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REFINEMENTS OF THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS

S.S. DRAGOMIR AND A. MCANDREW

SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS
VICTORIA UNIVERSITY OF TECHNOLOGY
PO BOX 14428, MELBOURNE VIC 8001, AUSTRALIA.
sever.dragomir@vu.edu.au
URL: http://rgmia.vu.edu.au/dragomir

Alasdair.Mcandrew@vu.edu.au
URL: http://sci.vu.edu.au/~amca/

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ABSTRACT. New refinements for the celebrated Hermite-Hadamard inequality for convex functions are obtained. Applications for special means are pointed out as well.

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1. INTRODUCTION

The following result is well known in the literature as the Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

provided that $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$.

The following refinements of the $H.-H.$ inequality were obtained in [2]

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \geq \left|\frac{1}{b-a}\int_a^b \left|\frac{f(x)+f(a+b-x)}{2}\right| dx - \left|f\left(\frac{a+b}{2}\right)\right|\right| \geq 0. \tag{1.2}$$

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and

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt \geq \begin{cases} \left|\left|f(a)\right| - \frac{1}{b-a}\int_a^b \left|f(x)\right| dx\right| & \text{if } f(a)=f(b) \\ \\ \left|\frac{1}{f(b)-f(a)}\int_{f(a)}^{f(b)} |x|\,dx - \frac{1}{b-a}\int_a^b \left|f(x)\right| dx\right| & \text{if } f(a)\neq f(b) \end{cases} \tag{1.3}$$

for the general case of convex functions $f:[a,b]\to\mathbb{R}$.

If one would assume differentiability of f on (a,b), then the following bounds in terms of its derivative holds (see [3, pp. 30-31])

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt \geq \max\{|A|,|B|,|C|\}\geq 0 \tag{1.4}$$

where

$$A := \frac{1}{b-a}\int_a^b \left|x-\frac{a+b}{2}\right| \left|f'(x)\right| dx - \frac{1}{4}\int_a^b \left|f'(x)\right| dx,$$
$$B := \frac{f(b)-f(a)}{4} - \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^b f(x)\,dx\right]$$

and

$$C := \frac{1}{b-a}\int_a^b \left(x-\frac{a+b}{2}\right)\left|f'(x)\right| dx.$$

A different approach considered in [1] led to the following lower bounds

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt \geq \max\{|D|,|E|,|F|\}\geq 0, \tag{1.5}$$

where

$$D := \frac{1}{b-a}\int_a^b \left|xf'(x)\right| dx - \frac{1}{b-a}\int_a^b \left|f'(x)\right| dx \cdot \frac{1}{b-a}\int_a^b |x|\,dx,$$
$$E := \frac{1}{b-a}\int_a^b x\left|f'(x)\right| dx - \frac{a+b}{2}\cdot\frac{1}{b-a}\int_a^b \left|f'(x)\right| x dx$$

and

$$F := \frac{1}{b-a}\int_a^b |x|\, f'(x)\,dx - \frac{f(b)-f(a)}{b-a}\cdot\frac{1}{b-a}\int_a^b |x|\,dx.$$

For other results connected to the $H.-H.$ inequality see the recent monograph on line [3].

In the present paper, we use a different method to obtain other refinements of the $H.-H.$ inequality. Applications for special means are pointed out as well.

2. The Results

The following refinement of the Hermite-Hadamard inequality for differentiable convex functions holds.

Theorem 2.1. *Assume that* $f : [a,b] \to \mathbb{R}$ *is differentiable convex on* (a,b) *. Then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \geq \left|\frac{1}{b-a}\int_a^b \left|f(x) - f\left(\frac{a+b}{2}\right)\right| dx - \frac{b-a}{4}\cdot\left|f'\left(\frac{a+b}{2}\right)\right|\right| \geq 0. \tag{2.1}$$

Proof. Since f is differentiable convex on (a,b), then for each $x,y \in (a,b)$ one has the inequality

$$f(x) - f(y) \geq (x-y) f'(y). \tag{2.2}$$

Using the properties of modulus, we have

$$\begin{aligned} f(x) - f(y) - (x-y) f'(y) &= |f(x) - f(y) - (x-y) f'(y)| \\ &\geq ||f(x) - f(y)| - |x-y| |f'(y)|| \end{aligned} \tag{2.3}$$

for each $x,y \in (a,b)$.

If we choose $y = \frac{a+b}{2}$ in (2.3) we get

$$\begin{aligned} f(x) - f\left(\frac{a+b}{2}\right) - \left(x - \frac{a+b}{2}\right) f'\left(\frac{a+b}{2}\right) \\ \geq \left|\left|f(x) - f\left(\frac{a+b}{2}\right)\right| - \left|x - \frac{a+b}{2}\right| \left|f'\left(\frac{a+b}{2}\right)\right|\right| \end{aligned} \tag{2.4}$$

for any $x \in (a,b)$.

Integrating (2.4) on $[a,b]$, dividing by $(b-a)$ and using the properties of modulus, we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) - f'\left(\frac{a+b}{2}\right)\cdot\frac{1}{b-a}\int_a^b \left(x - \frac{a+b}{2}\right) dx \\ &\quad\geq \frac{1}{b-a}\int_a^b \left|\left|f(x) - f\left(\frac{a+b}{2}\right)\right| - \left|x - \frac{a+b}{2}\right|\left|f'\left(\frac{a+b}{2}\right)\right|\right| dx \\ &\quad\geq \left|\frac{1}{b-a}\int_a^b \left|f(x) - f\left(\frac{a+b}{2}\right)\right| dx - \left|f'\left(\frac{a+b}{2}\right)\right| \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| dx\right| \end{aligned}$$

and since

$$\int_a^b \left(x - \frac{a+b}{2}\right) dx = 0, \quad \int_a^b \left|x - \frac{a+b}{2}\right| dx = \frac{(b-a)^2}{4}, \tag{2.5}$$

we deduce by (2.5) the desired result (2.1). □

The second result is embodied in the following theorem.

Theorem 2.2. *Assume that* $f : [a,b] \to \mathbb{R}$ *is differentiable convex on* (a,b) *. Then one has the inequality*

$$\begin{aligned} &\frac{1}{2}\left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\quad\geq \frac{1}{2}\left|\frac{1}{b-a}\int_a^b \left|f(x) - f\left(\frac{a+b}{2}\right)\right| dx - \frac{1}{b-a}\int_a^b \left|x - \frac{a+b}{2}\right| |f'(x)|\,dx\right| \geq 0. \end{aligned} \tag{2.6}$$

Proof. We choose $x = \frac{a+b}{2}$ in (2.3) to get

$$f\left(\frac{a+b}{2}\right) - f(y) - \left(\frac{a+b}{2} - y\right) f'(y) \geq \left| \left| f\left(\frac{a+b}{2}\right) - f(y)\right| - \left|\frac{a+b}{2} - y\right| |f'(y)| \right|. \tag{2.7}$$

Integrating (2.7) over y, dividing by $(b-a)$ and using the modulus properties, we get

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(y)\,dy - \int_a^b \left(\frac{a+b}{2} - y\right) f'(y)\,dy \geq \left| \frac{1}{b-a}\int_a^b \left| f\left(\frac{a+b}{2}\right) - f(y)\right| dy - \frac{1}{b-a}\int_a^b \left|\frac{a+b}{2} - y\right| |f'(y)|\,dy \right|. \tag{2.8}$$

Since

$$\int_a^b \left(y - \frac{a+b}{2}\right) f'(y)\,dy = \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt,$$

then by (2.8) we deduce

$$f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2} - \frac{2}{b-a}\int_a^b f(y)\,dy \geq \left| \frac{1}{b-a}\int_a^b \left| f(y) - f\left(\frac{a+b}{2}\right)\right| dy - \frac{1}{b-a}\int_a^b \left|y - \frac{a+b}{2}\right| |f'(y)|\,dy \right|$$

which is clearly equivalent to (2.6). □

The following result holding for the subclass of monotonic and convex functions is whort to mention.

Theorem 2.3. *Assume that* $f : [a,b] \to \mathbb{R}$ *is monotonic and convex on* (a,b). *Then we have:*

$$\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \geq \left| \frac{1}{4}[f(b) - f(a)] + \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(\frac{a+b}{2} - x\right) f(x)\,dx \right|. \tag{2.9}$$

Proof. Since the class of differentiable convex functions in (a,b) is dense in uniform topology in the class of all convex functions defined on (a,b), we may assume, without loss of generality, that f is differentiable convex and monotonic on (a,b).

Firstly, assume that f is monotonic nondecreasing on $[a,b]$. Then

$$\begin{aligned} \int_a^b \left| f(x) - f\left(\frac{a+b}{2}\right)\right| dx &= \int_a^{\frac{a+b}{2}} \left(f\left(\frac{a+b}{2}\right) - f(x)\right) dx + \int_{\frac{a+b}{2}}^b \left(f(x) - f\left(\frac{a+b}{2}\right)\right) dx \\ &= \int_{\frac{a+b}{2}}^b f(x)\,dx - \int_a^{\frac{a+b}{2}} f(x)\,dx, \end{aligned}$$

$$\begin{aligned}\int_a^b \left|x-\frac{a+b}{2}\right| \left|f'(x)\right| dx &= \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-x\right) f'(x)\,dx + \int_{\frac{a+b}{2}}^b \left(x-\frac{a+b}{2}\right) f'(x)\,dx \\ &= \left(\frac{a+b}{2}-x\right) f(x)\Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} f(x)\,dx \\ &\quad + \left(x-\frac{a+b}{2}\right) f(x)\Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b f(x)\,dx \\ &= -\frac{b-a}{2} f(a) + \int_a^{\frac{a+b}{2}} f(x)\,dx + \frac{b-a}{2} f(b) - \int_{\frac{a+b}{2}}^b f(x)\,dx.\end{aligned}$$

Using (2.6) we have

$$\begin{aligned}&\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\qquad \geq \frac{1}{2(b-a)}\left|\int_{\frac{a+b}{2}}^b f(x)\,dx - \int_a^{\frac{a+b}{2}} f(x)\,dx \right.\\ &\qquad\qquad \left. - \left[\frac{b-a}{2} f(b) - \frac{b-a}{2} f(a) + \int_a^{\frac{a+b}{2}} f(x)\,dx - \int_{\frac{a+b}{2}}^b f(x)\,dx\right]\right| \\ &\qquad = \frac{1}{2(b-a)}\left|2\int_{\frac{a+b}{2}}^b f(x)\,dx - 2\int_a^{\frac{a+b}{2}} f(x)\,dx - \frac{b-a}{2}\left[f(b)-f(a)\right]\right|,\end{aligned}$$

which is clearly equivalent to (2.9).

A similar argument may be done if f is monotonic nonincreasing and we omit the details. □

3. APPLICATIONS FOR SPECIAL MEANS

Let us recall the following means:

a) The *arithmetic mean*

$$A(a,b) := \frac{a+b}{2}, \quad a,b > 0,$$

b) The *geometric mean*

$$G(a,b) := \sqrt{ab}; \quad a,b \geq 0,$$

c) The *harmonic mean*

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}; \quad a,b > 0,$$

d) The *identric mean*

$$I(a,b) := \begin{cases} \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } \; b \neq a \\ a & \text{if } \; b = a \end{cases}; \quad a,b > 0$$

e) The *logarithmic mean*

$$L(a,b) := \begin{cases} \dfrac{b-a}{\ln b - \ln a} & \text{if} \quad b \neq a \\ a & \text{if} \quad b = a \end{cases} ; \quad a,b > 0$$

f) The $p-$*logarithmic mean*

$$L_p(a,b) := \begin{cases} \left(\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}} & \text{if} \quad b \neq a, \quad p \in \mathbb{R}\backslash\{-1,0\} \\ a & \text{if} \quad b = a \end{cases} ; \quad a,b > 0.$$

It is well known that, if, on denoting $L_{-1}(a,b) := L(a,b)$ and $L_0(a,b) := I(a,b)$, then the function $\mathbb{R} \ni p \to L_p(a,b)$ is strictly monotonic increasing and, in particular, the following classical inequalities are valid

$$\min\{a,b\} \leq H(a,b) \leq G(a,b) \leq L(a,b) \leq I(a,b) \leq A(a,b) \leq \max\{a,b\} \tag{3.1}$$

for any $a,b > 0$.

The following proposition holds:

Proposition 3.1. *Let* $0 < a < b < \infty$. *Then we have the following refinement for the inequality* $A \geq L$:

$$A - L \geq \frac{AL}{b-a}\left[\left(\frac{G}{A}\right)^2 - \ln\left(\frac{G}{A}\right)^2 - 1\right] \geq 0. \tag{3.2}$$

The proof follows by Theorem 2.1 on choosing $f : [a,b] \to (0,\infty)$, $f(t) = 1/t$ and we omit the details.

The following proposition contains a refinementof the following well known inequality

$$\frac{1}{2}\left(A^{-1} + H^{-1}\right) \geq L^{-1}.$$

Proposition 3.2. *With the above assumption for* a *and* b *we have*

$$\frac{1}{2}\left(A^{-1} + H^{-1}\right) - L^{-1} \geq \frac{1}{b-a}\left[\left(\frac{A}{G}\right)^2 - \ln\left(\frac{A}{G}\right)^2 - 1\right] \geq 0. \tag{3.3}$$

The proof follows by Theorem 2.3 for the same funcion $f : [a,b] \to (0,\infty)$, $f(t) = 1/t$, which is monotonic and convex on $[a,b]$, and the details are omitted.

One may state other similar results that improve classical inequalities for means by choosing appropriate convex functions f. However, they will not be stated below.

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