is null or all are effectively proportional, see [3, Proposition 188]. The first possibility would contradict one of the hypotheses; the alternative implies that for almost all v,

$$K(u,v)\phi^p(u)\left(\frac{v}{u}\right)^{-\frac{1}{p'\lambda}} = K(u,v)\psi^q(v)\left(\frac{u}{v}\right)^{-\frac{1}{q'\lambda}}$$

for almost all u. For such a v, $\phi^p(u) = Au^{-1}$ for some positive constant A, contradicting $\|\phi\|_p < \infty$. Thus the inequality in (3.3) is strict giving (3.1).

Finally we note that

$$\int_0^\infty \psi(v)\,dv\int_0^\infty K^\lambda(u,v)\phi(u)\,du < k^\lambda\,\|\phi\|_p\,\|\psi\|_q$$

for all $\psi \in L_q$. Equation (3.2) follows by the converse of Hölder's inequality, [3, Proposition 191].

We next extend Theorem 2.3 to non-conjugate p and q.

Theorem 3.3. *Suppose* $p > 1$, $q > 1$, $\frac{1}{p}+\frac{1}{q} \ge 1$, $\lambda = 2-\frac{1}{p}-\frac{1}{q}$ *and* $\alpha+\frac{1}{q'} > 0$*; suppose* $u^{p(\lambda-\alpha-2/q')}f^p(u)$ *is in* $P(0,\infty)$ *and* F *is the Laplace transform of* f*, then*

$$\left\|s^\alpha F(s)\right\|_{q'} < \lambda^{\alpha+\frac{1}{q'}}\,\Gamma^\lambda\left(\frac{\alpha}{\lambda}+\frac{1}{\lambda q'}\right)\left\|u^{\lambda-\alpha-\frac{2}{q'}}f(u)\right\|_p.$$

This follows from Theorem 3.2 by substituting

$$K(u,v) = e^{-u/v}u^{(\alpha/\lambda)+(2/\lambda q')-1}v^{-(\alpha/\lambda)-(2/\lambda q')}$$

and

$$\phi(u) = u^{-\alpha-(2/q')+\lambda}f(u)$$

giving $k = \Gamma\left(\frac{\alpha}{\lambda}+\frac{1}{\lambda q'}\right)$ and

$$\int_0^\infty K^\lambda(u,v)\phi(u)\,du = v^{-\alpha-\frac{2}{q'}}F\left(\frac{\lambda}{v}\right)$$

and then equation (3.2) gives the required inequality.

The case $q = p'$ gives $\lambda = 1$ and reduces to Theorem 2.3. Another known case occurs if $q = p$ (which requires $1 < p \le 2$) and $\alpha = 0$ giving $\lambda = 2/p'$ and

$$\|F\|_{p'} < \left(\frac{2\pi}{p'}\right)^{\frac{1}{p'}} \|f\|_p,$$

which is Proposition 352 of [3].

A third case of interest can be obtained by substituting $\alpha = \frac{1}{p'} - \frac{1}{q'}$ and introducing $r = q'$ with $r \ge p$. Then Theorem 3.3 gives

$$\left\| x^{1-\frac{1}{p}-\frac{1}{r}} F(x) \right\|_r < \lambda^{\frac{1}{p'}} \Gamma^{\lambda}\left(\frac{1}{p'\lambda}\right) \|f\|_p$$

where $\lambda = \frac{1}{p'} + \frac{1}{r}$. This is given in [3] as Proposition 360 except that the constant $\lambda^{\frac{1}{p'}}\Gamma^{\lambda}\left(\frac{1}{p'\lambda}\right)$ is not specified there.

We can now extend Theorem 2.1 to the case of non-conjugate parameters.

Theorem 3.4. *Suppose* $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} \ge 1$, $q \le r \le p'$, $b + \frac{1}{r'} > 0$ *and* $c + \frac{1}{r} > 0$*;* $u^{p(1/p'-1/r'-b)} f^p(u)$ *and* $v^{q(1/q'-1/r-c)} g^q(v)$ *are in* $P(0,\infty)$ *and suppose* F, G *are the Laplace transforms of* f *and* g *respectively, then*

$$\int_0^{\infty}\int_0^{\infty} \frac{f(u)g(v)}{(u+v)^{b+c+1}}\,dudv \le \frac{1}{\Gamma(b+c+1)} \left\|s^b F(s)\right\|_{r'} \|s^c G(s)\|_r \tag{3.4}$$

$$< C \left\| u^{\frac{1}{p'}-\frac{1}{r'}-b} f(u)\right\|_p \left\| v^{\frac{1}{q'}-\frac{1}{r}-c} g(v)\right\|_q, \tag{3.5}$$

where $\beta = \frac{1}{p'} + \frac{1}{r'}$, $\gamma = \frac{1}{q'} + \frac{1}{r}$ *and*

$$C = \beta^{b+\frac{1}{r'}} \gamma^{c+\frac{1}{r}} \Gamma^{\beta}\left(\frac{b}{\beta} + \frac{1}{r'\beta}\right) \Gamma^{\gamma}\left(\frac{c}{\gamma} + \frac{1}{r\gamma}\right) \Gamma^{-1}(b+c+1).$$

The proof of (3.4) proceeds as that of (2.1) except that p and q are replaced by r' and r respectively. We then apply Theorem 3.3 to $\|s^b F(s)\|_{r'}$. In that theorem replace α by b, q by r and $\lambda = \frac{1}{p'} + \frac{1}{q'}$ by $\beta = \frac{1}{p'} + \frac{1}{r'}$, giving

$$\|s^b F(s)\|_{r'} < \beta^{b+\frac{1}{r'}} \Gamma^{\beta}\left(\frac{b}{\beta} + \frac{1}{\beta r'}\right) \left\| u^{\frac{1}{p'}-\frac{1}{r'}-b} f(u)\right\|_p .$$

The condition $\frac{1}{p} + \frac{1}{q} \ge 1$ is satisfied because $r \le p'$. Alternatively if in Theorem 3.3 we replace α by c, p by q and q' by r with $r \ge q$ we obtain

$$\|s^c G(s)\|_r < \gamma^{c+\frac{1}{r}} \Gamma^{\gamma}\left(\frac{c}{\gamma} + \frac{1}{\gamma r}\right) \left\| u^{\frac{1}{q'}-\frac{1}{r}-c} g(u)\right\|_q .$$

Applying these to (3.4) gives (3.5) and completes the proof.

The question arises as to whether the constant C in (3.5) is better than Levin's $B^{\lambda}\left(\frac{1}{\lambda p'}, \frac{1}{\lambda q'}\right)$ in the case $b = \frac{1}{p'} - \frac{1}{r'}$, $c = \frac{1}{q'} - \frac{1}{r}$. For that case

$$C = \beta^{\frac{1}{p'}} \gamma^{\frac{1}{q'}} \Gamma^{\beta}\left(\frac{1}{\beta p'}\right) \Gamma^{\gamma}\left(\frac{1}{\gamma q'}\right) \Gamma^{-1}(\lambda)$$

where $\beta = \frac{1}{p'} + \frac{1}{r'}$ and $\gamma = \frac{1}{q'} + \frac{1}{r}$. Experiments with Maple suggest that Levin's constant is the better one.

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