Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 5, Article 137, 2005

ON VECTOR BOUNDARY VALUE PROBLEMS WITHOUT GROWTH RESTICTIONS

CHRISTOPHER C. TISDELL AND LIT HAU TAN

SCHOOL OF MATHEMATICS
THE UNIVERSITY OF NEW SOUTH WALES
SYDNEY 2052, AUSTRALIA
cct@maths.unsw.edu.au

lithau@maths.unsw.edu.au

Received 07 April, 2005; accepted 08 September, 2005
Communicated by D. Hinton

ABSTRACT. Herein, we consider the existence of solutions to second-order systems of two-point boundary value problems (BVPs). The methods used involve the topological transversality approach of Granas et. al. combined with a Bernstein-Nagumo condition from Gaines and Mawhin. The new results allow the treatment of systems of BVPs without growth restrictions in the third variable. The new results also are applicable to systems of BVPs that may have singularities in the right-hand side at the end-points of the interval of existence. Some examples are presented to illustrate the theory.

Key words and phrases: Boundary value problem, No growth restriction, Existence of solutions, *a priori* bounds.

2000 *Mathematics Subject Classification.* Primary 34B15.

1. INTRODUCTION

Consider the existence of solutions to the second-order, ordinary differential equation

$$x'' = F(t, x, x'), \qquad t \in [0, 1], \tag{1.1}$$

subject to some suitable boundary conditions.

Topological methods, used in proving the existence of solutions to boundary value problems, such as: the continuation method of Gaines and Mawhin [5], [6]; or the topological transversality method of Granas, Guenther and Lee [9], [10]; generally rely on guaranteeing *a priori* bounds on solutions (and their derivatives) to the BVP under consideration in such a way that the same *a priori* bounds apply to a certain family of BVPs.

ISSN (electronic): 1443-5756

This research was supported by the Australian Research Council's Discovery Projects DP0450752.

This paper is based on the talk given by the first author within the "International Conference of Mathematical Inequalities and their Applications, I", December 06-08, 2004, Victoria University, Melbourne, Australia [http://rgmia.vu.edu.au/conference].

113-05

A classical issue associated with the preceding discussion is the following question. How can we ensure an *a priori* bound on solutions' derivatives x' to (1.1) with the bound on x' being in terms of an *a priori* bound on possible solutions x? A sufficient condition that guarantees the desired *a priori* bound on x' is traditionally known as a "Bernstein-Nagumo condition".

For scalar-valued BVPs, many authors have formulated Bernstein-Nagumo conditions for (1.1), for example: [3], [18], [15], [14], [22], [1], [10], [16] and also see references therein.

However, for vector-valued BVPs (i.e. $F : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$), less is known about sufficient Bernstein-Nagumo conditions, perhaps to the Bernstein-Nagumo question becoming more difficult than in the scalar-valued situation (see [2, Remark 1.41] or [12] for more discussion and some examples.)

Authors such as: Hartman [11]; Schmitt and Thompson [21]; Gaines and Mawhin [6]; Fabry [4]; George and Sutton [7]; and George and York [8] have all presented interesting Bernstein-Nagumo conditions for vector BVPs. Their conditions involved growth-type conditions on F in x' or the existence of suitable Lyapunov functions.

Herein, we consider vector equations of the type

$$x'' = f(t, x, x'), \qquad t \in [0, 1], \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and (1.2) is subject to the following boundary conditions:

$$x'(0) = g_1(x(0)), \qquad x'(1) = g_2(x(1)), \quad \text{(where each} \quad g_i : \mathbb{R}^n \to \mathbb{R}^n). \tag{1.3}$$

Well-known special cases of the rather general boundary conditions (1.3) include: the Sturm-Liouville boundary conditions

$$\alpha x(0) - \beta x'(0) = C, \gamma x(1) + \delta x'(1) = D, \tag{1.4}$$
$$\alpha, \beta, \gamma, \delta \text{ are constants in } \mathbb{R}; C, D \text{ are constants in } \mathbb{R}^n;$$

and the homogenous Neumann boundary conditions

$$x'(0) = 0, \qquad x'(1) = 0; \tag{1.5}$$

plus variations of the above (see Remarks 2.5 and 2.7), including nonlinear boundary conditions.

In Section 2 we combine the topological transversality method of [10, Theorem 2.6] in conjunction with a general Berstein-Nagumo condition from [6, Proposition 5.1]. The combination leads to novel and quite general existence theorems for solutions to the above systems of BVPs. In particular, the new results extend the workings of [10] and [6] in the sense that the new results herein allow the treatment of certain classes of BVPs whereas the theorems of [10] and [6] may not directly apply.

In Section 3 we briefly consider systems of BVPs with singularities in the right-hand side.

Examples are presented throughout the paper to demonstrate the applicability of the new theorems. It appears that no existing theory in the literature is applicable to the examples given.

2. EXISTENCE RESULTS

To generate our new topological transversality-based existence theorems, we consider the following family of BVPs:

$$a(t)x'' + b(t)x' + c(t)x = \lambda g(t, x, x'), \quad t \in [0, 1], \tag{2.1}$$

$$a_0 x(0) + a_1 x'(0) + a_2 x(1) + a_3 x'(1) = \lambda \psi_1(x(0), x'(0), x(1), x'(1)), \tag{2.2}$$

$$b_0 x(0) + b_1 x'(0) + b_2 x(1) + b_3 x'(1) = \lambda \psi_2(x(0), x'(0), x(1), x'(1)), \tag{2.3}$$

where: $\lambda \in [0, 1]$; a, b, c are continuous functions with $a(t) \neq 0$ for any $t \in [0, 1]$; each a_i and b_i are constants; $g : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and each $\psi_i : \mathbb{R}^{4n} \to \mathbb{R}^n$.

Below we denote $\|\cdot\|$ as the usual Euclidean norm and $\langle\cdot,\cdot\rangle$ as the usual inner product on $\mathbb{R}^n$.

To streamline the proofs of our results, we will use the following existence theorem, a vector-variant of [10, Theorem 6.1, Chap.II].

Theorem 2.1. *Let g and each ψ_i be continuous and let $R > 0$ be a constant independent of λ. If:*

$$\text{the family (2.1)–(2.3) has only the zero solution for } \lambda = 0; \text{ and} \tag{2.4}$$

$$\text{for } \lambda \in (0,1] \text{ all possible solutions } x \in C^2([0,1];\mathbb{R}^n) \text{ to (2.1)–(2.3) satisfy} \tag{2.5}$$

$$\max\{\|x(t)\|,\ \|x'(t)\|,\ \|x''(t)\|\} < R, \quad t \in [0,1],$$

then for $\lambda = 1$ the BVP (2.1)–(2.3) has at least one solution.

Theorem 2.2. *Let f be continuous and let M, N be positive constants with*

$$N > \max\left\{\frac{\|C\| + |\alpha|M}{|\beta|},\quad \frac{\|D\| + |\gamma|M}{|\delta|}\right\}. \tag{2.6}$$

If

$$\alpha/\beta > 0,\ \ \gamma/\delta < 0,\ \ \alpha(\gamma+\delta) + \beta\gamma \neq 0; \tag{2.7}$$

and

$$\langle x, f(t,x,x')\rangle + \|x'\|^2 > 0, \quad \text{for} \quad t \in [0,1],\ \|x\| \geq M,\ \langle x, x'\rangle = 0; \tag{2.8}$$

and

$$\langle x', f(t,x,x')\rangle > 0, \quad \text{for} \quad t \in [0,1],\ \|x\| \leq M,\ \|x'\| = N, \tag{2.9}$$

then (1.2), (1.4) has at least one solution.

Proof. Consider the family of BVPs:

$$x'' = \lambda f(t,x,x'), \quad t \in [0,1], \tag{2.10}$$

$$\alpha x(0) - \beta x'(0) = \lambda C, \tag{2.11}$$

$$\gamma x(1) + \delta x'(0) = \lambda D, \tag{2.12}$$

for $\lambda \in [0,1]$ and see that this is in the form (2.1)–(2.2), with $g = f$.

Let x be a solution to (2.10)–(2.12). Since $\alpha(\gamma+\delta)+\beta\gamma \neq 0$, note that, for $\lambda = 0$, the above family of BVPs only has the zero solution by direct calculation.

We show that (2.8) and (2.7) imply

$$\|x(t)\| \leq M_1 := \max\left\{\frac{\|C\|}{|\alpha|},\ M,\ \frac{\|D\|}{|\gamma|}\right\}, \quad \text{for} \quad t \in [0,1]. \tag{2.13}$$

Consider $r_1(t) = \|x(t)\|^2$ for $t \in [0,1]$ and let $t_0 \in [0,1]$ be such that $r_1(t_0) = \max_{t\in[0,1]} r(t)$. If $r_1(t_0) = 0$ then $r_1(t) = 0$ for all $t \in [0,1]$ and obviously $\|x(t)\| = 0 < M$ for all $t \in [0,1]$ and all $M > 0$, so assume $r_1(t_0) > 0$ from now on.

If $t_0 = 0$ then

$$\begin{aligned} 0 \geq r_1'(t_0) &= 2\langle x(0), x'(0)\rangle \\ &= 2\langle x(0), \frac{\alpha x(0) - \lambda C}{\beta}\rangle \quad \text{from (2.11)} \\ &= 2\frac{\alpha}{\beta}\|x(0)\|^2 \left(1 - \frac{\langle x(0), \lambda C\rangle}{\alpha\|x(0)\|^2}\right). \end{aligned}$$

Thus, by (2.7),

$$1 \leq \frac{\langle x(0), \lambda C\rangle}{\alpha \|x(0)\|^2} \leq \frac{\|x(0)\|\|C\|}{|\alpha|\ \|x(0)\|^2},$$

gives $\|x(0)\| \leq \|C\|/|\alpha|$ and we must have $\|x(t)\| \leq \|C\|/|\alpha|$ for all $t \in [0,1]$.

If $t_0 = 1$ then $0 \leq r_1'(1)$ and a similar argument to the case $t_0 = 0$ gives $\|x(t)\| \leq \|D\|/|\gamma|$ for all $t \in [0,1]$.

If $t_0 \in (0,1)$ and $r_1(t_0) \geq M^2$ then $0 = r_1'(t_0) = 2\langle x(t_0), x'(t_0)\rangle$. We also have

$$\begin{aligned} 0 \geq r_1''(t_0) &= 2\left[\langle x(t_0), x''(t_0)\rangle + \|x'(t_0)\|^2\right] \\ &= 2\left[\langle x(t_0), \lambda f(t_0, x(t_0), x'(t_0))\rangle + \|x'(t_0)\|^2\right] \\ &\geq 2\lambda\left[\langle x(t_0), f(t_0, x(t_0), x'(t_0))\rangle + \|x'(t_0)\|^2\right] \\ &> 0, \end{aligned}$$

by (2.8), a contradiction. Hence we have $\|x(t_0)\| < M$ for all $t_0 \in (0,1)$.

Combining all of the above bounds we obtain (2.13).

Let $x \in C^2([0,1];\mathbb{R}^n)$ be a solution to (1.2) with $\|x(t)\| \leq M_1$ for $t \in [0,1]$. We now show that (2.6) and (2.9) imply $\|x'(t)\| < N$ for all $t \in [0,1]$.

Argue by contradiction by assuming $r(t_0) = \|x'(t_0)\|^2 - N^2 \geq 0$ for some $t_0 \in [0,1]$ such that $\max_{t\in[0,1]} r(t) = r(t_0)$. If $t_0 = 0$ then rearranging the boundary conditions we obtain

$$\|x'(0)\| = \left\|\frac{\lambda C - \alpha x(0)}{\beta}\right\| \leq \frac{\|C\| + |\alpha| M}{|\beta|} < N,$$

and thus $r(0) < 0$. Similarly,

$$\|x'(1)\| \leq \frac{\|D\| + |\gamma| M}{|\delta|} < N,$$

and thus $r(1) < 0$. So we see that $t_0 \in (0,1)$.

Now since $r(1) < 0$ and $r(t_0) \geq 0$ we must have a point $t_1 \in [t_0, 1)$ such that $r(t_1) = 0$ and

$$\begin{aligned} 0 \geq r'(t_1) &= 2\langle x'(t_1),\ x''(t_1)\rangle \\ &= 2\langle x'(t_1), \lambda f(t_1, x(t_1, x'(t_1))\rangle \\ &> 0, \end{aligned}$$

for all $\lambda \in (0,1]$ by (2.9), a contradiction.

It is clear to see that once bounds on x and x' are found, a bound on x'' follows naturally, as

$$\begin{aligned} \|x''(t)\| = \|\lambda f(t,x,x')\| &\leq \|f(t,x,x')\| \\ &\leq P \quad \text{for} \quad t \in [0,1],\ \|x\| \leq M,\ \|x'\| \leq N, \end{aligned}$$

for some $P \geq 0$.

So we see that there exists an $R > 0$ with

$$R = \max\left\{\max\left\{\frac{\|C\|}{|\alpha|},\ M,\ \frac{\|D\|}{|\gamma|}\right\},\ N,\ P\right\} + 1$$

such that (2.5) holds.

Thus, by Theorem 2.1, the family (2.10)–(2.12) has a solution for $\lambda = 1$. For $\lambda = 1$, (2.10)–(2.12) is equivalent to (1.2)–(1.4) and hence the result follows. □

Example 2.1. Let $x = (x_1, x_2)$ and $p = (p_1, p_2)$. Consider (1.2), (1.4) for $n = 2$, where

$$f(t, x, p) = f(t, x_1, x_2, p_1, p_2)$$

$$= \begin{pmatrix} x_1 e^{x_2 p_2} & + & x_1^2 p_1^3 & + & p_1 \\ x_2 e^{x_2 p_2} & + & x_2^2 p_2^3 & + & p_2 \end{pmatrix}, \quad t \in [0, 1],$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} - 2 \begin{pmatrix} x_1'(0) \\ x_2'(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} + 2 \begin{pmatrix} x_1'(1) \\ x_2'(1) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

There is no growth condition applicable to f and thus the theorems of [11], [21], [4] do not apply. We will apply Theorem 2.2.

Firstly, for $\|x\| \geq M$, with M to be chosen below, and $\langle x, p \rangle = 0$, consider

$$\begin{aligned} \langle x, f(t, x, p) \rangle &= x_1^2 e^{x_2 p_2} + (x_1 p_1)^3 + x_1 p_1 + x_2^2 e^{x_2 p_2} + (x_2 p_2)^3 + x_2 p_2 \\ &= e^{x_2 p_2} [x_1^2 + x_2^2] \quad \text{(since } x_1 p_1 = -x_2 p_2) \\ &> 0 \quad \text{for any positive choice of } M. \end{aligned}$$

For convenience, choose $M = 1$, thus (2.8) holds. Now, for $\|x\| \leq 1,\ \|p\| = N = 2$ we have

$$\begin{aligned} \langle p, f(t, x, p) \rangle &= p_1 x_1 e^{x_2 p_2} + x_1^2 p_1^4 + p_1^2 + p_2 x_2 e^{x_2 p_2} + x_2^2 p_2^4 + p_2^2 \\ &\geq 4 + e^{x_2 p_2} [p_1 x_1 + p_2 x_2] \\ &\geq 2 > 0 \quad \text{for} \quad \|x\| \leq 1,\ \|p\| = 2, \end{aligned}$$

thus (2.9) holds.

It is easy to see that (2.6) holds for our choice of $M = 1$ and $N = 2$ and for the given boundary conditions. Thus Theorem 2.2 is applicable and the BVP has a solution.

Theorem 2.3. *Let f be continuous and let M, N be positive constants with*

$$2N^2 \geq -\langle x, x' \rangle, \quad \textit{for} \quad \|x\| \leq M,\ \|x'\| = N. \tag{2.14}$$

If (2.8) and (2.9) hold then (1.2), (1.5) has at least one solution.

Proof. Consider the family of BVPs:

$$x'' - 2x' - x = \lambda[f(t, x, x') - 2x' - x], \quad t \in [0, 1], \tag{2.15}$$

$$x'(0) = 0, \tag{2.16}$$

$$x'(1) = 0, \tag{2.17}$$

for $\lambda \in [0, 1]$ and see that this is in the form (2.1)–(2.3) with $g(t, x, x') = f(t, x, x') - 2x' - x$.

Let x be a solution to (2.15)–(2.17). By direct calculation, the only solution to (2.15)–(2.17) for $\lambda = 0$ is $x = 0$, so (2.4) holds.

Now rearranging (2.15) we obtain

$$\begin{aligned} x'' &= \lambda f(t, x, x') + 2(1 - \lambda)x' + (1 - \lambda)x, \quad t \in [0, 1], \\ &:= q_\lambda(t, x, x'). \end{aligned} \tag{2.18}$$

We show that $\|x(t)\| < M$ for all $t \in [0, 1]$ and all $\lambda \in (0, 1]$. Consider $r(t) = \|x(t)\|^2$ for $t \in [0, 1]$ and let $t_0 \in [0, 1]$ be such that $r(t_0) = \max_{t \in [0,1]} r(t) \geq M^2$.

If $t_0 = 0$ then the boundary conditions give $\langle x(0), x'(0)\rangle = 0$. Therefore, by (2.8) we have

$$\begin{aligned} 0 &< 2\left[\langle x(0), f(0,x(0),x'(0))\rangle + \|x'(0)\|^2\right], \quad \text{and so} \\ 0 &< 2\lambda\left[\langle x(0), f(0,x(0),x'(0))\rangle + \|x'(0)\|^2\right], \quad \text{for } \lambda\in(0,1] \\ &\le 2\left[\langle x(0), \lambda f(0,x(0),x'(0))\rangle + 2(1-\lambda)\langle x(0),x'(0)\rangle + (1-\lambda)\|x(0)\|^2 + \|x'(0)\|^2\right] \\ &= 2\left[\langle x(0), q_\lambda(0,x(0),x'(0))\rangle + \|x'(0)\|^2\right] \\ &= 2\left[\langle x(0), x''(0)\rangle + \|x'(0)\|^2\right] \\ &= r''(0), \end{aligned}$$

so $r'(t)$ is strictly increasing for t near 0. Therefore $0 = r'(0) < r'(t)$ for t near 0. This means that $r(t)$ is increasing for t near 0, that is, $r(0) < r(t)$ and hence $r(0) \neq \max_{t\in[0,1]} r(t)$.

If $t_0 = 1$ then a similar argument to the case for $t_0 = 0$ gives $\|x(1)\| < M$.

If $t_0 \in (0,1)$ then an identical argument to the proof of Theorem 2.2 gives $\|x(t_0)\| < M$. Hence we have $\|x(t)\| < M$ for all $t\in[0,1]$.

Consider solutions $x \in C^2([0,1];\mathbb{R}^n)$ with $\|x(t)\| \le M$ for $t\in[0,1]$. We now show that (2.9) imply $\|x'(t)\| < N$ for all $t\in[0,1]$.

Argue by contradiction by assuming $r_1(t_0) = \|x'(t_0)\|^2 - N^2 \ge 0$ for some $t_0\in[0,1]$ such that $\max_{t\in[0,1]} r_1(t) = r_1(t_0)$. The boundary conditions give $r_1(0) < 0$ and $r_1(1) < 0$. So we see that $t_0 \in (0,1)$. Now since $r_1(1) < 0$ and $r_1(t_0) \ge 0$ we must have a point $t_1 \in [t_0, 1)$ such that $r_1(t_1) = 0$ and

$$\begin{aligned} 0 \ge r_1'(t_1) &= 2\langle x'(t_1), x''(t_1)\rangle \\ &= 2\langle x'(t_1), q_\lambda(x(t_1), x(t_1), x'(t_1))\rangle \\ &= \lambda\langle x'(t_1), f(t_1,x(t_1),x'(t_1))\rangle + 2(1-\lambda)\|x'(t_1)\|^2 + (1-\lambda)\langle x(t_1), x'(t_1)\rangle \\ &= \lambda\langle x'(t_1), f(t_1,x(t_1),x'(t_1))\rangle + (1-\lambda)[2N^2 - \langle x(t_1), x'(t_1)\rangle] \\ &> 0, \end{aligned}$$

for all $\lambda\in(0,1]$, a contradiction.

Hence we have $\|x'(t)\| < N$ for $t\in[0,1]$.

Since *a priori* bounds are now obtained on x and x', the *a priori* bound on x'' naturally follows as in the proof of Theorem 2.2.

Hence, by Theorem 2.1, the family (2.15)–(2.17) has a solution for $\lambda = 1$, which is identical to the BVP (1.2), (1.5) and hence the result follows. □

Example 2.2. Consider the scalar BVP (1.2), (1.5) where f is given by the right-hand side of

$$x'' = (x + 1 + x')e^{x'}, \quad t\in[0,1]. \tag{2.19}$$

It is not difficult to show that (2.19) satisfies (2.8), (2.9) and (2.14) for $M = 3/2$ and $N = 2$. Thus, by Theorem 2.3 we conclude that the scalar BVP (2.19), (1.5) has at least one solution.

Theorem 2.4. *Let f, g_1 and g_2 be continuous and let M, N be positive constants such that*

$$N > \max\left\{\max_{\|x\|\le M}\|g_1(x)\|, \max_{\|x\|\le M}\|g_2(x)\|\right\}. \tag{2.20}$$

If

$$\langle z, g_1(z)\rangle > 0,\ \langle z, g_2(z)\rangle < 0, \quad \textit{for all} \quad \|z\| \ge M, \tag{2.21}$$

and (2.8), (2.9) hold, then (1.2), (1.3) has at least one solution.

Proof. Consider the family of BVPs:

$$x'' - 2x' - x = \lambda[f(t, x, x') - 2x' - x], \quad t \in [0, 1], \tag{2.22}$$

$$x'(0) = \lambda g_1(x(0)), \tag{2.23}$$

$$x'(1) = \lambda g_2(x(0)), \tag{2.24}$$

for $\lambda \in [0, 1]$.

Let x be a solution to (2.22)–(2.24). See that, for $\lambda = 0$, the above family of BVPs only has the zero solution.

We show that $\|x(t)\| \leq M$, for $t \in [0, 1]$. Consider $r_1(t) = \|x(t)\|^2$ for $t \in [0, 1]$ and let $t_0 \in [0, 1]$ be such that $r_1(t_0) = \max_{t\in[0,1]} r_1(t) \geq M^2$.

If $t_0 = 0$ then

$$\begin{aligned} 0 \geq r_1'(t_0) &= 2\langle x(0), x'(0)\rangle \\ &= 2\langle x(0), \lambda g_1(x(0))\rangle \quad \text{from (2.11)} \\ &> 0 \end{aligned}$$

a contradiction.

If $t_0 = 1$ then $0 \leq r_1'(1)$ and a similar arguement to the case $t_0 = 0$ gives another contradiction.

If $t_0 \in (0, 1)$ such that $r_1(t_0) \geq M^2$ then $0 = r_1'(t_0) = 2\langle x(t_0), x'(t_0)\rangle$ and $0 \geq r_1''(t_0)$ with a contradiction arising by (2.8) as in the proof of Theorem 2.3. Hence we have $\|x(t_0)\| < M$ for all $t_0 \in (0, 1)$.

Consider solutions $x \in C^2([0, 1]; \mathbb{R}^n)$ to (2.22) with $\|x(t)\| \leq M$ for $t \in [0, 1]$. We now show that (2.20) and (2.9) imply $\|x'(t)\| < N$ for all $t \in [0, 1]$.

Argue by contradiction by assuming $r(t_0) = \|x'(t_0)\|^2 - N^2 \geq 0$ for some $t_0 \in [0, 1]$ such that $\max_{t\in[0,1]} r(t) = r(t_0)$. By (2.20) we have $r(0) < 0$ and $r(1) < 0$ in a similar fashion to the argument in the proof of Theorem 2.2. So we see that $t_0 \in (0, 1)$. Now since $r(1) < 0$ and $r(t_0) \geq 0$ we must have a point $t_1 \in [t_0, 1)$ such that $r(t_1) = 0$ and $0 \geq r'(t_1)$ with a contradiction being reached as in the proof of Theorem 2.3.

Hence we have $\|x'(t)\| < N$ for $t \in [0, 1]$.

Since *a priori* bounds are now obtained on x and x', the *a priori* bound on x'' naturally follows as in the proof of Theorem 2.2.

Hence, by Theorem 2.1, the family (2.15)–(2.17) has a solution for $\lambda = 1$, which is just the BVP (1.2), (1.4) and hence the result follows. □

Remark 2.5. Theorem 2.4 may be generalised to treat (1.2) subject to

$$x'(0) = g_3(x(0), x'(0), x(1), x'(1)) \tag{2.25}$$

$$x'(1) = g_4(x(0), x'(0), x(1), x'(1)) \tag{2.26}$$

in the following way.

Let f, g_3 and g_4 be continuous and let M, N be positive constants. Suppose each $g_i(h, i, j, k)$ is bounded on the set D, where

$$D := \{(h, i, j, k) \in \mathbb{R}^{4n} : \|h\| \leq M,\ \|j\| \leq M,\ (i, k) \in \mathbb{R}^{2n}\}$$

and

$$N > \max\left\{\sup_{(h,i,j,k)\in D} \|g_3(h, i, j, k)\|, \sup_{(h,i,j,k)\in D} \|g_4(h, i, j, k)\|\right\}. \tag{2.27}$$

If (2.8), (2.9) hold and

$$\langle h, g_3(h,i,j,k)\rangle > 0, \quad \text{for} \quad h \neq 0,\ (i,j,k) \in \mathbb{R}^{3n}$$
$$\langle j, g_4(h,i,j,k)\rangle < 0, \quad \text{for} \quad j \neq 0,\ (h,i,k) \in \mathbb{R}^{3n}.$$

then the BVP (1.2), (2.25), (2.26) has at least one solution.

Remark 2.6. It is also clear that by combining the relevant bounding inequalities used in each of the Theorems in this section, the treatment of (1.2) subject to any of the following boundary conditions is possible:

$$x'(0) = 0, \quad \gamma x(1) + \delta x'(1) = D, \quad \gamma/\delta < 0,$$
$$\alpha x(0) - \beta x'(0) = C, \quad x'(0) = 0, \quad \alpha/\beta > 0,$$
$$x'(0) = 0, \quad x'(1) = g_2(x(0)),$$
$$x'(0) = g_1(x(0)), \quad x'(1) = 0,$$

and so on.

Remark 2.7. In Theorems 2.2–2.4 the inequality (2.9) could be reversed and the existence theorems would still hold. However, for brevity we omit the statement of these new results.

3. On BVPs with Singularities

In this final section we consider systems of BVPs that may have singularities in the right-hand side. Consider

$$x'' = \eta(t) f(t, x, x'), \quad t \in [0,1], \tag{3.1}$$

subject to any of the boundary conditions (1.3)–(1.5). Here $1/\eta : [0,1] \to [0,\infty)$ is continuous with $\eta > 0$ on $(0,1)$, η is integrable on $[0,1]$ and η may be singular at $t = 0$ or at $t = 1$. Probably the most famous type of BVP involving singularities in the right-hand side is the Thomas-Fermi equation, ($n(t) = 1/\sqrt{t}$, $f(t,x,x') = x^{3/2}$) which appears in the study of electron distribution in an atom [13].

In view of the proof of [17, Theorem 1.5] and [17, Theorem 0.1], in order to prove the existence of solutions to (3.1) subject to (2.2), (2.3), it is sufficient to show that:
(i) all solutions to

$$a(t)x'' + b(t)x' + c(t)x = \lambda \eta(t) g(t, x, x'), \quad t \in [0,1], \tag{3.2}$$
$$a_0 x(0) + a_1 x'(0) + a_2 x(1) + a_3 x'(1) = \lambda \psi_1(x(0), x'(0), x(1), x'(1)), \tag{3.3}$$
$$b_0 x(0) + b_1 x'(0) + b_2 x(1) + b_3 x'(1) = \lambda \psi_2(x(0), x'(0), x(1), x'(1)), \tag{3.4}$$

satisfy

$$\max\{\|x(t)\|, \|x'(t)\|\} < R, \quad t \in [0,1],$$

for some $R > 0$, independent of $\lambda \in (0,1]$; and
(ii) that for $\lambda = 0$, the family (3.2)–(3.4) has only the zero solution. Then, for $\lambda = 1$, the BVP (3.2)–(3.4) will have at least one solution in $C^1([0,1];\mathbb{R}^n)$. (This solution will also be in $C^2((0,1);\mathbb{R}^n)$ because solutions to (3.2) are absolutely continuous on $[0,1]$ and satisfy (3.2) almost everywhere.) Above, g and each ψ_i are as in Section 2.

There are only minor modifications needed in the proofs of Section 2 to obtain the *a priori* bounds on solutions to (3.1) and therefore we only present the statement of the new theorems for brevity.

Theorem 3.1. *Let η be as above and let f be continuous. Let M, N be positive constants satisfying* (2.6) *If* (2.7), (2.8) (2.9) *hold then* (3.1), (1.4) *has at least one solution.*

Theorem 3.2. *Let η be as above and let f be continuous. Let M, N be positive constants satisfying* (2.14)*. If* (2.8) *and* (2.9) *hold then* (3.1), (1.5) *has at least one solution.*

Theorem 3.3. *Let η be as above and let f, g_1 and g_2 be continuous. Let M, N be positive constants such* (2.20) *holds. If* (2.21) *holds and* (2.8), (2.9) *hold, then* (3.1), (1.3) *has at least one solution.*

Example 3.1. Let f and the boundary conditions be defined as in Example 2.1. Consider $\eta(t) = 1/\sqrt{t}$. Then by Theorem 3.1 the singular BVP under consideration has at least one solution $x \in C^1([0,1];\mathbb{R}^n) \cap C^2((0,1);\mathbb{R}^n)$.

Example 3.2. Let $x = (x_1, x_2)$ and $p = (p_1, p_2)$. Consider (3.1), (1.4) for $n = 2$, where $\eta(t) = 1/\sqrt{t}$

$$f(t,x,p) = f(t,x_1,x_2,p_1,p_2)$$

$$= \begin{pmatrix} (x_1+p_1)k(t,x_1,x_2,p_1,p_2) \\ \\ (x_2+p_2)k(t,x_1,x_2,p_1,p_2) \end{pmatrix}, \quad t \in [0,1].$$

There is no growth condition applicable to f and thus the theorems of [11], [21], [4], [19], [20] do not apply.

We claim that the singular BVP has a solution if k is continuous and satisfies

$$k(t,x_1,x_2,p_1,p_2) > 0, \ \text{ for all } (t,x_1,x_2,p_1,p_2) \in [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2,$$

for some $M \le N$ such that (2.6) and (2.7) hold.

We will apply Theorem 3.1.

Firstly, for $\|x\| \ge M$ and $\langle x, p\rangle = 0$, consider

$$\begin{aligned} \langle x, f(t,x,p)\rangle &= k(t,x_1,x_2,p_1,p_2)\left[x_1^2 + x_2^2 + x_1p_1 + x_2p_2\right] \\ &= k(t,x_1,x_2,p_1,p_2)[x_1^2 + x_2^2] \quad (\text{since } x_1p_1 = -x_2p_2) \\ &> 0 \quad \text{for any positive choice of } M. \end{aligned}$$

Thus (2.8) holds. Now, for $\|x\| \le M, \ \|p\| = N$ we have

$$\begin{aligned} \langle p, f(t,x,p)\rangle &= k(t,x_1,x_2,p_1,p_2)\left[p_1x_1 + p_2x_2 + p_1^2 + p_2^2\right] \\ &\ge k(t,x_1,x_2,p_1,p_2)\left[N^2 + p_1x_1 + p_2x_2\right] \\ &> 0 \quad \text{for} \quad \|x\| \le M, \ \|p\| = N, \end{aligned}$$

for any choice of N such that $N \ge M$, thus (2.9) holds.

Thus by Theorem 3.1 the singular BVP under consideration has at least one solution $x \in C^1([0,1];\mathbb{R}^n) \cap C^2((0,1);\mathbb{R}^n)$.

REFERENCES

[1] K. AKÔ, Subfunctions for ordinary differential equations. V., *Funkcial. Ekvac.*, **12** (1969/1970), 239–249.

[2] S.R. BERNFELD AND V. LAKSHMIKANTHAM, An introduction to nonlinear boundary value problems, *Mathematics in Science and Engineering*, Vol. 109, Academic Press, Inc. (1974), 386.

[3] S.N. BERNSTEIN, Sur les équations du calcul des varations, *Ann. Sci. Ecole Norm. Sup.*, **29** (1912), 431–485.

[4] C. FABRY, Nagumo conditions for systems of second-order differential equations, *J. Math. Anal. Appl.*, **107** (1985), 132–143.

[5] R.E. GAINES AND J.L. MAWHIN, Coincidence degree and nonlinear differential equations, *Lecture Notes in Mathematics*, Vol. 568. Springer-Verlag, Berlin-New York, 1977.

[6] R.E. GAINES AND J.L. MAWHIN, Ordinary differential equations with nonlinear boundary conditions, *J. Differential Equations*, **26**(2) (1977), 200–222.

[7] J.H. GEORGE AND W.G. SUTTON, Application of Liapunov theory to boundary value problems, *Proc. Amer. Math. Soc.*, **25** (1970), 666–671.

[8] J.H. GEORGE AND R.J. YORK, Application of Liapunov theory to boundary value problems. II., *Proc. Amer. Math. Soc.*, **37** (1973), 207–212.

[9] A. GRANAS, R. GUENTHER AND J. LEE, Nonlinear boundary value problems for some classes of ordinary differential equations, *Rocky Mountain J. Math.*, **10**(1) (1980), 35–58.

[10] A. GRANAS, R. GUENTHER AND J. LEE, Nonlinear boundary value problems for ordinary differential equations, *Dissertationes Math.* (Rozprawy Mat.), **244** (1985).

[11] P. HARTMAN, On boundary value problems for systems of ordinary, nonlinear, second order differential equations, *Trans. Amer. Math. Soc.*, **96** (1960), 493–509.

[12] E. HEINZ, On certain nonlinear elliptic differential equations and univalent mappings, *J. Analyse Math.*, 5 (1956/1957), 197–272.

[13] E. HILLE, On the Thomas-Fermi equation, *Proc. Natl. Acad. Sci. U.S.A.*, **62**(1) (1969), 7–10.

[14] L.K. JACKSON, Subfunctions and second-order ordinary differential inequalities, *Advances in Math.*, **2** (1968), 307–363.

[15] H.-W. KNOBLOCH, Comparison theorems for nonlinear second order differential equations, *J. Differential Equations*, **1** (1965), 1–26.

[16] Ph. KORMAN, Remarks on Nagumo's condition, *Portugal. Math.*, **55** (1998), 1–9.

[17] J. LEE AND D. O'REGAN, Existence results for differential delay equations - II, *Nonlin. Anal.*, **17** (1991), 683–702.

[18] M. NAGUMO, Über die differential gleichung $y'' = f(x, y, y')$, *Proc. Phys. Math. Soc. Japan*, **19** (1937), 861–866.

[19] D. O'REGAN, Existence results for some singular higher order boundary value problems, *Utilitas Math.*, **39** (1991), 97–117.

[20] D. O'REGAN, Second and higher order systems of boundary value problems, *J. Math. Anal. Appl.*, **156** (1991), 120–149.

[21] K. SCHMITT AND R. THOMPSON, Boundary value problems for infinite systems of second-order differential equations, *J. Differential Equations*, **18** (1975), 277–295.

[22] K. SCHMITT, Boundary value problems for non-linear second order differential equations, *Monatsh. Math.*, **72** (1968), 347–354.