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INEQUALITIES BETWEEN THE SUM OF SQUARES AND THE EXPONENTIAL OF SUM OF A NONNEGATIVE SEQUENCE

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ABSTRACT. Using a standard argument, the following inequality between the sum of squares and the exponential of sum of a nonnegative sequence is established:

$$\frac{e^2}{4}\sum_{i=1}^{n} x_i^2 \le \exp\left(\sum_{i=1}^{n} x_i\right),$$

where $n \ge 2$, $x_i \ge 0$ for $1 \le i \le n$, and the constant $\frac{e^2}{4}$ is the best possible.

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1. INTRODUCTION

In the 2004 Master Graduate Admission Examination of Mathematical Analysis of the Beijing Institute of Technology, the following inequality, which was brought up by one of the author's students, was asked to be shown: For $(x, y) \in [0, \infty) \times [0, \infty)$, show

$$\frac{x^2 + y^2}{4} \le \exp(x + y - 2). \tag{1.1}$$

The aim of this paper is to give a generalization of inequality (1.1).

For our own convenience, we introduce the following notations:

$$[0, \infty)^n \triangleq \underbrace{[0, \infty) \times [0, \infty) \times \cdots \times [0, \infty)}_{n \text{ times}} \tag{1.2}$$

and

$$(0, \infty)^n \triangleq \underbrace{(0, \infty) \times (0, \infty) \times \cdots \times (0, \infty)}_{n \text{ times}} \tag{1.3}$$

for $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers.

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The main results of this paper are the following theorems.

Theorem 1.1. *For $(x_1, x_2, \dots, x_n) \in [0,\infty)^n$ and $n \geq 2$, inequality*

$$\frac{e^2}{4}\sum_{i=1}^{n} x_i^2 \leq \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1.4}$$

is valid. Equality in (1.4) *holds if $x_i = 2$ for some given $1 \leq i \leq n$ and $x_j = 0$ for all $1 \leq j \leq n$ with $j \neq i$. So, the constant $\frac{e^2}{4}$ in* (1.4) *is the best possible.*

Theorem 1.2. *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. Then*

$$\frac{e^2}{4}\sum_{i=1}^{\infty} x_i^2 \leq \exp\left(\sum_{i=1}^{\infty} x_i\right). \tag{1.5}$$

Equality in (1.5) *holds if $x_i = 2$ for some given $i \in \mathbb{N}$ and $x_j = 0$ for all $j \in \mathbb{N}$ with $j \neq i$. So, the constant $\frac{e^2}{4}$ in* (1.5) *is the best possible.*

Remark 1.3. Taking $n = 2$ and $(x_1, x_2) = (x, y)$ in (1.4) easily leads to inequality (1.1).

Taking $x_i = x$ and $x_j = y$ for some given $i, j \in \mathbb{N}$ and $x_k = 0$ for all $k \in \mathbb{N}$ with $k \neq i$ and $k \neq j$ in inequality (1.5) also clearly leads to inequality (1.1).

Remark 1.4. Inequality (1.4) can be rewritten as

$$\frac{e^2}{4}\sum_{i=1}^{n} x_i^2 \leq \prod_{i=1}^{n} e^{x_i} \tag{1.6}$$

or

$$\frac{e^2}{4}\|\boldsymbol{x}\|_2^2 \leq \exp\|\boldsymbol{x}\|_1, \tag{1.7}$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\|\cdot\|_p$ denotes the p-norm.

Remark 1.5. Inequality (1.5) can be rewritten as

$$\frac{e^2}{4}\sum_{i=1}^{\infty} x_i^2 \leq \prod_{i=1}^{\infty} e^{x_i} \tag{1.8}$$

which is equivalent to inequality (1.7) for $\boldsymbol{x} = (x_1, x_2, \dots) \in [0,\infty)^{\infty}$.

Remark 1.6. Taking $x_i = \frac{1}{i}$ for $i \in \mathbb{N}$ in (1.4) and rearranging gives

$$2 - 2\ln 2 + \ln\left(\sum_{i=1}^{n}\frac{1}{i^2}\right) \leq \sum_{i=1}^{n}\frac{1}{i}. \tag{1.9}$$

Taking $x_i = \frac{1}{i^s}$ for $i \in \mathbb{N}$ and $s > 1$ in (1.5) and rearranging gives

$$2 - 2\ln 2 + \ln\left(\sum_{i=1}^{\infty}\frac{1}{i^{2s}}\right) = 2 - 2\ln 2 + \ln[\zeta(2s)] \leq \sum_{i=1}^{\infty}\frac{1}{i^s} = \zeta(s), \tag{1.10}$$

where ζ denotes the well known Riemann Zeta function.

2. PROOFS OF THEOREMS

Now we are in a position to prove our theorems.

Proof of Theorem 1.1. Let

$$f(x_1, x_2, \dots, x_n) = \ln\left(\sum_{i=1}^{n} x_i^2\right) - \sum_{i=1}^{n} x_i \tag{2.1}$$

for $(x_1, x_2, \dots, x_n) \in [0, \infty)^n \setminus \{(0, 0, \dots, 0)\}$. Simple calculation results in

$$\frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_k} = \frac{2x_k}{\sum_{i=1}^{n} x_i^2} - 1, \tag{2.2}$$

$$\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_k^2} = \frac{2\left(\sum_{i\neq k}^{n} x_i^2 - x_k^2\right)}{\left(\sum_{i=1}^{n} x_i^2\right)^2}, \tag{2.3}$$

$$\frac{\partial^2 f(x_1, x_2, \dots, x_n)}{\partial x_\ell \partial x_m} = -\frac{4x_\ell x_m}{\left(\sum_{i=1}^{n} x_i^2\right)^2}, \tag{2.4}$$

where $1 \le k, \ell, m \le n$ and $\ell \neq m$. The system of equations

$$\frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_k} = 0 \quad \text{for} \quad 1 \le k \le n, \tag{2.5}$$

which is equivalent to

$$\sum_{i\neq k} x_i^2 + (x_k - 1)^2 = 1 \quad \text{for} \quad 1 \le k \le n, \tag{2.6}$$

has a unique nonzero solution $x_i = \frac{2}{n}$ for $1 \le i \le n$. Thus, the point $\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)$ is a unique critical point of the function $f(x_1, x_2, \dots, x_n)$, which is located in the interior of $[0, \infty)^n \setminus \{(0, 0, \dots, 0)\}$.

Straightforward computation gives us

$$\begin{aligned} \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_k^2} &= \frac{n-2}{2}, \\ \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_\ell \partial x_m} &= -1, \end{aligned} \tag{2.7}$$

$$D_i = \begin{vmatrix} \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_1^2} & \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_1 \partial x_i} \\ \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_2 \partial x_1} & \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_2^2} & \cdots & \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_2 \partial x_i} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_i \partial x_1} & \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_i \partial x_2} & \cdots & \frac{\partial^2 f\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)}{\partial x_i^2} \end{vmatrix} \tag{2.8}$$

$$
= \begin{vmatrix} \frac{n-2}{2} & -1 & \cdots & -1 \\ -1 & \frac{n-2}{2} & \cdots & -1 \\ \cdots & \cdots & \cdots & \cdots \\ -1 & -1 & \cdots & \frac{n-2}{2} \end{vmatrix}
$$
$$
= \left[\frac{n-2}{2} + (i-1)(-1)\right]\left[\frac{n-2}{2} - (-1)\right]^{i-1}
$$
$$
= \left(\frac{n}{2} - i\right)\left(\frac{n}{2}\right)^{i-1}.
$$

Since

$$
D_i \begin{cases} > 0, & \text{if } i < \frac{n}{2}, \\ = 0, & \text{if } i = \frac{n}{2}, \\ < 0, & \text{if } i > \frac{n}{2}, \end{cases} \tag{2.9}
$$

it is affirmed that the critical point $\left(\frac{2}{n}, \frac{2}{n}, \dots, \frac{2}{n}\right)$ located in the interior of $[0,\infty)^n \setminus \{(0,0,\dots,0)\}$ is not an extremal point of the function $f(x_1, x_2, \dots, x_n)$.

The boundary of $[0,\infty)^n \setminus \{(0,0,\dots,0)\}$ is $\cup_{i=0}^{n-1} [0,\infty)^i \times \{0\} \times [0,\infty)^{n-i-1}$.

On the set $[0,\infty)^{n-1} \times \{0\} \setminus \{(0,0,\dots,0)\}$, it is concluded that

$$
f(x_1, \dots, x_{n-1}, 0) = \ln\left(\sum_{k=1}^{n-1} x_k^2\right) - \sum_{k=1}^{n-1} x_k. \tag{2.10}
$$

By the same standard argument as above, it is deduced that the unique critical point, located in the interior of $[0,\infty)^{n-1} \times \{0\} \setminus \{(0,0,\dots,0)\}$, of $f(x_1, \dots, x_{n-1}, 0)$ is $\left(\frac{2}{n-1}, \dots, \frac{2}{n-1}, 0\right)$ which is not an extremal point of $f(x_1, \dots, x_{n-1}, 0)$.

By induction, in the interior of the set $[0,\infty)^i \times \underbrace{\{0\} \times \cdots \times \{0\}}_{n-i \text{ times}} \setminus \{(0,0,\dots,0)\}$ for $2 \le i \le n$, there is no extremal point of $f(x_1, \dots, x_i, 0, \dots, 0)$.

On the set $(0,\infty) \times \underbrace{\{0\} \times \cdots \times \{0\}}_{n-1 \text{ times}}$, it is easy to obtain that the function

$$
f(x_1, 0, \dots, 0) = 2\ln x_1 - x_1
$$

has a maximal point $x_1 = 2$ and the maximal value equals $f(2, 0, \dots, 0) = 2\ln 2 - 2$.

Considering that the function $f(x_1, x_2, \dots, x_n)$ is symmetric with respect to all permutations of the n variables x_i for $1 \le i \le n$ and by induction, we obtain the following conclusion: The maximal value of the function $f(x_1, \dots, x_n)$ on the set $[0,\infty)^n \setminus \{(0,0,\dots,0)\}$ is $2\ln 2 - 2$. Therefore, it follows that

$$
f(x_1, x_2, \dots, x_n) = \ln\left(\sum_{i=1}^{n} x_i^2\right) - \sum_{i=1}^{n} x_i \le 2\ln 2 - 2, \tag{2.11}
$$

which is equivalent to inequality (1.4), on the set $[0,\infty)^n \setminus \{(0,0,\dots,0)\}$.

It is clear that inequality (1.4) holds also at the point $(0, \dots, 0)$. Hence, the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. This can be concluded by letting $n \to \infty$ in Theorem 1.1. □

3. OPEN PROBLEMS

Finally, the following problems can be proposed.

Open Problem 1. *For* $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ *and* $n \geq 2$, *determine the best possible constants* $\alpha_n, \lambda_n \in \mathbb{R}$ *and* $0 < \beta_n, \mu_n < \infty$ *such that*

$$\beta_n \sum_{i=1}^{n} x_i^{\alpha_n} \leq \exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}. \tag{3.1}$$

Open Problem 2. *What is the integral analogue of the double inequality* (3.1)*?*

Open Problem 3. *Can one find applications and practical meanings in mathematics for inequality* (3.1) *and its integral analogues?*