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SOME P.D.F.-FREE UPPER BOUNDS FOR THE DISPERSION $\sigma(X)$ AND THE QUANTITY $\sigma^2(X) + (x - EX)^2$

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ABSTRACT. In comparison with Theorems 2.1 and 2.4 in [1], we provide some *p.d.f.*-free upper bounds for the dispersion $\sigma(X)$ and the quantity $\sigma^2(X) + (x - EX)^2$ taking only into account the endpoints of the given finite interval.

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1. INTRODUCTION AND RESULTS

Let $f : [a, b] \subset \mathbb{R} \to [0, \infty)$ be the *p.d.f.* (probability density function) of a random variable X whose expectation and dispersion are respectively given by

$$EX = \int_a^b t f(t)\, dt$$

and

$$\sigma(X) = \sqrt{\int_a^b (t - EX)^2 f(t)\, dt} = \sqrt{\int_a^b t^2 f(t)\, dt - (EX)^2}.$$

In [1], Theorems 2.1 and 2.4, the following upper bounds were obtained for the dispersion $\sigma(X)$

$$\sigma(X) \le \begin{cases} \frac{\sqrt{3}(b-a)^2}{6} \|f\|_\infty & \text{if} \quad f \in L_\infty[a, b] \\ \frac{\sqrt{2}(b-a)^{1+q^{-1}}}{2[(q+1)(2q+1)]^{\frac{2}{q}}} \|f\|_p & \text{if} \quad f \in L_p[a, b],\ p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{\sqrt{2}(b-a)}{2} & \text{if} \quad f \in L_1[a, b] \end{cases}$$

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and the quantity $\sigma^2(X) + (x - EX)^2$

$$\sigma^2(X) + (x - EX)^2 \le \begin{cases} (b-a)\left[\frac{(b-a)^2}{12} + \left(x - \frac{b+a}{2}\right)^2\right]\sqrt{\|f\|_\infty} & \text{if} \quad f \in L_\infty[a,b] \\ \left[\frac{(b-x)^{2q+1} + (x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{2q}} \sqrt{\|f\|_p} & \text{if} \quad f \in L_p[a,b], \; p > 1, \; \frac{1}{p} + \frac{1}{q} = 1 \\ \left(\frac{b-a}{2} + \left|x - \frac{b+a}{2}\right|\right)^2 & \text{if} \quad f \in L_1[a,b] \end{cases}$$

for all $x \in [a,b]$.

In this communication we intend to make free from the *p.d.f.* the above upper bounds for the dispersion $\sigma(X)$ and the quantity $\sigma^2(X) + (x - EX)^2$ taking only into account the endpoints of the given finite interval.

Theorem 1.1. *Under the above restriction on the p.d.f. we have*

$$\sigma(X) \le \min\{\max\{|a|, |b|\}, \; b - a\}.$$

Proof. First, for any number $t \in [a,b]$ we note (via $f(t) \ge 0$) that $af(t) \le tf(t) \le bf(t)$ leading to $a \le EX \le b$. Consequently,

$$0 \le EX - a \le b - a \quad \text{and} \quad 0 \le b - EX \le b - a. \tag{1.1}$$

We point out that the function $g : [a,b] \to [0,\infty)$, defined by $g(t) = (t - EX)^2$, is a bounded convex function which assumes the minimum at point $(EX, 0)$. Thus

$$\begin{aligned} \beta &:= \sup\left\{(t - EX)^2 : t \in [a,b]\right\} \\ &= \max\left\{(a - EX)^2, \; (b - EX)^2\right\} \\ &\le (b-a)^2, \end{aligned}$$

by taking into consideration (1.1). Now, it can be easily seen that

$$\sigma(X) = \sqrt{\int_a^b (t - EX)^2 f(t)\,dt} \le \sqrt{\beta \int_a^b f(t)\,dt} = \sqrt{\beta} \le b - a.$$

Next, using the facts that function $h(t) = t^2$ decreases on $(-\infty, \; 0)$ and increases on $(0, \; \infty)$ on the one hand and,

$$\sigma(X) = \sqrt{\int_a^b t^2 f(t)\,dt - (EX)^2} \le \sqrt{\int_a^b t^2 f(t)\,dt}$$

on the other, we can easily check that

$$\sigma^2(X) \le \int_a^b t^2 f(t)\,dt \le \begin{cases} b^2 & \text{if} \quad a \ge 0 \\ \max\{a^2, b^2\} & \text{if} \quad a < 0 \text{ and } b > 0 \\ a^2 & \text{if} \quad b \le 0, \end{cases}$$

so that $\sigma^2(X) \le \max\{a^2, b^2\}$. Therefore, we can conclude on the validity of the argument. □

Theorem 1.2. *Under the above restriction on the p.d.f. we have*

$$\sqrt{\sigma^2(X) + (x - EX)^2} \le 2\min\{\max\{|a|, |b|\}, \; b - a\}$$

for all $x \in [a, \; b]$.

Proof. We recall the identity

$$\sigma^2(X) + (x - EX)^2 = \int_a^b (t-x)^2 f(t)\, dt, \qquad x \in [a,b],$$

from the proof of Theorem 2.4 in [1]. Clearly,

$$\int_a^b (t-x)^2 f(t)\, dt \le \max\left\{(t-x)^2 : t,\ x \in [a,b]\right\},$$

so that

$$\sqrt{\sigma^2(X) + (x - EX)^2} \le \max\{|t-x| : t,\ x \in [a,b]\}.$$

It is obvious that $0 \le t - a \le b - a$ and $0 \le x - a \le b - a$, since $t,\ x \in [a,\ b]$. We note that we can estimate from above the quantity $|t-x|$ in two ways:

$$|t-x| \le |t-a| + |a-x| \le 2(b-a)$$

and

$$|t-x| \le |t| + |x| \le 2\max\{|a|, |b|\}.$$

Consequently,

$$\max\{|t-x| : t,\ x \in [a,b]\} \le 2\min\{\max\{|a|, |b|\},\ b-a\}.$$

This leads to the desired result. □

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