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COMPANION INEQUALITIES TO JENSEN'S INEQUALITY FOR m-CONVEX AND (α, m)-CONVEX FUNCTIONS

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ABSTRACT. General companion inequalities related to Jensen's inequality for the classes of m-convex and (α, m)-convex functions are presented. We show how Jensen's inequality for these two classes, as well as Slater's inequality, can be obtained from these general companion inequalities as special cases. We also present several variants of the converse Jensen's inequality, weighted Hermit-Hadamard's inequalities and inequalities of Giaccardi and Petrović for these two classes of functions.

Key words and phrases: m-convex functions, (α, m)-convex functions, Jensen's inequality, Slater's inequality, Hermite-Hadamard's inequalities, Giaccardi's inequality, Fejér's inequalities.

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1. INTRODUCTION

Let $[0, b]$, $b > 0$, be an interval of the real line $\mathbb{R}$, and let $K(b)$ be the class of all functions $f : [0, b] \to \mathbb{R}$ which are continuous and nonnegative on $[0, b]$ and such that $f(0) = 0$. We

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define the mean function F of the function $f \in K(b)$ as

$$F(x) = \begin{cases} \frac{1}{x}\int_0^x f(t)\,dt, & x \in (0,b] \\ 0, & x = 0 \end{cases} .$$

We say that the function f is *convex* on $[0,b]$ if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$$

holds for all $x, y \in [0,b]$ and $t \in [0,1]$. Let $K_C(b)$ denote the class of all functions $f \in K(b)$ convex on $[0,b]$, and let $K_F(b)$ be the class of all functions $f \in K(b)$ *convex in mean* on $[0,b]$, i.e., the class of all functions $f \in K(b)$ for which $F \in K_C(b)$. Let $K_S(b)$ denote the class of all functions $f \in K(b)$ which are *starshaped* with respect to the origin on $[0,b]$, i.e., the class of all functions f with the property that

$$f(tx) \le tf(x)$$

holds for all $x \in [0,b]$ and $t \in [0,1]$. In the paper [1], Bruckner and Ostrow, among others, proved that

$$K_C(b) \subset K_F(b) \subset K_S(b).$$

In the paper [10] G. Toader defined the *m-convexity*: another intermediate between the usual convexity and starshaped convexity.

Definition 1.1. The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0,1]$, if we have

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$. We say that f is m-concave if $-f$ is m-convex.
Denote by $K_m(b)$ the class of all m-convex functions on $[0,b]$ for which $f(0) \le 0$.

Obviously, for $m = 1$ Definition 1.1 recaptures the concept of standard convex functions on $[0,b]$, and for $m = 0$ the concept of starshaped functions.

The following lemmas hold (see [11]).

Lemma A. *If f is in the class $K_m(b)$, then it is starshaped.*

Lemma B. *If f is in the class $K_m(b)$ and $0 < n < m \le 1$, then f is in the class $K_n(b)$.*

From Lemma A and Lemma B it follows that

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0,1)$. Note that in the class $K_1(b)$ we have only the convex functions $f : [0,b] \to \mathbb{R}$ for which $f(0) \le 0$, i.e., $K_1(b)$ is a proper subclass of the class of convex functions on $[0,b]$.

It is interesting to point out that for any $m \in (0,1)$ there are continuous and differentiable functions which are m-convex, but which are not convex in the standard sense. Furthermore, in the paper [12], the following theorem was proved.

Theorem A. *For each $m \in (0,1)$ there is an m-convex polynomial f such that f is not n-convex for any $m < n \le 1$.*

For instance, $f : [0,\infty) \to \mathbb{R}$ defined as

$$f(x) = \frac{1}{12}\left(x^4 - 5x^3 + 9x^2 - 5x\right)$$

is $\frac{16}{17}$-convex, but it is not m-convex for any $m \in \left(\frac{16}{17}, 1\right]$ (see [7]).

It is well known (see for example [8, p. 5]) that the function $f : (a, b) \to \mathbb{R}$ is convex iff there is at least one line support for f at each point $x_0 \in (a, b)$, i.e.,

$$f(x_0) \leq f(x) + \lambda(x_0 - x),$$

for all $x \in (a, b)$, where $\lambda \in \mathbb{R}$ depends on x_0 and is given by $\lambda = f'(x_0)$ when $f'(x_0)$ exists, and $\lambda \in [f'_-(x_0), f'_+(x_0)]$ when $f'_-(x_0) \neq f'_+(x_0)$.

The following Lemma [3] gives an analogous result for m-convex functions.

Lemma C. *If f is differentiable, then f is m-convex iff*

$$f(x) \leq mf(y) + f'(x)(x - my)$$

for all $x, y \in [0, b]$.

The notion of m-convexity can be further generalized via introduction of another parameter $\alpha \in [0, 1]$ in the definition of m-convexity. The class of (α, m)-convex functions was first introduced in [6] and it is defined as follows.

Definition 1.2. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha) f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.
Denote by $K_m^\alpha(b)$ the class of all (α, m)-convex functions on $[0, b]$ for which $f(0) \leq 0$.

It can be easily seen that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex functions. Note that in the class $K_1^1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$, i.e., $K_1^1(b)$ is a proper subclass of the class of all convex functions on $[0, b]$. The interested reader can find more about partial ordering of convexity in [8, p. 8, 280].

Lemma C for m-convex functions has its analogue for the class of (α, m)-convex functions, as it is stated below (see [6]).

Lemma D. *If f is differentiable, then f is (α, m)-convex on $[0, b]$ iff we have*

$$f'(x)(x - my) \geq \alpha(f(x) - mf(y)),$$

for all $x, y \in [0, b]$.

The paper is organized as follows.

In Section 2 we first prove a general companion inequality related to Jensen's inequality for m-convex functions in its integral and discrete form. We show that Jensen's inequality for m-convex functions, as well as Slater's inequality, can be obtained from this general inequality as two special cases. In this section we also present two converse Jensen's inequalities for m-convex functions.

In Section 3 we use results from Section 2 to prove several more inequalities for m-convex functions: weighted Hermite-Hadamard's inequalities and inequalities of Giaccardi and Petrović.

In Section 4 we give a selection of the results presented in Sections 2 and 3, but for the class of (α, m)-convex functions.

2. COMPANION INEQUALITIES TO JENSEN'S INEQUALITY FOR m-CONVEX FUNCTIONS

Theorem 2.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $f : [0,b] \to \mathbb{R},\ b > 0,$ be a differentiable $m-$convex function on $[0,b]$ with $m \in (0,1]$. If $u : \Omega \to [0,b]$ is a measurable function such that $f' \circ u$ is in $L^1(\mu)$, then for any $\xi, \eta \in [0,b]$ we have*

$$\frac{f(\xi)}{m} + f'(\xi)\left(\frac{1}{\mu(\Omega)}\int_\Omega u d\mu - \frac{\xi}{m}\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega (f \circ u)\, d\mu \leq mf(\eta) + \frac{1}{\mu(\Omega)}\int_\Omega (u - m\eta)(f' \circ u)\, d\mu. \tag{2.1}$$

Proof. First observe that since u is measurable and bounded we have $u \in L^\infty(\mu)$ and since f is differentiable we also know that $f \circ u \in L^1(\mu)$ (moreover, it is in $L^\infty(\mu)$). On the other hand, by the assumption we have $f' \circ u \in L^1(\mu)$, so it also follows that $u \cdot (f' \circ u) \in L^1(\mu)$.

From Lemma C we know that the inequalities

$$f(x) + f'(x)(my - x) \leq mf(y), \tag{2.2}$$

$$f(y) \leq mf(x) + f'(y)(y - mx), \tag{2.3}$$

hold for all $x, y \in [0,b]$. If in (2.2) we let $x = \xi$ and $y = u(t)$, $t \in \Omega$, we get

$$f(\xi) + f'(\xi)(mu(t) - \xi) \leq m(f \circ u)(t), \quad t \in \Omega.$$

Integrating over Ω we obtain

$$\mu(\Omega) f(\xi) + f'(\xi)\left(m\int_\Omega u d\mu - \xi\mu(\Omega)\right) \leq m\int_\Omega (f \circ u)\, d\mu,$$

from which the left hand side of (2.1) immediately follows.

In order to obtain the right hand side of (2.1) we proceed in a similar way: if in (2.3) we let $x = \eta$ and $y = u(t)$, $t \in \Omega$, we get

$$(f \circ u)(t) \leq mf(\eta) + (f' \circ u)(t)(u(t) - m\eta), \quad t \in \Omega,$$

so after integration over Ω we obtain

$$\int_\Omega (f \circ u)\, d\mu \leq m\mu(\Omega) f(\eta) + \int_\Omega (u - m\eta)(f' \circ u)\, d\mu,$$

from which the right hand side of (2.1) easily follows. □

If $m = 1$, Theorem 2.1 gives an analogous result for convex functions which was proved in [5].

The following theorem is a variant of Theorem 2.1 for the class of starshaped functions.

Theorem 2.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $f : [0,b] \to \mathbb{R},\ b > 0,$ be a differentiable starshaped function. If $u : \Omega \to [0,b]$ is a measurable function such that $f' \circ u$ is in $L^1(\mu)$, then we have*

$$\frac{1}{\mu(\Omega)}\int_\Omega (f \circ u)\, d\mu \leq \frac{1}{\mu(\Omega)}\int_\Omega u(f' \circ u)\, d\mu.$$

Proof. As a special case of Lemma D for $m = 0$ we obtain

$$f(x) \leq x f'(x).$$

After putting

$$x = u(t), \qquad t \in \Omega,$$

we follow the same idea as in the proof of the previous theorem. □

Our next corollary is the discrete version of Theorem 2.1.

Corollary 2.3. *Let $f : [0, b] \to \mathbb{R}$, $b > 0$, be a differentiable $m-$convex function on $[0, b]$ with $m \in (0, 1]$. Let $p_1, ..., p_n$ be nonnegative real numbers such that $P_n = \sum_{i=1}^{n} p_i \neq 0$ and let $x_i \in [0, b]$ be given real numbers. Then for any $\xi, \eta \in [0, b]$ we have*

$$\frac{f(\xi)}{m} + f'(\xi)\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - \frac{\xi}{m}\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \leq m f(\eta) + \frac{1}{P_n}\sum_{i=1}^{n} p_i (x_i - m\eta) f'(x_i).$$

Proof. This is a direct consequence of Theorem 2.1: we simply choose

$$\Omega = \{1, 2, ..., n\},$$
$$\mu(\{i\}) = p_i, \quad i = 1, 2, ..., n,$$
$$u(i) = x_i, \quad i = 1, 2, ..., n.$$

□

Now we give an estimation of the difference between the first two inequalities in (2.1). The obtained inequality incorporates the integral version of the Dragomir-Goh result [2] for convex functions defined on an open interval in $\mathbb{R}$.

Corollary 2.4. *Let all the assumptions of Theorem 2.1 be satisfied. We have*

$$0 \leq \frac{1}{\mu(\Omega)}\int_{\Omega}(f \circ u)\, d\mu - \frac{1}{m} f\left(\frac{m}{\mu(\Omega)}\int_{\Omega} u\, d\mu\right) \leq \frac{m^2 - 1}{m} f\left(\frac{m}{\mu(\Omega)}\int_{\Omega} u\, d\mu\right) + \frac{1}{\mu(\Omega)}\int_{\Omega}\left(u - \frac{m^2}{\mu(\Omega)}\int_{\Omega} u\, d\mu\right)(f' \circ u)\, d\mu.$$

Proof. If we let ξ and η in (2.1) be defined as

$$\xi = \eta = \frac{m}{\mu(\Omega)}\int_{\Omega} u\, d\mu \in [0, b],$$

we obtain

$$\frac{1}{m} f\left(\frac{m}{\mu(\Omega)}\int_{\Omega} u\, d\mu\right) \leq \frac{1}{\mu(\Omega)}\int_{\Omega}(f \circ u)\, d\mu \leq m f\left(\frac{m}{\mu(\Omega)}\int_{\Omega} u\, d\mu\right) + \frac{1}{\mu(\Omega)}\int_{\Omega}\left(u - \frac{m^2}{\mu(\Omega)}\int_{\Omega} u\, d\mu\right)(f' \circ u)\, d\mu.$$

□

Our next result is the integral Jensen's inequality for m-convex functions.

Corollary 2.5. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space with* $0 < \mu(\Omega) < \infty$ *and let* $f : [0, b] \to \mathbb{R}$, $b > 0$, *be a differentiable* $m-$*convex function on* $[0, b]$ *with* $m \in (0, 1]$. *If* $u : \Omega \to [0, b]$ *is a measurable function, then we have*

$$\frac{1}{m} f\left(\frac{m}{\mu(\Omega)} \int_\Omega u d\mu\right) \leq \frac{1}{\mu(\Omega)} \int_\Omega (f \circ u)\, d\mu.$$

The following theorem gives Slater's inequality for m-convex functions.

Theorem 2.6. *Let all the assumptions of Theorem 2.1 be satisfied. If*

$$\int_\Omega (f' \circ u)\, d\mu \neq 0, \qquad \frac{\int_\Omega u (f' \circ u)\, d\mu}{m \int_\Omega (f' \circ u)\, d\mu} \in [0, b], \tag{2.4}$$

then we have

$$\frac{1}{\mu(\Omega)} \int_\Omega (f \circ u)\, d\mu \leq m f\left(\frac{\int_\Omega u (f' \circ u)\, d\mu}{m \int_\Omega (f' \circ u)\, d\mu}\right).$$

Proof. If the conditions (2.4) are satisfied, then in Theorem 2.1 we may choose

$$\eta = \frac{\int_\Omega u (f' \circ u)\, d\mu}{m \int_\Omega (f' \circ u)\, d\mu},$$

so from the right hand side of the inequality (2.1) we obtain

$$\frac{1}{\mu(\Omega)} \int_\Omega (f \circ u)\, d\mu \leq m f\left(\frac{\int_\Omega u (f' \circ u)\, d\mu}{m \int_\Omega (f' \circ u)\, d\mu}\right),$$

since in this case

$$\int_\Omega (u - m\eta) (f' \circ u)\, d\mu = \int_\Omega \left(u - \frac{\int_\Omega u (f' \circ u)\, d\mu}{\int_\Omega (f' \circ u)\, d\mu}\right) (f' \circ u)\, d\mu = 0.$$

□

If $m = 1$ Theorem 2.6 recaptures Slater's result from [9]: if f is convex and increasing and if $\int_\Omega (f' \circ u)\, d\mu \neq 0$ we have

$$\frac{\int_\Omega u (f' \circ u)\, d\mu}{\int_\Omega (f' \circ u)\, d\mu} = \frac{1}{\nu(\Omega)} \int_\Omega u d\nu \in [0, b],$$

where the positive measure ν is defined as $d\nu = (f' \circ u)\, d\mu$.

In the next two theorems we give converses of the integral Jensen's inequality for m-convex functions.

Theorem 2.7. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space with* $0 < \mu(\Omega) < \infty$ *and let* $f : [0, \infty) \to \mathbb{R}$ *be an* $m-$*convex function on* $[0, \infty)$ *with* $m \in (0, 1]$. *If* $u : \Omega \to [a, b]$, $0 \leq a < b < \infty$, *is a measurable function such that* $f \circ u$ *is in* $L^1(\mu)$, *then we have*

$$\begin{aligned} &\frac{1}{\mu(\Omega)} \int_\Omega (f \circ u)\, d\mu \\ &\quad \leq \min\left\{\frac{b - \overline{u}}{b - a} f(a) + m \frac{\overline{u} - a}{b - a} f\left(\frac{b}{m}\right), m \frac{b - \overline{u}}{b - a} f\left(\frac{a}{m}\right) + \frac{\overline{u} - a}{b - a} f(b)\right\}, \end{aligned} \tag{2.5}$$

where

$$\overline{u} = \frac{1}{\mu(\Omega)} \int_\Omega u d\mu.$$

Proof. We may write

$$(f \circ u)(t) = f\left(\frac{b-u(t)}{b-a}a + m\frac{u(t)-a}{b-a}\frac{b}{m}\right), \qquad t \in \Omega.$$

Since f is m-convex on $[0,\infty)$ we have

$$(f \circ u)(t) \le \frac{b-u(t)}{b-a}f(a) + m\frac{u(t)-a}{b-a}f\left(\frac{b}{m}\right), \qquad t \in \Omega,$$

and after integration over Ω we get

$$\int_\Omega (f \circ u)\, d\mu \le \mu(\Omega)\left[\frac{b-\overline{u}}{b-a}f(a) + m\frac{\overline{u}-a}{b-a}f\left(\frac{b}{m}\right)\right]. \tag{2.6}$$

In the similar way we obtain

$$\int_\Omega (f \circ u)\, d\mu \le \mu(\Omega)\left[m\frac{b-\overline{u}}{b-a}f\left(\frac{a}{m}\right) + \frac{\overline{u}-a}{b-a}f(b)\right],$$

so (2.5) immediately follows. □

Theorem 2.8. *Let all the assumptions of Theorem 2.7 be satisfied and suppose also that the function u is symmetric about $\frac{a+b}{2}$. Then we have*

$$\frac{1}{\mu(\Omega)}\int_\Omega (f \circ u)\, d\mu \le \min\left\{\frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{mf\left(\frac{a}{m}\right) + f(b)}{2}\right\}.$$

Proof. Since u is symmetric about $\frac{a+b}{2}$ we have

$$u(t) = a + b - u(t) = \frac{u(t)-a}{b-a}a + m\frac{b-u(t)}{b-a}\frac{b}{m}$$

from which we get

$$\int_\Omega (f \circ u)\, d\mu \le \mu(\Omega)\left[\frac{\overline{u}-a}{b-a}f(a) + m\frac{b-\overline{u}}{b-a}f\left(\frac{b}{m}\right)\right]. \tag{2.7}$$

If we add (2.6) to (2.7) and then divide the sum by $2\mu(\Omega)$ we obtain

$$\begin{aligned}\frac{1}{\mu(\Omega)}\int_\Omega (f \circ u)\, d\mu &\le \frac{1}{2}\left[\frac{b-\overline{u}}{b-a}f(a) + m\frac{\overline{u}-a}{b-a}f\left(\frac{b}{m}\right)\right.\\ &\qquad \left. + \frac{\overline{u}-a}{b-a}f(a) + m\frac{b-\overline{u}}{b-a}f\left(\frac{b}{m}\right)\right]\\ &= \frac{f(a) + mf\left(\frac{b}{m}\right)}{2}.\end{aligned}$$

Analogously we obtain

$$\frac{1}{\mu(\Omega)}\int_\Omega (f \circ u)\, d\mu \le \frac{mf\left(\frac{a}{m}\right) + f(b)}{2}.$$

□

Corollary 2.9. *Let $f : [0,\infty) \to \mathbb{R}$ be an $m-$convex function on $[0,\infty)$ with $m \in (0,1]$. Let $p_1, ..., p_n$ be nonnegative real numbers such that $P_n = \sum_{i=1}^n p_i \neq 0$ and let $x_i \in [a,b]$, $0 \le a < b < \infty$, be given real numbers. Than we have*

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \le \min\left\{\frac{b-\overline{x}}{b-a}f(a) + m\frac{\overline{x}-a}{b-a}f\left(\frac{b}{m}\right), m\frac{b-\overline{x}}{b-a}f\left(\frac{a}{m}\right) + \frac{\overline{x}-a}{b-a}f(b)\right\},$$

where

$$\overline{x}=\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i.$$

Proof. The proof is analogous to the proof of Corollary 2.3. □

3. SOME FURTHER RESULTS

In this section we first show that the Fejér inequalities [4] (i.e., weighted Hermite-Hadamard's inequalities) for m-convex functions presented in [13, Th7, Th8] can be obtained as special cases of Theorem 2.1 and Theorem 2.7.

Corollary 3.1. *Let $f:[0,b]\to\mathbb{R}$ be an $m-$convex function on $[0,b]$ with $m\in(0,1]$ and let $g:[a,b]\to[0,\infty)$ be integrable and symmetric about $\frac{a+b}{2}$, where $0\le a<b<\infty$. If f is differentiable and f' is in $L^1([a,b])$, then*

$$\left[\frac{f(mb)}{m}-\frac{b-a}{2}f'(mb)\right]\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le \int_a^b\left[(x-ma)\,f'(x)+mf(a)\right]g(x)\,dx.$$

Proof. This is a simple consequence of Theorem 2.1. We just choose μ to be the Lebesgue measure defined as $d\mu=g(x)\,dx$, $\Omega=[a,b]$, $u(x)=x$ for all $x\in[a,b]$, $\xi=mb$ and $\eta=a$. Note that in this case we have

$$\frac{1}{\mu(\Omega)}\int_\Omega u\,d\mu=\frac{\int_a^b xg(x)\,dx}{\int_a^b g(x)\,dx}=\frac{a+b}{2}.$$

□

Corollary 3.2. *Let $f:[0,\infty)\to\mathbb{R}$ be an $m-$convex function on $[0,\infty)$ with $m\in(0,1]$ and let $g:[a,b]\to[0,\infty)$ be integrable and symmetric about $\frac{a+b}{2}$, where $0\le a<b<\infty$. If f is in $L^1([a,b])$, then*

$$\int_a^b f(x)\,g(x)\,dx\le\min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{mf\left(\frac{a}{m}\right)+f(b)}{2}\right\}\int_a^b g(x)\,dx. \tag{3.1}$$

If f is also differentiable, then

$$\frac{1}{m}f\left(m\frac{a+b}{2}\right)\int_a^b g(x)\,dx\le\int_a^b f(x)\,g(x)\,dx. \tag{3.2}$$

Proof. If we choose μ to be the Lebesgue measure defined as $d\mu=g(x)\,dx$, $\Omega=[a,b]$, $u(x)=x$ for all $x\in[a,b]$, then (3.1) is obtain directly from Theorem 2.7. Similarly, the inequality (3.2) is obtained from Corollary 2.5. □

In two following theorems we prove inequalities of Giaccardi and Petrović for m-convex functions.

Theorem 3.3. *Let $f:[0,\infty)\to\mathbb{R}$ be an $m-$convex function on $[0,\infty)$ with $m\in(0,1]$. Let $x_0,\ x_i$ and p_i $(i=1,...,n)$ be nonnegative real numbers. If*

$$(x_i-x_0)(\widetilde{x}-x_i)\ge 0\quad(i=1,2,\dots,n),\quad \widetilde{x}\ne x_0, \tag{3.3}$$

where $\widetilde{x} = \sum_{k=1}^{n} p_k x_k$, *then*

$$\sum_{k=1}^{n} p_k f(x_k) \le \min\left\{ mAf\left(\frac{\widetilde{x}}{m}\right) + (P_n - 1) Bf(x_0), Af(\widetilde{x}) + m(P_n - 1) Bf\left(\frac{x_0}{m}\right) \right\}, \tag{3.4}$$

where

$$A = \frac{\sum_{k=1}^{n} p_k (x_k - x_0)}{\widetilde{x} - x_0}, \qquad B = \frac{\widetilde{x}}{\widetilde{x} - x_0}.$$

Proof. From the condition (3.3) we may deduce that

$$x_0 \le x_i \le \widetilde{x}, \quad (i = 1, ..., n),$$

or

$$\widetilde{x} \le x_i \le x_0, \quad (i = 1, ..., n).$$

Suppose that the first conclusion holds true. We may apply Corollary 2.9 to obtain

$$\frac{1}{P_n} \sum_{k=1}^{n} p_k f(x_k) \le \min\left\{ \frac{\widetilde{x} - \overline{x}}{\widetilde{x} - x_0} f(x_0) + m \frac{\overline{x} - x_0}{\widetilde{x} - x_0} f\left(\frac{\widetilde{x}}{m}\right), m \frac{\widetilde{x} - \overline{x}}{\widetilde{x} - x_0} f\left(\frac{x_0}{m}\right) + \frac{\overline{x} - x_0}{\widetilde{x} - x_0} f(\widetilde{x}) \right\}.$$

Since

$$P_n \left[\frac{\widetilde{x} - \overline{x}}{\widetilde{x} - x_0} f(x_0) + m \frac{\overline{x} - x_0}{\widetilde{x} - x_0} f\left(\frac{\widetilde{x}}{m}\right) \right] = (P_n - 1) \frac{\widetilde{x}}{\widetilde{x} - x_0} f(x_0) + m \frac{\sum_{k=1}^{n} p_k (x_k - x_0)}{\widetilde{x} - x_0} f\left(\frac{\widetilde{x}}{m}\right),$$

and

$$P_n \left[m \frac{\widetilde{x} - \overline{x}}{\widetilde{x} - x_0} f\left(\frac{x_0}{m}\right) + \frac{\overline{x} - x_0}{\widetilde{x} - x_0} f(\widetilde{x}) \right] = m(P_n - 1) \frac{\widetilde{x}}{\widetilde{x} - x_0} f\left(\frac{x_0}{m}\right) + \frac{\sum_{k=1}^{n} p_k (x_k - x_0)}{\widetilde{x} - x_0} f(\widetilde{x}),$$

the inequality (3.4) is proved.

The other case is similar. □

Corollary 3.4. *Let* $f : [0, \infty) \to \mathbb{R}$ *be an* $m-$*convex function on* $[0, \infty)$ *with* $m \in (0, 1]$. *Let* x_i *and* p_i $(i = 1, ..., n)$ *be nonnegative real numbers. If*

$$0 \ne \widetilde{x} = \sum_{k=1}^{n} p_k x_k \ge x_i \qquad (i = 1, ..., n),$$

then

$$\sum_{k=1}^{n} p_k f(x_k) \le \min\left\{ mf\left(\frac{\widetilde{x}}{m}\right) + (P_n - 1) f(0), f(\widetilde{x}) + m(P_n - 1) f(0) \right\}.$$

Proof. This is a special case of Theorem 3.3 for $x_0 = 0$. □

4. INEQUALITIES FOR (α, m)-CONVEX FUNCTIONS

In this section we first give a general companion inequality to Jensen's inequality for (α, m) $-$convex functions.

Theorem 4.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $f : [0, b] \to \mathbb{R}$, $b > 0$, be a differentiable (α, m) $-$convex function on $[0, b]$ with $(\alpha, m) \in (0, 1]^2$. If $u : \Omega \to [0, b]$ is a measurable function such that $f' \circ u$ is in $L^1(\mu)$, then for any $\xi, \eta \in [0, b]$ we have*

$$\frac{f(\xi)}{m} + \frac{f'(\xi)}{\alpha}\left(\frac{1}{\mu(\Omega)}\int_\Omega u d\mu - \frac{\xi}{m}\right) \leq \frac{1}{\mu(\Omega)}\int_\Omega (f \circ u) d\mu \leq mf(\eta) + \frac{1}{\alpha\mu(\Omega)}\int_\Omega (u - m\eta)(f' \circ u) d\mu. \tag{4.1}$$

Proof. From Lemma D we know that the inequalities

$$\alpha f(x) + f'(x)(my - x) \leq \alpha m f(y), \tag{4.2}$$

$$\alpha f(y) \leq \alpha m f(x) + f'(y)(y - mx), \tag{4.3}$$

hold for all $x, y \in [0, b]$. If in (4.2) we let $x = \xi$ and $y = u(t)$, $t \in \Omega$, we get

$$\alpha f(\xi) + f'(\xi)(mu(t) - \xi) \leq \alpha m (f \circ u)(t), \quad t \in \Omega.$$

Integrating over Ω we obtain

$$\mu(\Omega)\alpha f(\xi) + f'(\xi)\left(m\int_\Omega u d\mu - \xi\mu(\Omega)\right) \leq \alpha m \int_\Omega (f \circ u) d\mu,$$

from which the left hand side of (4.1) immediately follows.

In order to obtain the right hand side of (4.1) we proceed in a similar way: if in (4.3) we let $x = \eta$ and $y = u(t)$, $t \in \Omega$, we get

$$\alpha (f \circ u)(t) \leq \alpha m f(\eta) + (f' \circ u)(t)(u(t) - m\eta), \quad t \in \Omega,$$

so after integration over Ω we obtain

$$\alpha \int_\Omega (f \circ u) d\mu \leq \alpha m \mu(\Omega) f(\eta) + \int_\Omega (u - m\eta)(f' \circ u) d\mu,$$

from which the right hand side of (4.1) easily follows. □

The following theorem is a variant of Theorem 4.1 for the class of α-starshaped functions (i.e. $(\alpha, 0)$-convex functions).

Theorem 4.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $f : [0, b] \to \mathbb{R}$, $b > 0$, be a differentiable α-starshaped function. If $u : \Omega \to [0, b]$ is a measurable function such that $f' \circ u$ is in $L^1(\mu)$, then we have*

$$\frac{1}{\mu(\Omega)}\int_\Omega (f \circ u) d\mu \leq \frac{1}{\alpha\mu(\Omega)}\int_\Omega u (f' \circ u) d\mu.$$

Proof. Similarly to the proof of Theorem 2.2. □

Our next corollary gives the integral version of the Dragomir-Goh result [2] for the class of (α, m)-functions.

Corollary 4.3. *Let all the assumptions of Theorem 4.1 be satisfied. We have*

$$\begin{aligned}0 &\le \frac{1}{\mu(\Omega)}\int_\Omega (f\circ u)\,d\mu - \frac{1}{m}f\left(\frac{m}{\mu(\Omega)}\int_\Omega u d\mu\right)\\ &\le \frac{m^2-1}{m}f\left(\frac{m}{\mu(\Omega)}\int_\Omega u d\mu\right) + \frac{1}{\alpha\mu(\Omega)}\int_\Omega\left(u-\frac{m^2}{\mu(\Omega)}\int_\Omega u d\mu\right)(f'\circ u)\,d\mu.\end{aligned}$$

Proof. If we let ξ and η in (4.1) be defined as

$$\xi=\eta=\frac{m}{\mu(\Omega)}\int_\Omega u d\mu\in[0,b],$$

we obtain

$$\begin{aligned}\frac{1}{m}f\left(\frac{m}{\mu(\Omega)}\int_\Omega u d\mu\right) &\le \frac{1}{\mu(\Omega)}\int_\Omega (f\circ u)\,d\mu\\ &\le mf\left(\frac{m}{\mu(\Omega)}\int_\Omega u d\mu\right)\\ &\quad+\frac{1}{\alpha\mu(\Omega)}\int_\Omega\left(u-\frac{m^2}{\mu(\Omega)}\int_\Omega u d\mu\right)(f'\circ u)\,d\mu.\end{aligned}$$

□

It may be interesting to note here that the variants of Jensen's inequality and Slater's inequality for the class of (α,m)-convex functions are the same as the variants for the class of m-convex functions $(\alpha\neq 0)$, i.e., they do not depend on α.

In the next theorem we give a converse of the integral Jensen's inequality for (α,m)-convex functions.

Theorem 4.4. *Let $(\Omega,\mathcal{A},\mu)$ be a measure space with $0<\mu(\Omega)<\infty$ and let $f:[0,\infty)\to\mathbb{R}$ be an (α,m)-convex function on $[0,\infty)$, with $\alpha\in(0,1)$ and $m\in(0,1]$. If $u:\Omega\to[a,b]$, $0\le a<b<\infty$, is a measurable function such that $f\circ u$ is in $L^1(\mu)$, then we have*

$$\begin{aligned}&\frac{1}{\mu(\Omega)}\int_\Omega (f\circ u)\,d\mu\\ &\le \min\left\{mf\left(\frac{b}{m}\right)+\frac{1}{\mu(\Omega)}\left[f(a)-mf\left(\frac{b}{m}\right)\right]\int_\Omega\left(\frac{b-u(t)}{b-a}\right)^\alpha d\mu,\right.\\ &\qquad\left. mf\left(\frac{a}{m}\right)+\frac{1}{\mu(\Omega)}\left[f(b)-mf\left(\frac{a}{m}\right)\right]\int_\Omega\left(\frac{u(t)-a}{b-a}\right)^\alpha d\mu\right\}.\end{aligned}$$

If additionally we have $f(a)-mf\left(\frac{b}{m}\right)\ge 0$, then

$$\begin{aligned}\frac{1}{\mu(\Omega)}\int_\Omega (f\circ u)\,d\mu &\le mf\left(\frac{b}{m}\right)+\left[f(a)-mf\left(\frac{b}{m}\right)\right]\left(\frac{b-\overline{u}}{b-a}\right)^\alpha\\ &\le mf\left(\frac{b}{m}\right)+\left[f(a)-mf\left(\frac{b}{m}\right)\right]\left[1-\alpha\frac{\overline{u}-a}{b-a}\right],\end{aligned}$$

or symmetrically, if we have $f(b)-mf\left(\frac{a}{m}\right)\ge 0$, then

$$\begin{aligned}\frac{1}{\mu(\Omega)}\int_\Omega (f\circ u)\,d\mu &\le mf\left(\frac{a}{m}\right)+\left[f(b)-mf\left(\frac{a}{m}\right)\right]\left(\frac{\overline{u}-a}{b-a}\right)^\alpha\\ &\le mf\left(\frac{a}{m}\right)+\left[f(b)-mf\left(\frac{a}{m}\right)\right]\left[1-\alpha\frac{b-\overline{u}}{b-a}\right].\end{aligned}$$

Proof. We can write

$$(f\circ u)(t)=f\left(\frac{b-u(t)}{b-a}a+m\left[1-\frac{b-u(t)}{b-a}\right]\frac{b}{m}\right),\quad t\in\Omega.$$

Since f is (α,m)-convex we have

$$\begin{aligned}(f\circ u)(t)&\le\left(\frac{b-u(t)}{b-a}\right)^{\alpha}f(a)+m\left[1-\left(\frac{b-u(t)}{b-a}\right)^{\alpha}\right]f\left(\frac{b}{m}\right)\\&=mf\left(\frac{b}{m}\right)+\left[f(a)-mf\left(\frac{b}{m}\right)\right]\left(\frac{b-u(t)}{b-a}\right)^{\alpha},\quad t\in\Omega,\end{aligned}$$

so after integration over Ω we obtain

$$\frac{1}{\mu(\Omega)}\int_{\Omega}(f\circ u)\,d\mu\le mf\left(\frac{b}{m}\right)+\frac{1}{\mu(\Omega)}\left[f(a)-mf\left(\frac{b}{m}\right)\right]\int_{\Omega}\left(\frac{b-u(t)}{b-a}\right)^{\alpha}d\mu. \tag{4.4}$$

Analogously, from

$$(f\circ u)(t)=f\left(\frac{u(t)-a}{b-a}b+m\left[1-\frac{u(t)-a}{b-a}\right]\frac{a}{m}\right),\quad t\in\Omega,$$

we obtain

$$\frac{1}{\mu(\Omega)}\int_{\Omega}(f\circ u)\,d\mu\le mf\left(\frac{a}{m}\right)+\frac{1}{\mu(\Omega)}\left[f(b)-mf\left(\frac{a}{m}\right)\right]\int_{\Omega}\left(\frac{u(t)-a}{b-a}\right)^{\alpha}d\mu. \tag{4.5}$$

Suppose now that $f(a)-mf\left(\frac{b}{m}\right)\ge 0$. We know that the function $\varphi:[0,\infty)\to\mathbb{R}$ defined as $\varphi(x)=x^{\alpha}$, where $\alpha\in(0,1]$ is fixed, is concave on $[0,\infty)$, so from the integral Jensen's inequality we have

$$\frac{1}{\mu(\Omega)}\int_{\Omega}\left(\frac{b-u(t)}{b-a}\right)^{\alpha}d\mu\le\left(\frac{1}{\mu(\Omega)}\int_{\Omega}\frac{b-u(t)}{b-a}d\mu\right)^{\alpha}=\left(\frac{b-\overline{u}}{b-a}\right)^{\alpha}.$$

Using that, from (4.4) we obtain

$$\frac{1}{\mu(\Omega)}\int_{\Omega}(f\circ u)\,d\mu\le mf\left(\frac{b}{m}\right)+\left[f(a)-mf\left(\frac{b}{m}\right)\right]\left(\frac{b-\overline{u}}{b-a}\right)^{\alpha}.$$

On the other hand, from the generalized Bernoulli's inequality we have

$$\left(\frac{b-\overline{u}}{b-a}\right)^{\alpha}\le 1-\alpha\left(1-\frac{b-\overline{u}}{b-a}\right)=1-\alpha\frac{\overline{u}-a}{b-a},$$

so from (4.4) we may deduce

$$\begin{aligned}&mf\left(\frac{b}{m}\right)+\left[f(a)-mf\left(\frac{b}{m}\right)\right]\left(\frac{b-\overline{u}}{b-a}\right)^{\alpha}\\&\qquad\le mf\left(\frac{b}{m}\right)+\left[f(a)-mf\left(\frac{b}{m}\right)\right]\left[1-\alpha\frac{\overline{u}-a}{b-a}\right].\end{aligned}$$

Analogously, if $f(b)-mf\left(\frac{a}{m}\right)\ge 0$, from (4.5) we obtain

$$\frac{1}{\mu(\Omega)}\int_{\Omega}(f\circ u)\,d\mu\le mf\left(\frac{a}{m}\right)+\left[f(b)-mf\left(\frac{a}{m}\right)\right]\left(\frac{\overline{u}-a}{b-a}\right)^{\alpha},$$

and

$$mf\left(\frac{a}{m}\right)+\left[f(b)-mf\left(\frac{a}{m}\right)\right]\left(\frac{\overline{u}-a}{b-a}\right)^{\alpha} \leq mf\left(\frac{a}{m}\right)+\left[f(b)-mf\left(\frac{a}{m}\right)\right]\left[1-\alpha\frac{b-\overline{u}}{b-a}\right].$$

This completes the proof. □

Remark 4.5. It can be easily seen that the assertion of Theorem 4.4 remains valid for $\alpha = 1$, since in this case we directly have

$$\frac{1}{\mu(\Omega)}\int_{\Omega}\frac{b-u(t)}{b-a}d\mu=\frac{b-\overline{u}}{b-a},$$
$$\frac{1}{\mu(\Omega)}\int_{\Omega}\frac{u(t)-a}{b-a}d\mu=\frac{\overline{u}-a}{b-a}.$$

This means that in this case the conditions $f(a)-mf\left(\frac{b}{m}\right)\geq 0$ and $f(b)-mf\left(\frac{a}{m}\right)\geq 0$ can be omitted, which implies that for $\alpha = 1$ Theorem 4.4 gives the previously obtained result for m-convex functions given in Theorem 2.7.

At the end of this section we give two variants of the weighted Hadamard inequality for (α, m)-convex functions and also a variant of Hadamard's inequality.

Corollary 4.6. *Let $f:[0,b]\to\mathbb{R}$ be an (α,m)-convex function on $[0,b]$ with $(\alpha,m)\in(0,1]^2$ and let $g:[a,b]\to[0,\infty)$ be integrable and symmetric about $\frac{a+b}{2}$, where $0\leq a<b<\infty$. If f is differentiable and f' is in $L^1([a,b])$, then*

$$\left[\frac{f(mb)}{m}-\frac{b-a}{2\alpha}f'(mb)\right]\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \int_a^b\left[\frac{x-ma}{\alpha}f'(x)+mf(a)\right]g(x)\,dx.$$

Proof. The proof is similar to that of Corollary 3.1. □

Corollary 4.7. *Let $f:[0,\infty)\to\mathbb{R}$ be an (α,m)-convex function on $[0,\infty)$, with $\alpha\in(0,1)$ and $m\in(0,1]$, and let $g:[a,b]\to[0,\infty)$ be integrable and symmetric about $\frac{a+b}{2}$, where $0\leq a<b<\infty$. If f is in $L^1([a,b])$, then*

$$\int_a^b f(x)g(x)\,dx \leq \min\left\{mf\left(\frac{b}{m}\right)\int_a^b g(x)\,dx+\left[f(a)-mf\left(\frac{b}{m}\right)\right]\int_a^b\left(\frac{b-x}{b-a}\right)^{\alpha}g(x)\,dx,\right.$$
$$\left.mf\left(\frac{a}{m}\right)\int_a^b g(x)\,dx+\left[f(b)-mf\left(\frac{a}{m}\right)\right]\int_a^b\left(\frac{x-a}{b-a}\right)^{\alpha}g(x)\,dx\right\}.$$

If additionally we have $f(a)-mf\left(\frac{b}{m}\right)\geq 0$, then

$$\int_a^b f(x)g(x)\,dx\leq\left[mf\left(\frac{b}{m}\right)+\frac{1}{2^{\alpha}}\left(f(a)-mf\left(\frac{b}{m}\right)\right)\right]\int_a^b g(x)\,dx,$$

or symmetrically, if we have $f(b)-mf\left(\frac{a}{m}\right)\geq 0$, then

$$\int_a^b f(x)g(x)\,dx\leq\left[mf\left(\frac{a}{m}\right)+\frac{1}{2^{\alpha}}\left(f(b)-mf\left(\frac{a}{m}\right)\right)\right]\int_a^b g(x)\,dx.$$

If f is differentiable, then we also have

$$\frac{1}{m}f\left(m\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx. \tag{4.6}$$

Proof. If we choose μ to be the Lebesgue measure defined as $d\mu = g(x)\,dx$, $\Omega = [a,b]$, $u(x) = x$ for all $x \in [a,b]$, then (3.1) is obtained directly from Theorem 4.4. Note that in this case we have

$$\left(\frac{\overline{u}-a}{b-a}\right)^{\alpha} = \left(\frac{b-\overline{u}}{b-a}\right)^{\alpha} = \frac{1}{2^{\alpha}}.$$

The inequality (4.6) is a simple consequence of Corollary 4.3. □

Corollary 4.8. *Let $f : [0,\infty) \to \mathbb{R}$ be an (α,m)-convex function on $[0,\infty)$, with $\alpha \in (0,1)$ and $m \in (0,1]$, and let $0 \le a < b < \infty$. If f is in $L^1([a,b])$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \min\left\{mf\left(\frac{b}{m}\right) + \frac{1}{\alpha+1}\left[f(a) - mf\left(\frac{b}{m}\right)\right],\right.$$
$$\left. mf\left(\frac{a}{m}\right) + \frac{1}{\alpha+1}\left[f(b) - mf\left(\frac{a}{m}\right)\right]\right\}.$$

If f is differentiable, then we also have

$$\frac{1}{m}f\left(m\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Proof. Directly from Corollary 4.7. We simply choose the function g to be the constant function 1, and in that case we have

$$\int_a^b \left(\frac{b-x}{b-a}\right)^{\alpha} g(x)\,dx = \int_a^b \left(\frac{x-a}{b-a}\right)^{\alpha} g(x)\,dx = \frac{b-a}{\alpha+1}.$$

□

Variants of other inequalities, which were proved for the class of m-convex functions in Sections 2 and 3, can be also stated for this class of mappings, but we omit the details.

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