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BOUNDS FOR ZETA AND RELATED FUNCTIONS

P. CERONE

SCHOOL OF COMPUTER SCIENCE AND MATHEMATICS
VICTORIA UNIVERSITY OF TECHNOLOGY
PO BOX 14428, MELBOURNE VIC 8001, AUSTRALIA.
pc@csm.vu.edu.au
URL: http://rgmia.vu.edu.au/cerone

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ABSTRACT. Sharp bounds are obtained for expressions involving Zeta and related functions at a distance of one apart. Since Euler discovered in 1736 a closed form expression for the Zeta function at the even integers, a comparable expression for the odd integers has not been forthcoming. The current article derives sharp bounds for the Zeta, Lambda and Eta functions at a distance of one apart. The methods developed allow an accurate approximation of the function values at the odd integers in terms of the neighbouring known function at even integer values. The Dirichlet Beta function which has explicit representation at the odd integer values is also investigated in the current work.

Čebyšev functional bounds are utilised to obtain tight upper bounds for the Zeta function at the odd integers.

Key words and phrases: Euler Zeta function, Dirichlet beta, eta and lambda functions, Sharp bounds, Čebyšev functional.

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1. INTRODUCTION

The present paper represents in part a review of the recent work of the author in obtaining sharp bounds for expressions involving functions at a distance of one apart. The main motivation for the work stems from the fact that Zeta and related functions are explicitly known at either even function values (Zeta, Lambda and Eta) or at odd function values as for the Dirichlet Beta function.

The approach of the current paper is to investigate integral identities for the secant slope for $\eta(x)$ and $\beta(x)$ from which sharp bounds are procured. The results for $\eta(x)$ of Section 3 are extended to the $\zeta(x)$ and $\lambda(x)$ functions because of the relationship between them. The sharp bounds procured in the $\eta(x)$ for $\zeta(x)$ are obtained, it is argued, in a more straightforward

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fashion than in the earlier work of Alzer [2]. Some numerical illustration of the results relating to the approximation of the Zeta function at odd integer values is undertaken in Section 4.

The technique for obtaining the $\eta(x)$ bounds is also adapted to developing the bounds for $\beta(x)$ in Section 5.

The final Section 6 of the paper investigates the use of bounds for the Čebyšev function in extracting upper bounds for the odd Zeta functional values that are tighter than those obtained in the earlier sections. However, this approach does not seem to be able to provide lower bounds.

2. THE EULER ZETA AND RELATED FUNCTIONS

The Zeta function

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}, \qquad x > 1 \tag{2.1}$$

was originally introduced in 1737 by the Swiss mathematician Leonhard Euler (1707-1783) for real x who proved the identity

$$\zeta(x) := \prod_{p} \left(1 - \frac{1}{p^x}\right)^{-1}, \qquad x > 1, \tag{2.2}$$

where p runs through all primes. It was Riemann who allowed x to be a complex variable z and showed that even though both sides of (2.1) and (2.2) diverge for $\mathrm{Re}(z) \le 1$, the function has a continuation to the whole complex plane with a simple pole at $z = 1$ with residue 1. The function plays a very significant role in the theory of the distribution of primes (see [2], [4], [5], [15] and [16]). One of the most striking properties of the zeta function, discovered by Riemann himself, is the functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z) \tag{2.3}$$

that can be written in symmetric form to give

$$\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \pi^{-\left(\frac{1-z}{2}\right)} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z). \tag{2.4}$$

In addition to the relation (2.3) between the zeta and the gamma function, these functions are also connected via the integrals [13]

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1} dt}{e^t - 1}, \qquad x > 1, \tag{2.5}$$

and

$$\zeta(x) = \frac{1}{C(x)} \int_0^{\infty} \frac{t^{x-1} dt}{e^t + 1}, \qquad x > 0, \tag{2.6}$$

where

$$C(x) := \Gamma(x) \left(1 - 2^{1-x}\right) \quad \text{and} \quad \Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt. \tag{2.7}$$

In the series expansion

$$\frac{t e^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}, \tag{2.8}$$

where $B_m(x)$ are the Bernoulli polynomials (after Jacob Bernoulli), $B_m(0) = B_m$ are the Bernoulli numbers. They occurred for the first time in the formula [1, p. 804]

$$\sum_{k=1}^{m} k^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}, \qquad n, m = 1, 2, 3, \dots . \tag{2.9}$$

One of Euler's most celebrated theorems discovered in 1736 (Institutiones Calculi Differentialis, Opera (1), Vol. 10) is

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_{2n}; \qquad n = 1, 2, 3, \dots . \tag{2.10}$$

The result may also be obtained in a straight forward fashion from (2.6) and a change of variable on using the fact that

$$B_{2n} = (-1)^{n-1} \cdot 4n \int_0^\infty \frac{t^{2n-1}}{e^{2\pi t} - 1} dt \tag{2.11}$$

from Whittaker and Watson [25, p. 126].

We note here that

$$\zeta(2n) = A_n \pi^{2n},$$

where

$$A_n = (-1)^{n-1} \cdot \frac{n}{(2n+1)!} + \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{(2j+1)!} A_{n-j}$$

and $A_1 = \frac{1}{3!}$.

Further, the Zeta function for even integers satisfy the relation (Borwein et al. [4], Srivastava [21])

$$\zeta(2n) = \left(n + \frac{1}{2}\right)^{-1} \sum_{j=1}^{n-1} \zeta(2j)\zeta(2n-2j), \quad n \in \mathbb{N} \backslash \{1\} .$$

Despite several efforts to find a formula for $\zeta(2n+1)$, (for example [22, 23]), there seems to be no elegant closed form representation for the zeta function at the odd integer values. Several series representations for the value $\zeta(2n+1)$ have been proved by Srivastava and co-workers in particular.

From a long list of these representations, [22, 23], we quote only a few

$$\begin{aligned} \zeta(2n+1) = (-1)^{n-1}\pi^{2n} \Bigg[&\frac{H_{2n+1} - \log \pi}{(2n+1)!} \\ &+ \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} + 2\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \Bigg], \end{aligned} \tag{2.12}$$

$$\zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n}}{n(2^{2n+1}-1)} \left[\sum_{k=1}^{n-1} \frac{(-1)^{k-1}k}{(2n-2k)!} \frac{\zeta(2k)}{\pi^{2k}} + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \right], \tag{2.13}$$

and

$$\begin{aligned} \zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n}}{(2n-1)2^{2n}+1} \Bigg[&\sum_{k=1}^{n-1} \frac{(-1)^{k-1}k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ &+ \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \Bigg], \qquad n = 1, 2, 3, \dots . \end{aligned} \tag{2.14}$$

There is also an integral representation for $\zeta(n+1)$ namely,

$$\zeta(2n+1) = (-1)^{n+1} \cdot \frac{(2\pi)^{2n+1}}{2\delta (n+1)!} \int_0^{\delta} B_{2n+1}(t) \cot(\pi t)\, dt, \tag{2.15}$$

where $\delta = 1$ or $\frac{1}{2}$ ([1, p. 807]). Recently, Cvijović and Klinkowski [12] have given the integral representations

$$\zeta(2n+1) = (-1)^{n+1} \cdot \frac{(2\pi)^{2n+1}}{2\delta (1-2^{-2n})(2n+1)!} \int_0^{\delta} B_{2n+1}(t) \tan(\pi t)\, dt, \tag{2.16}$$

and

$$\zeta(2n+1) = (-1)^{n} \cdot \frac{\pi^{2n+1}}{4\delta (1-2^{-(2n+1)})(2n)!} \int_0^{\delta} E_{2n}(t) \csc(\pi t)\, dt. \tag{2.17}$$

Both the series representations (2.12) – (2.14) and the integral representations (2.15) – (2.16) are however both somewhat difficult in terms of computational aspects and time considerations.

We note that there are functions that are closely related to $\zeta(x)$. Namely, the Dirichlet $\eta(\cdot)$ and $\lambda(\cdot)$ functions given by

$$\eta(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t+1} dt, \quad x > 0 \tag{2.18}$$

and

$$\lambda(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x} = \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{t^{x-1}}{e^t - e^{-t}} dt, \quad x > 0. \tag{2.19}$$

These are related to $\zeta(x)$ by

$$\eta(x) = \left(1-2^{1-x}\right)\zeta(x) \quad \text{and} \quad \lambda(x) = \left(1-2^{-x}\right)\zeta(x) \tag{2.20}$$

satisfying the identity

$$\zeta(x) + \eta(x) = 2\lambda(x). \tag{2.21}$$

It should be further noted that explicit expressions for both of $\eta(2n)$ and $\lambda(2n)$ exist as a consequence of the relation to $\zeta(2n)$ via (2.20).

The Dirichlet beta function or Dirichlet $L-$function is given by [14]

$$\beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^x}, \quad x > 0 \tag{2.22}$$

where $\beta(2) = G$, Catalan's constant.

It is readily observed from (2.19) that $\beta(x)$ is the alternating version of $\lambda(x)$, however, it cannot be directly related to $\zeta(x)$. It is also related to $\eta(x)$ in that only the odd terms are summed.

The beta function may be evaluated explicitly at positive odd integer values of x, namely,

$$\beta(2n+1) = (-1)^n \frac{E_{2n}}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1}, \tag{2.23}$$

where E_n are the Euler numbers generated by

$$\operatorname{sech}(x) = \frac{2e^x}{e^{2x}+1} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

The Dirichlet beta function may be analytically continued over the whole complex plane by the functional equation

$$\beta(1-z) = \left(\frac{2}{\pi}\right)^z \sin\left(\frac{\pi z}{2}\right)\Gamma(z)\,\beta(z).$$

The function $\beta(z)$ is defined everywhere in the complex plane and has no singularities, unlike the Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$, which has a simple pole at $s = 1$.

The Dirichlet beta function and the zeta function have important applications in a number of branches of mathematics, and in particular in Analytic number theory. See for example [3], [13] – [17].

Further, $\beta(x)$ has an alternative integral representation [14, p. 56]. Namely,

$$\beta(x) = \frac{1}{2\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{\cosh(t)}dt, \quad x > 0.$$

That is,

$$\beta(x) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{e^t + e^{-t}}dt, \quad x > 0. \tag{2.24}$$

The function $\beta(x)$ is also connected to prime number theory [14] which may perhaps be best summarised by

$$\beta(x) = \prod_{\substack{p \text{ prime} \\ p\equiv 1 \bmod 4}} \left(1 - p^{-x}\right)^{-1} \cdot \prod_{\substack{p \text{ prime} \\ p\equiv 3 \bmod 4}} \left(1 + p^{-x}\right)^{-1} = \prod_{\substack{p \text{ odd} \\ \text{prime}}} \left(1 - (-1)^{\frac{p-1}{2}}\,p^{-x}\right)^{-1},$$

where the rearrangement of factors is permitted because of absolute convergence.

Cerone et al. [8] developed the identity given in the following lemma and the bounds in Theorem 2.2 which are used to obtain approximations to the odd zeta function values in terms of the even function values.

Lemma 2.1. *The following identity involving the Zeta function holds. Namely,*

$$\int_0^\infty \frac{t^x}{(e^t+1)^2}dt = C(x+1)\,\zeta(x+1) - xC(x)\,\zeta(x), \qquad x > 0, \tag{2.25}$$

where $C(x)$ is as given by (2.7).

Theorem 2.2. *The Zeta function satisfies the bounds*

$$(1-b(x))\,\zeta(x) + \frac{b(x)}{8} \le \zeta(x+1) \le (1-b(x))\,\zeta(x) + \frac{b(x)}{2}, \qquad x > 0, \tag{2.26}$$

where

$$b(x) := \frac{1}{2^x - 1}. \tag{2.27}$$

Theorem 2.3. *The Zeta function satisfies the bounds*

$$(1-b(x))\,\zeta(x) + \frac{b(x)}{8} \le \zeta(x+1) \le (1-b(x))\,\zeta(x) + \frac{b(x)}{2\theta(\lambda^*, x)} := U^*(x) \tag{2.28}$$

where $b(x)$ is as given by (2.27),

$$\theta(\lambda, x) = \lambda^{x-1}\left(\frac{\lambda}{1-\lambda}\right)^{2(1-\lambda)}$$

and

$$\lambda^* = \frac{1}{z}$$

with z the solution of

$$z = 1 + e^{-\frac{x+1}{2} \cdot z}.$$

The $\frac{1}{2}$ on the right hand side is the best constant. The best constant for the lower bound was shown to be $\ln 2 - \frac{1}{2}$ by Alzer [2]*, on making use of Lemma 2.1 and Theorem 2.2, rather than $\frac{1}{8}$.*

3. An Identity and Bounds Involving the Eta and Related Functions

The following lemma was developed in Cerone [5] to obtain sharp bounds for the eta function, $\eta(x)$ as given in Theorem 2.3.

Lemma 3.1. *The following identity for the eta function holds. Namely,*

$$Q(x) := \frac{1}{\Gamma(x+1)} \int_0^\infty \frac{t^x}{(e^t+1)^2} dt = \eta(x+1) - \eta(x), \quad x > 0. \tag{3.1}$$

Proof. From (2.18),

$$\begin{aligned} x\Gamma(x)\eta(x) &= \int_0^\infty \frac{xt^{x-1}}{e^t+1} dt, \qquad x > 0 \\ &= \lim_{T\to\infty} \frac{T^x}{e^T+1} + \int_0^\infty \frac{t^x e^t}{(e^t+1)^2} dt \end{aligned}$$

and so we have

$$\Gamma(x+1)\eta(x) = \int_0^\infty \frac{e^t t^x}{(e^t+1)^2} dt. \tag{3.2}$$

Thus, from (2.18) and (3.2),

$$\Gamma(x+1)[\eta(x+1) - \eta(x)] = \int_0^\infty \frac{t^x}{e^t+1}\left[1 - \frac{e^t}{e^t+1}\right] dt = \int_0^\infty \frac{t^x}{(e^t+1)^2} dt,$$

giving (3.1). □

The following theorem presents sharp bounds for the secant slope $\eta(x)$ for a distance of one apart.

Theorem 3.2. *For real numbers $x > 0$, we have*

$$\frac{c}{2^{x+1}} < \eta(x+1) - \eta(x) < \frac{d}{2^{x+1}} \tag{3.3}$$

with the best possible constants

$$c = 2\ln 2 - 1 = 0.3862943\ldots \quad \textit{and} \quad d = 1. \tag{3.4}$$

Proof. Let $x > 0$. We first establish the first inequality in (3.3). From the identity (3.1) proved in Lemma 3.1, it is readily evident that $0 < Q(x)$. We further consider

$$J = \int_0^\infty \frac{dt}{(e^t+1)^2} = \int_0^\infty \frac{e^{-2t}}{(e^{-t}+1)^2} dt. \tag{3.5}$$

Thus, after some obvious simplifications

$$J = \int_0^1 \frac{u}{(u+1)^2} du = \int_1^2 \frac{u-1}{u^2} du = \ln 2 - \frac{1}{2}. \tag{3.6}$$

Now, let us examine

$$2^{x+1} Q(x) - (2\ln 2 - 1).$$

That is, from (3.1), (3.5) and (3.6),

$$\begin{aligned}(3.7)\qquad \Gamma(x+1)\left[2^{x+1}Q(x)-2J\right] &= 2^{x+1}\int_0^\infty \frac{t^x}{(e^t+1)^2}dt - 2\cdot\Gamma(x+1)\int_0^\infty \frac{dt}{(e^t+1)^2} \\ &= 2\int_0^\infty \frac{e^{-2t}\left[(2t)^x-\Gamma(x+1)\right]}{(1+e^{-t})^2}dt \\ &= \int_0^\infty \frac{e^{-u}\left[u^x-\Gamma(x+1)\right]}{\left(1+e^{-\frac{u}{2}}\right)^2}du \\ &= \int_0^\infty u(t,x)\,v(t)\,dt,\end{aligned}$$

where

$$u(t,x)=e^{-t}\left[t^x-\Gamma(x+1)\right],\quad v(t)=\left(1+e^{-\frac{t}{2}}\right)^{-2}. \tag{3.8}$$

The function $v(t)$ is strictly increasing for $t\in(0,\infty)$.

Now, let $t_0=(\Gamma(x+1))^{\frac{1}{x}}$, then for $0<t<t_0$, $u(t,x)<0$ and $v(t)<v(t_0)$. Also, for $t>t_0$, $u(t,x)>0$ and $v(t)>v(t_0)$. Hence we have that $u(t,x)v(t)>u(t,x)v(t_0)$ for $t>0$ and $t\neq t_0$. This implies that

$$\int_0^\infty u(t,x)\,v(t)\,dt > v(t_0)\int_0^\infty e^{-t}\left[t^x-\Gamma(x+1)\right]dt=0.$$

Hence from (3.7) and (3.6)

$$Q(x)>\frac{2\ln 2-1}{2^{x+1}},\quad x>0. \tag{3.9}$$

Now for the right inequality.

We have from (3.4) that

$$\begin{aligned}\Gamma(x+1)\left[1-2^{x+1}Q(x)\right] &= \Gamma(x+1)-2\int_0^\infty \frac{(2t)^x e^{-2t}}{(1+e^{-t})^2}dt \\ &= \int_0^\infty e^{-t}t^x\left[1-v(t)\right]dt,\end{aligned}$$

where $v(t)$ is as given by (3.8). We make the observation that $e^{-t}t^x$ is positive and $1-v(t)$ is strictly decreasing and positive for $t\in(0,\infty)$, which naturally leads to the conclusion that

$$Q(x)<\frac{1}{2^{x+1}},\quad x>0. \tag{3.10}$$

In summary we note that (3.9) and (3.11) provide lower and upper bounds respectively for $Q(x)$. That the constants in (3.3) are best possible remains to be shown.

Since (3.3) holds for all positive x, we have

$$c<2^{x+1}Q(x)<d. \tag{3.11}$$

Now, from (3.1), we have

$$2^{x+1}Q(x)=\frac{2^{x+1}}{\Gamma(x+1)}\int_0^\infty \frac{e^{-2t}t^x}{(1+e^{-t})^2}dt \tag{3.12}$$

and so

$$\lim_{x\to 0}2^{x+1}Q(x)=2\int_0^\infty \frac{e^{-2t}}{(1+e^{-t})}dt=2\cdot J=2\ln 2-1, \tag{3.13}$$

where the permissable interchange of the limit and integration has been undertaken and we have used (3.5) – (3.6).

Now, since for $0 < w < 1$ the elementary inequality $1 - 2w < (1+w)^{-2} < 1$ holds, then we have

$$1 - 2e^{-t} < \frac{1}{(1+e^{-t})^2} < 1.$$

Thus, from (3.12),

$$1 - 2 \cdot \left(\frac{2}{3}\right)^{x+1} < 2^{x+1} Q(x) < 1, \tag{3.14}$$

where we have utilised the fact that,

$$\int_0^\infty e^{-st} t^x dt = \frac{\Gamma(x+1)}{s^{x+1}}. \tag{3.15}$$

Finally, from (3.14) we conclude that

$$\lim_{x \to \infty} 2^{x+1} Q(x) = 1. \tag{3.16}$$

From (3.11), (3.13) and (3.16) we have $c \le 2\ln 2 - 1$ and $d \ge 1$ which implies that the best possible constants in (3.3) are given by $c = 2\ln 2 - 1$ and $d = 1$. □

Corollary 3.3. *The bound*

$$\left| \eta(x+1) - \eta(x) - \frac{d+c}{2^{x+2}} \right| < \frac{d-c}{2^{x+2}}, \quad x > 0 \tag{3.17}$$

holds, where $c = 2\ln 2 - 1$ *and* $d = 1$.

Proof. From (3.3), let

$$L(x) = \eta(x) + \frac{c}{2^{x+1}} \quad \text{and} \quad U(x) = \eta(x) + \frac{d}{2^{x+1}}$$

then

$$L(x) < \eta(x+1) < U(x)$$

and so

$$-\frac{U(x) - L(x)}{2} < \eta(x+1) - \frac{U(x) + L(x)}{2} < \frac{U(x) - L(x)}{2}.$$

□

Remark 3.4. The form of (3.17) is very useful since we may write

$$\eta(x+1) = \eta(x) + \frac{d+c}{2^{x+2}} + E(x),$$

where $|E(x)| < \varepsilon$ for

$$x > x^* := \frac{\ln\left(\frac{d-c}{4\varepsilon}\right)}{\ln 2}. \tag{3.18}$$

Corollary 3.5. *The eta function satisfies the bounds*

$$L_2(x) < \eta(x+1) < U_2(x), \quad x > 0, \tag{3.19}$$

where

$$L_2(x) = \eta(x+2) - \frac{d}{2^{x+2}} \quad \textit{and} \quad U_2(x) = \eta(x+2) - \frac{c}{2^{x+2}}. \tag{3.20}$$

Proof. From (3.6)

$$-\frac{d}{2^{x+1}} < \eta(x) - \eta(x+1) < -\frac{c}{2^{x+1}}.$$

Replace x by $x+1$ and rearrange to give (3.19) – (3.20). □

Remark 3.6. We note that $L(\cdot)$ and $U(\cdot)$ will be used to denote the lower and upper bounds respectively. If the bounds involve a previous value at a distance of one away from the function that is bounded, then no subscript is used. If it involves a subsequent value then a subscript of 2 is used. This is shown in Corollaries 3.3 and 3.5 above for the eta function. No distinction in the notation is used when referring to other functions.

Given the sharp inequalities for $\eta(x+1) - \eta(x)$ in (3.3) – (3.4), then we may readily obtain sharp bounds for expressions involving the zeta function and the lambda function at a distance of one apart.

Corollary 3.7. *For real numbers $x > 0$ we have*

$$\left(\ln 2 - \frac{1}{2}\right) b(x) < \zeta(x+1) - (1 - b(x))\zeta(x) < \frac{b(x)}{2}, \tag{3.21}$$

where

$$b(x) = \frac{1}{2^x - 1}. \tag{3.22}$$

Proof. From Theorem 3.2 and (2.20) giving a relationship between $\eta(x)$ and $\zeta(x)$ we have

$$\eta(x+1) - \eta(x) = \left(1 - 2^{-x}\right)\zeta(x+1) - \left(1 - 2^{1-x}\right)\zeta(x)$$

and so from (3.3) and (3.4)

$$\frac{c}{2} \cdot b(x) < \zeta(x+1) - (1 - b(x))\zeta(x) < \frac{d}{2} \cdot b(x).$$

□

Remark 3.8. Cerone et al. [8] obtained the upper bound in (3.21) and a coarser lower bound of $\frac{b(x)}{8}$ as presented in (2.26). Alzer [3] demonstrated that the constants $\ln 2 - \frac{1}{2}$ and $\frac{1}{2}$ in (3.21) are sharp. The sharpness of the constant $\frac{1}{2}$ was obtained by Alzer on utilising a different approach, other than the sharpness of the constant $d = 1$ in (3.4) via the eta function and hence $\frac{1}{2}$ in (3.21).

Corollary 3.9. *For real $x > 0$ we have*

$$\begin{aligned}\left(\ln 2 - \frac{1}{2}\right) b(x)\left(1 - 2^{-(x+1)}\right) &< \lambda(x+1) - \left(\frac{1 - b(x)}{1 - b(x+1)}\right)\lambda(x) \\ &< \frac{b(x)}{2} \cdot \left(1 - 2^{-(x+1)}\right),\end{aligned} \tag{3.23}$$

where $b(x)$ is as given by (3.22).

Proof. Again utilising Theorem 3.2 and from (2.20) and (2.21) we have, after some algebra,

$$\eta(x) = (1 - b(x))\lambda(x) \tag{3.24}$$

and so from (3.3) and (3.4)

$$\begin{aligned}\frac{2\ln 2 - 1}{2^{x+1}} &< \eta(x+1) - \eta(x) \\ &= (1 - b(x+1))\lambda(x+1) - (1 - b(x))\lambda(x) < \frac{1}{2^{x+1}}.\end{aligned}$$

Division by $1 - b(x+1)$ and some simplification readily produces (3.23). □

The advantage of having sharp inequalities such as (3.3), (3.21) and (3.23) involving function values at a distance of one apart is that if we place $x = 2n,$ then since $\zeta(2n)$ is known explicitly, we may approximate $\zeta(2n+1)$ and provide explicit bounds. This is so for $\eta(\cdot)$ and $\lambda(\cdot)$ as well because of their relationship to $\zeta(\cdot)$ via (2.20) – (2.21).

4. Some Zeta Related Numerics

In what follows, we investigate some numerical results associated with bounding the unknown $\zeta(2n+1)$ by expressions involving the explicitly known $\zeta(2n)$. The following corollaries hold.

Corollary 4.1. *The bound*

$$\left|\zeta(x+1) - (1-b(x))\zeta(x) - \frac{\ln 2}{2} b(x)\right| \le \frac{1-\ln 2}{2} b(x), \quad x > 0 \tag{4.1}$$

holds, where $b(x)$ *is as given by (3.2).*

Proof. Let

$$L(x) = (1-b(x))\zeta(x) + \left(\ln 2 - \frac{1}{2}\right) b(x), \quad \text{and} \tag{4.2}$$

$$U(x) = (1-b(x))\zeta(x) + \frac{b(x)}{2}$$

then from (3.21) we have

$$L(x) \le \zeta(x+1) \le U(x).$$

Hence

$$-\frac{U(x)-L(x)}{2} \le \zeta(x+1) - \frac{U(x)+L(x)}{2} \le \frac{U(x)-L(x)}{2}$$

which may be expressed as the stated result (4.1) on noting the obvious correspondences and simplification. ☐

Remark 4.2. The form (4.1) is a useful one since we may write

$$\zeta(x+1) = (1-b(x))\zeta(x) + \frac{\ln 2}{2} b(x) + E(x), \tag{4.3}$$

where

$$|E(x)| < \varepsilon$$

for

$$x > x^* := \ln\left(1 + \frac{1-\ln 2}{2\varepsilon}\right) \Big/ \ln 2.$$

That is, we may approximate $\zeta(x+1)$ by $(1-b(x))\zeta(x) + \frac{\ln 2}{2} b(x)$ within an accuracy of ε for $x > x^*$.

We note that both the result of Corollary 3.7 and Corollary 4.1 as expressed in (3.21) and (4.1) respectively rely on approximating $\zeta(x+1)$ in terms of $\zeta(x)$. The following result involves approximating $\zeta(x+1)$ in terms of $\zeta(x+2)$, the subsequent zeta values within a distance of one rather than the former zeta values.

Theorem 4.3. *The zeta function satisfies the bounds*

$$L_2(x) \le \zeta(x+1) \le U_2(x), \tag{4.4}$$

where

$$L_2(x) = \frac{\zeta(x+2) - \frac{b(x+1)}{2}}{1 - b(x+1)} \text{ and } U_2(x) = \frac{\zeta(x+2) - \left(\ln 2 - \frac{1}{2}\right) b(x+1)}{1 - b(x+1)}. \tag{4.5}$$

Proof. From (3.21) we have

$$0 \le \left(\ln 2 - \frac{1}{2}\right) b(x) \le \zeta(x+1) - (1 - b(x))\zeta(x) \le \frac{b(x)}{2}$$

and so

$$-\frac{b(x)}{2} \le (1 - b(x))\zeta(x) - \zeta(x+1) \le -\left(\ln 2 - \frac{1}{2}\right) b(x)$$

to produce

$$\zeta(x+1) - \frac{b(x)}{2} \le (1 - b(x))\zeta(x) \le \zeta(x+1) - \left(\ln 2 - \frac{1}{2}\right) b(x).$$

A rearrangement and change of x to $x+1$ produces the stated result (4.4) – (4.5). □

Remark 4.4. Some experimentation using the Maple computer algebra package indicates that the lower bound $L_2(x)$ is better than the lower bound $L(x)$ for $x > x_* = 1.30467865\ldots$ and vice versa for $x < x_*$. In a similar manner the upper bound $U_2(x)$ is better than $U(x)$ for $x < x^* = 3.585904878\ldots$ and vice versa for $x > x^*$.

The following corollary is valid in which $\zeta(x+1)$ may be approximated in terms of $\zeta(x+2)$ and an explicit bound is provided for the error.

Corollary 4.5. *The bound*

$$\left| \zeta(x+1) - \frac{\zeta(x+2) - \left(\ln 2 - \frac{1}{2}\right) b(x+1)}{1 - b(x+1)} \right| \le \frac{1 - \ln 2}{2} \cdot \frac{b(x+1)}{1 - b(x+1)} \tag{4.6}$$

holds, where $b(x)$ is as defined by (3.22).

Proof. The proof is straight forward and follows that of Corollary 4.1 with $L(x)$ and $U(x)$ replaced by $L_2(x)$ and $U_2(x)$ as defined by (4.5). □

Corollary 4.6. *The zeta function satisfies the bounds*

$$\max\{L(x), L_2(x)\} \le \zeta(x+1) \le \min\{U(x), U_2(x)\}, \tag{4.7}$$

where $L(x), U(x)$ are given by (4.2) and $L_2(x), U_2(x)$ by (4.5).

Table 1 provides lower and upper bounds for $\zeta(2n+1)$ for $n = 1, \ldots, 5$, utilising Corollaries 3.9 and 4.5 for $x = 2n$. We notice that $L_2(2n)$ is better than $L(2n)$ and $U_2(2n)$ is better than $U(2n)$ only for $n = 1$ (see also Remark 4.4). Tables 2 and 3 give the use of Corollaries 4.1 and 4.5 for $x = 2n$. Thus, the table provides $\zeta(2n+1)$, its approximation and the bound on the error.

n	$\mathbf{L}(2n)$	$\mathbf{L}_2(2n)$	$\zeta(2n+1)$	$\mathbf{U}(2n)$	$\mathbf{U}_2(2n)$
1	1.161005104	1.179377107	1.202056903	1.263289378	1.230519243
2	1.023044831	1.034587831	1.036927755	1.043501685	1.044816259
3	1.004260588	1.008077971	1.008349277	1.009131268	1.010513311
4	1.000897239	1.001976919	1.002008393	1.002100583	1.002578591
5	1.000204892	1.000490588	1.000494189	1.000504847	1.000640564

Table 1. Table of $L(2n)$, $L_2(2n)$, $\zeta(2n+1)$, $U(2n)$ and $U_2(2n)$ as given by (4.2) and (4.5) for $n = 1, \ldots, 5$.

n	$\zeta(2n+1)$	$\frac{U(2n)+L(2n)}{2}$	$\frac{U(2n)-L(2n)}{2}$
1	1.202056903	1.212147241	0.0511421366
2	1.036927755	1.033273258	0.010228842731
3	1.008349277	1.006695928	0.002435339836
4	1.002008393	1.001498911	0.000601672195
5	1.000494189	1.000354870	0.0001499769401

Table 2. Table of $\zeta(2n+1)$, its approximation $\frac{U(2n)+L(2n)}{2}$ and its bound $\frac{U(2n)-L(2n)}{2}$ for $n = 1, \ldots, 5$ where $U(2n)$ and $L(2n)$ are given by (4.2).

n	$\zeta(2n+1)$	$\frac{U_2(2n)+L_2(2n)}{2}$	$\frac{U_2(2n)-L_2(2n)}{2}$
1	1.202056903	1.202056903	0.02557106828
2	1.036927755	1.039702045	0.00511421366
3	1.008349277	1.009295641	0.001217669918
4	1.002008393	1.002277755	0.0003008360975
5	1.000494189	1.000565576	0.0000749884700

Table 3. Table of $\zeta(2n+1)$, its approximation $\frac{U_2(2n)+L_2(2n)}{2}$ and its bound $\frac{U_2(2n)-L_2(2n)}{2}$ for $n = 1, \ldots, 5$ where $U_2(2n)$ and $L_2(2n)$ are given by (4.5).

5. An Identity and Bounds Involving the Beta Function

The following lemma was developed in Cerone [5] to obtain sharp bounds for the Dirichlet beta function, $\beta(x)$ at a distance of one apart as presented in Theorem 5.2.

The techniques closely follow those presented in Section 3 for the eta function.

Lemma 5.1. *The following identity for the Dirichlet beta function holds. Namely,*

$$P(x) := \frac{2}{\Gamma(x+1)} \int_0^\infty \frac{e^{-t}}{(e^t + e^{-t})^2} \cdot t^x dt = \beta(x+1) - \beta(x). \tag{5.1}$$

The following theorem produces sharp bounds for the secant slope of $\beta(x)$.

Theorem 5.2. *For real numbers $x > 0$, we have*

$$\frac{c^*}{3^{x+1}} < \beta(x+1) - \beta(x) < \frac{d^*}{3^{x+1}}, \tag{5.2}$$

with the best possible constants

$$c^* = 3\left(\frac{\pi}{4} - \frac{1}{2}\right) = 0.85619449\ldots \quad \textit{and} \quad d^* = 2. \tag{5.3}$$

The following corollaries were also given in Cerone [5] which prove useful in approximating $\beta(2n)$ in terms of known $\beta(2n+1)$. This is so since (5.2) may be written as

$$L(x) < \beta(x+1) < U(x), \tag{5.4}$$

where

$$L(x) = \beta(x) + \frac{c^*}{3^{x+1}} \quad \text{and} \quad U(x) = \beta(x) + \frac{d^*}{3^{x+1}}. \tag{5.5}$$

Corollary 5.3. *The bound*

$$\left|\beta(x+1) - \beta(x) - \frac{d^* + c^*}{2 \cdot 3^{x+1}}\right| < \frac{d^* - c^*}{2 \cdot 3^{x+1}} \tag{5.6}$$

holds where $c^* = 3\left(\frac{\pi}{4} - \frac{1}{2}\right)$ and $d^* = 2$.

Remark 5.4. The form (5.6) is useful since we may write

$$\beta(x+1) = \beta(x) + \frac{d^* + c^*}{2 \cdot 3^{x+1}} + E(x),$$

where $|E(x)| < \varepsilon$ for

$$x > x^* := \frac{\ln\left(\frac{d^* - c^*}{2 \cdot \varepsilon}\right)}{\ln(3)} - 1.$$

Corollary 5.5. *The Dirichlet beta function satisfies the bounds*

$$L_2(x) < \beta(x+1) < U_2(x), \tag{5.7}$$

where

$$L_2(x) = \beta(x+2) - \frac{d^*}{3^{x+2}} \text{ and } U_2(x) = \beta(x+2) - \frac{c^*}{3^{x+2}}. \tag{5.8}$$

Remark 5.6. Some experimentation with the Maple computer algebra package indicates that the lower bound $L_2(x)$ is better than $L(x)$ for $x > x_* \approx 0.65827$ and vice versa for $x < x_*$. Similarly, $U(x)$ is better than $U_2(x)$ for $x > x^* \approx 3.45142$ and vice versa for $x < x^*$.

Corollary 5.7. *The Dirichlet beta function satisfies the bounds*

$$\max\{L(x), L_2(x)\} < \beta(x+1) < \min\{U(x), U_2(x)\},$$

where $L(x), U(x)$ *are given by (5.5) and* $L_2(x), U_2(x)$ *by (5.8).*

Remark 5.8. Table 4 provides lower and upper bounds for $\beta(2n)$ for $n = 1, \ldots, 5$ utilising Theorem 5.2 and Corollary 5.5 with $x = 2n - 1$. That is, the bounds are in terms of $\beta(2n-1)$ and $\beta(2n+1)$ where these may be obtained explicitly using the result (2.23).

n	$L(2n-1)$	$L_2(2n-1)$	$\beta(2n)$	$U(2n-1)$	$U_2(2n-1)$
1	.8805308843	.8948720722	.9159655942	1.007620386	.9372352393
2	.9795164487	.9879273754	.9889445517	.9936375043	.9926343940
3	.9973323061	.9986400132	.9986852222	.9989013123	.9991630153
4	.9996850054	.9998480737	.9998499902	.9998593395	.9999061850
5	.9999641840	.9999830849	.9999831640	.9999835544	.9999895417

Table 4: Table of $L(2n-1), L_2(2n-1), \beta(2n), U(2n-1)$ and $U_2(2n-1)$ as given by (5.5) and (5.8) for $n = 1, \ldots, 5$.

6. ZETA BOUNDS VIA ČEBYŠEV

It is instructive to introduce some techniques for approximating and bounding integrals of the product of functions.

The weighted Čebyšev functional defined by

$$T(f, g; p) := \mathcal{M}(fg; p) - \mathcal{M}(f; p)\mathcal{M}(g; p), \tag{6.1}$$

where

$$P \cdot \mathcal{M}(f; p) := \int_a^b p(x) h(x)\, dx, \qquad P = \int_a^b p(x)\, dx \tag{6.2}$$

the weighted integral mean, has been extensively investigated in the literature with the view of determining its bounds.

There has been much activity in procuring bounds for $T(f,g;p)$ and the interested reader is referred to [9]. The functional $T(f,g;p)$ is known to satisfy a number of identities. Included amongst these, are identities of Sonin type, namely

$$P\cdot T(f,g;p)=\int_a^b p(t)\left[f(t)-\gamma\right]\left[g(t)-\mathcal{M}(g;p)\right]dt,\quad \text{for } \gamma \text{ a constant.} \tag{6.3}$$

The constant $\gamma\in\mathbb{R}$ but in the literature some of the more popular values have been taken as

$$0,\ \frac{\Delta+\delta}{2},\ f\left(\frac{a+b}{2}\right) \text{ and } \mathcal{M}(f;p),$$

where $-\infty<\delta\le f(t)\le\Delta<\infty$ for $t\in[a,b]$.

An identity attributed to Korkine viz

$$P^2\cdot T(f,g;p)=\frac{1}{2}\int_a^b\int_a^b p(x)p(y)(f(x)-f(y))(g(x)-g(y))\,dxdy \tag{6.4}$$

may also easily be shown to hold.

Here we shall mainly utilize the following results bounding the Čebyšev functional to determine bounds on the Zeta function. (See [6] for more general applications to special functions).

From (6.1) and (6.3) we note that

$$P\cdot|T(f,g;p)|=\left|\int_a^b p(x)(f(x)-\gamma)(g(x)-\mathcal{M}(g;p))\,dx\right| \tag{6.5}$$

to give

$$P\cdot|T(f,g;p)|\le\begin{cases}\inf\limits_{\gamma\in\mathbb{R}}\|f(\cdot)-\gamma\|\int_a^b p(x)\,|g(x)-\mathcal{M}(g;p)|\,dx,\\[2ex] \left(\int_a^b p(x)(f(x)-\mathcal{M}(f;p))^2dx\right)^{\frac{1}{2}}\\ \qquad\times\left(\int_a^b p(x)(g(x)-\mathcal{M}(g;p))^2dx\right)^{\frac{1}{2}},\end{cases} \tag{6.6}$$

where

$$\int_a^b p(x)(h(x)-\mathcal{M}(h;p))^2dx=\int_a^b p(x)h^2(x)\,dx-P\cdot\mathcal{M}^2(h;p) \tag{6.7}$$

and it may be easily shown by direct calculation that,

$$P\cdot\inf_{\gamma\in\mathbb{R}}\left[\int_a^b p(x)(f(x)-\gamma)^2dx\right]=\int_a^b p(x)(f(x)-\mathcal{M}(f;p))^2dx. \tag{6.8}$$

The following result was obtained by the author [7] by utilising the above Čebyšev functional bounds.

Theorem 6.1. *For $\alpha>1$ the Zeta function satisfies the inequality*

$$\left|\zeta(\alpha)-2^{\alpha-1}\cdot\frac{\pi^2}{6}\right|\le\kappa\cdot2^{\alpha-1}\left[\frac{2\Gamma(2\alpha-1)}{\Gamma^2(\alpha)}-1\right]^{\frac{1}{2}}, \tag{6.9}$$

where

$$\kappa=\left[\pi^2\left(1-\frac{\pi^2}{72}\right)-7\zeta(3)\right]^{\frac{1}{2}}=0.319846901\ldots \tag{6.10}$$

Theorem 6.2. *For $\alpha > 1$ and $m = \lfloor \alpha \rfloor$ the zeta function satisfies the inequality*

$$\left|\Gamma(\alpha+1)\zeta(\alpha+1) - 2^{\alpha-m}\Gamma(m+1)\zeta(m+1)\zeta(\alpha-m+1)\right| \le 2^{\left(\alpha-m+\frac{1}{2}\right)} \cdot E \cdot \left[\Gamma(2\alpha-2m+1) - \Gamma^2(\alpha-m+1)\right]^{\frac{1}{2}}, \tag{6.11}$$

where

$$E^2 = 2^{2m}\Gamma(2m+1)\left[\lambda(2m) - \lambda(2m+1)\right] - \frac{1}{2}\Gamma^2(m+1)\zeta^2(m+1), \tag{6.12}$$

with $\lambda(\cdot)$ given by (2.19).

Proof. Let

$$\begin{aligned}\tau(\alpha) = \Gamma(\alpha+1)\zeta(\alpha+1) &= \int_0^\infty \frac{x^\alpha}{e^x - 1}dx \\ &= \int_0^\infty e^{-\frac{x}{2}} \frac{x^m}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} \cdot x^{\alpha-m}dx, \quad \alpha > 1\end{aligned} \tag{6.13}$$

where $m = \lfloor \alpha \rfloor$.

Make the associations

$$p(x) = e^{-\frac{x}{2}}, \quad f(x) = \frac{x^m}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}, \quad g(x) = x^{\alpha-m} \tag{6.14}$$

then we have from (6.6)

$$\begin{cases} P = \int_0^\infty e^{-\frac{x}{2}}dx = 2, \\ \mathcal{M}(f;p) = \frac{1}{2}\int_0^\infty \frac{e^{-\frac{x}{2}}x^m}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}dx = \frac{1}{2}\Gamma(m+1)\zeta(m+1), \\ \mathcal{M}(g;p) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}}x^{\alpha-m}dx = 2^{\alpha-m}\Gamma(\alpha-m+1). \end{cases} \tag{6.15}$$

Thus, from (6.1) – (6.3), we have

$$\begin{aligned}P \cdot T(f,g;p) &= \Gamma(\alpha+1)\zeta(\alpha+1) - 2^{\alpha-m}\Gamma(m+1)\zeta(m+1)\zeta(\alpha-m+1) \\ &= \int_0^\infty e^{-\frac{x}{2}}\left(x^{\alpha-m} - \gamma\right)\left(\frac{x^m}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} - \frac{\Gamma(m+1)\zeta(m+1)}{2}\right)dx.\end{aligned}$$

Now, from (6.6) and (6.7), the best value for γ when utilising the Euclidean norm is the integral mean and so we have from (6.6),

$$\begin{aligned}&\left|\Gamma(\alpha+1)\zeta(\alpha+1) - 2^{\alpha-m}\Gamma(m+1)\zeta(m+1)\zeta(\alpha-m+1)\right| \\ &\quad \le \left(\int_0^\infty e^{-\frac{x}{2}}\left(x^{\alpha-m} - 2^{\alpha-m}\Gamma(\alpha-m+1)\right)^2 dx\right)^{\frac{1}{2}} \\ &\quad\quad \times \left(\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x^m}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} - \frac{\Gamma(m+1)\zeta(m+1)}{2}\right)^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

That is, on using (6.7), we have

$$\left|\Gamma(\alpha+1)\zeta(\alpha+1)-2^{\alpha-m}\Gamma(m+1)\zeta(m+1)\zeta(\alpha-m+1)\right| \\ \leq E_m^2\left[\int_0^\infty e^{-\frac{x}{2}}x^{2(\alpha-m)}dx-2^{2(\alpha-m)+1}\Gamma^2(\alpha-m+1)\right]^{\frac{1}{2}}, \tag{6.16}$$

where

$$E_m^2=\int_0^\infty e^{-\frac{x}{2}}\frac{x^{2m}}{\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)^2}dx-2\left(\frac{\Gamma(m+1)\zeta(m+1)}{2}\right)^2. \tag{6.17}$$

Now

$$\begin{aligned}\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x^m}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\right)^2dx &= \int_0^\infty e^{-\frac{3}{2}x}x^{2m}\left(1+2e^{-x}+3e^{-2x}+\cdots\right)dx \\ &= \sum_{n=1}^\infty n\int_0^\infty e^{\left(\frac{2n+1}{2}\right)x}x^{2m}dx \\ &= \sum_{n=1}^\infty n\frac{2^{2m+1}\Gamma(2m+1)}{(2n+1)^{2m+1}} \\ &= 2^{2m}\Gamma(2m+1)\sum_{n=1}^\infty\frac{2n}{(2n+1)^{2m+1}} \\ &= 2^{2m}\Gamma(2m+1)\left[\lambda(2m)-\lambda(2m+1)\right],\end{aligned} \tag{6.18}$$

where $\lambda(\cdot)$ is as given by (2.19), where we have used (3.15) and have undertaken the permissable interchange of summation and integration.

Substitution of (6.18) into (6.17) and using (6.16) gives the stated results (6.11) and (6.12) after some simplification. □

The following corollary provides upper bounds for the zeta function at odd integers.

Corollary 6.3. *The inequality*

$$\Gamma(2m+1)\left[2\cdot\left(2^{2m}-1\right)\zeta(2m)-\left(2^{2m+1}-1\right)\zeta(2m+1)\right] \\ -\Gamma^2(m+1)\zeta^2(m+1)>0 \tag{6.19}$$

holds for $m=1,2,\ldots$.

Proof. From equation (6.12) of Theorem 6.2, we have $E^2>0$. Utilising the relationship between $\lambda(\cdot)$ and $\zeta(\cdot)$ given by (2.20) readily gives the inequality (6.19). □

Remark 6.4. In (6.19), if m is odd, then $2m$ and $m+1$ are even so that an expression in the form

$$\alpha(m)\zeta(2m)-\beta(m)\zeta(2m+1)-\gamma(m)\zeta^2(m+1)>0, \tag{6.20}$$

results, where

$$\begin{aligned}&\alpha(m)=2\left(2^{2m}-1\right)\Gamma(2m+1), \\ &\beta(m)=\left(2^{2m+1}-1\right)\Gamma(2m+1) \quad \text{and} \\ &\gamma(m)=\Gamma^2(m+1).\end{aligned} \tag{6.21}$$

Thus for m **odd** we have

$$\zeta\left(2m+1\right) < \frac{\alpha\left(m\right)\zeta\left(2m\right)-\gamma\left(m\right)\zeta^2\left(m+1\right)}{\beta\left(m\right)}. \tag{6.22}$$

That is, for $m = 2k-1$, we have from (6.22)

$$\zeta\left(4k-1\right) < \frac{\alpha\left(2k-1\right)\zeta\left(4k-2\right)-\gamma\left(2k-1\right)\zeta^2\left(2k\right)}{\beta\left(2k-1\right)} \tag{6.23}$$

giving for $k = 1, 2, 3$, for example,

$$\zeta\left(3\right) < \frac{\pi^2}{7}\left(1-\frac{\pi^2}{72}\right) = 1.21667148,$$
$$\zeta\left(7\right) < \frac{2\pi^6}{1905}\left(1-\frac{\pi^2}{2160}\right) = 1.00887130,$$
$$\zeta\left(11\right) < \frac{62\pi^{10}}{5803245}\left(1-\frac{\pi^2}{492150}\right) = 1.00050356,$$

Guo [15] obtained $\zeta\left(3\right) < \frac{\pi^4}{72}$ and the above bound for $\zeta\left(3\right)$ was obtained previously by the author in [7] from (6.10). (See also [18] and [19]).

If m is **even** then for $m = 2k$ we have from (6.22)

$$\zeta\left(4k+1\right) < \frac{\alpha\left(2k\right)\zeta\left(4k\right)-\gamma\left(2k\right)\zeta^2\left(2k+1\right)}{\beta\left(2k\right)}, \quad k = 1, 2, \ldots. \tag{6.24}$$

We notice that in (6.24), or equivalently (6.20) with $m = 2k$ there are two zeta functions with odd arguments. There are a number of possibilities for resolving this, but firstly it should be noticed that $\zeta\left(x\right)$ is monotonically decreasing for $x > 1$ so that $\zeta\left(x_1\right) > \zeta\left(x_2\right)$ for $1 < x_1 < x_2$.

Firstly, we may use a lower bound obtained in Section 4 as given by (4.2) or (4.5). But from Table 1, it seems that $L_2\left(x\right) > L\left(x\right)$ for positive integer x and so we have from (6.24)

$$\zeta_L\left(4k+1\right) < \frac{\alpha\left(2k\right)\zeta\left(2k\right)-\gamma\left(2k\right)L_2^2\left(2k\right)}{\beta\left(2k\right)}, \tag{6.25}$$

where we have used the fact that $L_2\left(x\right) < \zeta\left(x+1\right)$.

Secondly, since the even argument $\zeta\left(2k+2\right) < \zeta\left(2k+1\right)$, then from (6.24) we have

$$\zeta_E\left(4k+1\right) < \frac{\alpha\left(2k\right)\zeta\left(4k\right)-\gamma\left(2k\right)\zeta^2\left(2k+2\right)}{\beta\left(2k\right)}. \tag{6.26}$$

Finally, we have that $\zeta\left(m+1\right) > \zeta\left(2m+1\right)$ so that from (6.20) we have, with $m = 2k$ on solving the resulting quadratic equation that

$$\zeta_Q\left(4k+1\right) < \frac{-\beta\left(2k\right)+\sqrt{\beta^2\left(2k\right)+4\gamma\left(2k\right)\alpha\left(2k\right)\zeta\left(4k\right)}}{2\gamma\left(2k\right)}. \tag{6.27}$$

For $k = 1$ we have from (6.25) – (6.27) that

$$\zeta_L\left(5\right) < \frac{\pi^4}{93}-\frac{1}{186}\left(\frac{7\pi^4}{540}-\frac{1}{12}\right)^2 = 1.039931461,$$
$$\zeta_E\left(5\right) < \frac{\pi^4}{93}\left(1-\frac{\pi^4}{16200}\right) = 1.041111605,$$
$$\zeta_Q\left(5\right) < -93+\sqrt{8649+2\pi^4} = 1.04157688;$$

and for $k = 2$

$$\zeta_L(9) < \frac{17}{160965}\pi^8 - \frac{1}{35770}\left(\frac{31}{28350}\pi^6 - \frac{1}{60}\right)^2 = 1.002082506,$$

$$\zeta_E(9) < \frac{17}{160965}\pi^8\left(1 - \frac{\pi^4}{337650}\right) = 1.0020834954,$$

$$\zeta_Q(9) < -17885 + \frac{1}{3}\sqrt{2878859025 + 34\pi^8} = 1.00208436.$$

It should be noted that the above results give tighter upper bounds for the odd zeta function evaluations than were possible using the methodology developed earlier in the paper, the numerics of which are presented in Table 1.

Numerical experimentation using Maple seems to indicate that the upper bounds for

$$\zeta_L(4k+1), \zeta_E(4k+1) \quad \text{and} \quad \zeta_Q(4k+1)$$

are in increasing order.

REFERENCES

[1] M. ABRAMOWITZ AND I.A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1970.

[2] H. ALZER, Remark on a double-inequality for the Euler zeta function, *Expositiones Mathematicae*, **23**(4) (2005), 349–352.

[3] T.M. APOSTOL, *Analytic Number Theory*, Springer, New York, 1976.

[4] J.M. BORWEIN, D.M. BRADLEY and R.E. CRANDALL, Computational strategies for the Riemann zeta function, *J. of Comput. and Applied Math.*, **121** (2000), 247–296.

[5] P. CERONE, On a double inequality for the Dirichlet beta function, *Acta Mathematica Hungarica*, submitted.

[6] P. CERONE, On applications of the integral of products of functions and its bounds, *RGMIA Res. Rep. Coll.*, **6**(4) (2003), Article 4. [ONLINE `http://rgmia.vu.edu.au/v6n4.html`]

[7] P. CERONE, On odd zeta and other special functions bounds, *Inequality Theory and Applications*, Nova Science Publishers, NY.

[8] P. CERONE, M.A. CHAUDHRY, G. KORVIN AND A. QADIR, New inequalities involving the zeta function, *J. Inequal. Pure Appl. Math.*, **5**(2) (2004), Art. 43. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=392`]

[9] P. CERONE AND S.S. DRAGOMIR, New upper and lower bounds for the Čebyšev functional, *J. Inequal. Pure Appl. Math.*, **3**(5) (2002), Art. 77, 1-11. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=229`]

[10] P. CERONE AND C. LENARD, On integral forms of generalised Mathieu series, *J. Inequal. Pure Appl. Math.*, **4**(5) (2003), Art. 100, 1-11. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=341`]

[11] J.B. CONREY, The Riemann hypothesis, *Notices of the AMS* (2003), 341–353.

[12] D. CVIJOVIĆ AND J. KLINOWSKI, Integral representations of the Riemann zeta function for odd-integer arguments, *J. of Comput. and Applied Math.*, **142**(2) (2002). 435–439.

[13] H.M. EDWARDS, *Riemann's Zeta Function*, Academic Press, New York, 1974.

[14] S.R. FINCH, *Mathematical Constants*, Cambridge Univ. Press, Cambridge, 2003.

[15] B.N. GUO, Note on Mathieu's inequality, *RGMIA Res. Rep. Coll.*, **3**(3) (2000), Article 5. [ONLINE `http://rgmia.vu.edu.au/v3n3.html`]

[16] J. HAVIL, *Gamma: Exploring Euler's constant*, Princeton University Press, New Jersey, 2003.

[17] A. IVIĆ, *The Riemann Zeta Function*, Wiley, New York, 1985.

[18] Q.-M. LUO, B.-N. GUO AND F. QI, On evaluation of Riemann zeta function $\zeta(s)$, *Adv. Stud. Contemp. Math*. (Kyungshang), **7**(2) (2003), 135–144.

[19] Q.-M. LUO, Z.-L. WEI AND F. QI, Lower and upper bounds of $\zeta(3)$, *Adv. Stud. Contemp. Math*. (Kyungshang), **6**(1) (2003), 47–51.

[20] E.C. TITCHMARSH, *The Theory of the Riemann Zeta Function,* Oxford Univ. Press, London, 1951.

[21] H.M. SRIVASTAVA, Certain classes of series associated with the Zeta and related functions, *Appl. Math. & Comput.*, **141** (2003), 13–49.

[22] H.M. SRIVASTAVA, Some rapidly converging series for $\zeta(2n+1)$, *Proc. Amer. Math. Soc*., **127** (1999), 385–3996.

[23] H.M. SRIVASTAVA, Some families of rapidly convergent series representation for the zeta function, *Taiwanese J. Math.*, **4** (2000), 569–596.

[24] H.M. SRIVASTAVA AND J. CHOI, Series associated with the zeta and related functions, Kluwer Acad. Publ., Dordrecht/Boston/London (2001), pp. 388.

[25] E.T. WHITTAKER AND G.N. WATSON, *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 1978.