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AN EXTENSION OF THE REGION OF VARIABILITY OF A SUBCLASS OF UNIVALENT FUNCTIONS

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ABSTRACT. We show that for $\alpha \in (0,2]$, if $f \in \mathcal{A}$ with $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies the condition

$$(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec F(z),$$

then f is univalent in $\mathbb{E}$, where F is the conformal mapping of the unit disk $\mathbb{E}$ with $F(0) = 1$ and

$$F(\mathbb{E}) = \mathbb{C} \setminus \left\{w \in \mathbb{C} : \Re\, w = \alpha,\ |\Im\, w| \geq \sqrt{\alpha(2-\alpha)}\right\}.$$

Our result extends the region of variability of the differential operator

$$(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right),$$

implying univalence of $f \in \mathcal{A}$ in $\mathbb{E}$, for $0 < \alpha \leq 2$.

Key words and phrases: Analytic Function, Univalent function, Starlike function, Differential subordination.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be the class of functions analytic in $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N}$ (set of natural numbers), let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}$ be the class of functions f, analytic in $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let f be analytic in $\mathbb{E}$, g analytic and univalent in $\mathbb{E}$ and $f(0) = g(0)$. Then, by the symbol $f(z) \prec g(z)$ (f subordinate to g) in $\mathbb{E}$, we shall mean $f(\mathbb{E}) \subset g(\mathbb{E})$.

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Let $\psi : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$, with $(p(z), zp'(z)) \in \mathbb{C} \times \mathbb{C}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$, then the function p is said to satisfy first order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z), \qquad \psi(p(0), 0) = h(0). \tag{1.1}$$

A univalent function q is called a dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of $\mathbb{E}$.

Denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively, the classes of starlike functions of order α and convex functions of order α, which are analytically defined as follows:

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathbb{E},\ 0 \leq \alpha < 1 \right\},$$

and

$$\mathcal{K}(\alpha) = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in \mathbb{E},\ 0 \leq \alpha < 1 \right\}.$$

We write $\mathcal{S}^* = \mathcal{S}^*(0)$, the class of univalent starlike convex functions (w.r.t. the origin) and $\mathcal{K}(0) = \mathcal{K}$, the class of univalent convex functions.

A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number $\alpha, -\pi/2 < \alpha < \pi/2$, and a convex function g (not necessarily normalized) such that

$$\Re\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0, \qquad z \in \mathbb{E}.$$

It is well-known that every close-to-convex function is univalent. In 1934/35, Noshiro [4] and Warchawski [8] obtained a simple but interesting criterion for univalence of analytic functions. They proved that if an analytic function f satisfies the condition $\Re f'(z) > 0$ for all z in $\mathbb{E}$, then f is close-to-convex and hence univalent in $\mathbb{E}$.

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $\Re\, \phi'(0) > 0$, then, the function $f \in \mathcal{A}$ is said to be ϕ-like in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0, \qquad z \in \mathbb{E}.$$

This concept was introduced by Brickman [2]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is ϕ-like for some ϕ. Later, Ruscheweyh [5] investigated the following general class of ϕ-like functions:

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. Then the function $f \in \mathcal{A}$ is called ϕ-like with respect to a univalent function $q, q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \qquad z \in \mathbb{E}.$$

Let $\mathcal{H}_\alpha(\beta)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \beta, \qquad z \in \mathbb{E},$$

where α and β are pre-assigned real numbers. Al-Amiri and Reade [1], in 1975, have shown that for $\alpha \leq 0$ and for $\alpha = 1$, the functions in $\mathcal{H}_\alpha(0)$ are univalent in $\mathbb{E}$. In 2005, Singh, Singh and Gupta [7] proved that for $0 < \alpha < 1$, the functions in $\mathcal{H}_\alpha(\alpha)$ are also univalent. In 2007, Singh, Gupta and Singh [6] proved that the functions in $\mathcal{H}_\alpha(\beta)$ satisfy the differential inequality $\Re\ f'(z) > 0,\ z \in \mathbb{E}$. Hence they are univalent for all real numbers α and β satisfying

$\alpha \leq \beta < 1$ and the result is sharp in the sense that the constant β cannot be replaced by any real number less than α.

The main objective of this paper is to extend the region of variability of the operator

$$(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right),$$

implying univalence of $f \in \mathcal{A}$ in $\mathbb{E}$, for $0 < \alpha \leq 2$. We prove a subordination theorem and as applications of the main result, we find the sufficient conditions for $f \in \mathcal{A}$ to be univalent, starlike and ϕ-like.

To prove our main results, we need the following lemma due to Miller and Mocanu.

Lemma 1.1 ([3, p.132, Theorem 3.4 h]). *Let q be univalent in $\mathbb{E}$ and let θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$.*
Set $Q(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either

(i) *h is convex, or*
(ii) *Q is starlike.*
In addition, assume that
(iii) $\Re \frac{zh'(z)}{Q(z)} > 0,\ z \in \mathbb{E}$.
If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p \prec q$ and q is the best dominant.

2. MAIN RESULT

Theorem 2.1. *Let $\alpha \neq 0$ be a complex number. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$$\Re\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] > \max\left\{0, \Re\left(\frac{\alpha-1}{\alpha}q(z)\right)\right\}. \tag{2.1}$$

If $p,\ p(z) \neq 0, z \in \mathbb{E}$, satisfies the differential subordination

$$(1-\alpha)(p(z)-1) + \alpha\frac{zp'(z)}{p(z)} \prec (1-\alpha)(q(z)-1) + \alpha\frac{zq'(z)}{q(z)}, \tag{2.2}$$

then $p \prec q$ and q is the best dominant.

Proof. Let us define the functions θ and ϕ as follows:

$$\theta(w) = (1-\alpha)(w-1),$$

and

$$\phi(w) = \frac{\alpha}{w}.$$

Obviously, the functions θ and ϕ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$ in $\mathbb{D}$.

Now, define the functions Q and h as follows:

$$Q(z) = zq'(z)\phi(q(z)) = \alpha\frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = (1-\alpha)(q(z)-1) + \alpha\frac{zq'(z)}{q(z)}.$$

Then in view of condition (2.1), we have
(1) Q is starlike in $\mathbb{E}$ and

(2) $\Re \frac{z\ h'(z)}{Q(z)} > 0,\ z \in \mathbb{E}$.
Thus conditions (ii) and (iii) of Lemma 1.1, are satisfied.

In view of (2.2), we have

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof, now, follows from Lemma 1.1. □

3. APPLICATIONS TO UNIVALENT FUNCTIONS

On writing $p(z) = f'(z)$ in Theorem 2.1, we obtain the following result.

Theorem 3.1. *Let $\alpha \neq 0$ be a complex number. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ and satisfy the condition (2.1) of Theorem 2.1. If $f \in \mathcal{A},\ f'(z) \neq 0,\ z \in \mathbb{E}$, satisfies the differential subordination*

$$(1-\alpha)(f'(z)-1) + \alpha \frac{zf''(z)}{f'(z)} \prec (1-\alpha)(q(z)-1) + \alpha \frac{zq'(z)}{q(z)},$$

then $f'(z) \prec q(z)$ and q is the best dominant.

On writing $p(z) = \frac{zf'(z)}{f(z)}$ in Theorem 2.1, we obtain the following result.

Theorem 3.2. *Let $\alpha \neq 0$ be a complex number. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ and satisfy the condition (2.1) of Theorem 2.1. If $f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential subordination*

$$(1-2\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)q(z) + \alpha \frac{zq'(z)}{q(z)},$$

then $\frac{zf'(z)}{f(z)} \prec q(z)$ and q is the best dominant.

By taking $p(z) = \frac{zf'(z)}{\phi(f(z))}$ in Theorem 2.1, we obtain the following result.

Theorem 3.3. *Let $\alpha \neq 0$ be a complex number. Let $q,\ q(z) \neq 0$, be a univalent function in $\mathbb{E}$ and satisfy the condition (2.1) of Theorem 2.1. If $f \in \mathcal{A},\ \frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, satisfies the differential subordination*

$$(1-\alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}\right) \prec (1-\alpha)q(z) + \alpha \frac{zq'(z)}{q(z)},$$

where ϕ is analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$, then $\frac{zf'(z)}{\phi(f(z))} \prec q(z)$ and q is the best dominant.

Remark 1. When we select the dominant $q(z) = \frac{1+z}{1-z},\ z \in \mathbb{E}$, then

$$Q(z) = \frac{\alpha z q'(z)}{q(z)} = \frac{2\alpha z}{1-z^2},$$

and

$$\frac{zQ'(z)}{Q(z)} = \frac{1+z^2}{1-z^2}.$$

Therefore, we have

$$\Re \frac{zQ'(z)}{Q(z)} > 0, \qquad z \in \mathbb{E},$$

and hence Q is starlike. We also have

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{1-\alpha}{\alpha}q(z)=\frac{1+z^2}{1-z^2}+\frac{1-\alpha}{\alpha}\frac{1+z}{1-z}.$$

Thus, for any real number $0<\alpha\leq 2$, we obtain

$$\Re\left[1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{1-\alpha}{\alpha}q(z)\right]>0,\qquad z\in\mathbb{E}.$$

Therefore, $q(z)=\frac{1+z}{1-z},\ z\in\mathbb{E}$, satisfies the conditions of Theorem 3.1, Theorem 3.2 and Theorem 3.3.

Moreover,

$$(1-\alpha)(q(z)-1)+\alpha\frac{zq'(z)}{q(z)}=2(1-\alpha)\frac{z}{1-z}+2\alpha\frac{z}{1-z^2}=F(z).$$

For $0<\alpha\leq 2$, we see that F is the conformal mapping of the unit disk $\mathbb{E}$ with $F(0)=0$ and

$$F(\mathbb{E})=\mathbb{C}\setminus\left\{w\in\mathbb{C}:\Re\,w=\alpha-1,\ |\Im\,w|\geq\sqrt{\alpha(2-\alpha)}\right\}.$$

In view of the above remark, on writing $q(z)=\frac{1+z}{1-z}$ in Theorem 3.1, we have the following result.

Corollary 3.4. *If $f\in\mathcal{A}$, $f'(z)\neq 0,\ z\in\mathbb{E}$, satisfies the differential subordination*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec 1+2(1-\alpha)\frac{z}{1-z}+2\alpha\frac{z}{1-z^2},$$

where $0<\alpha\leq 2$ is a real number, then $\Re\,f'(z)>0,\ z\in\mathbb{E}$. Therefore, f is close-to-convex and hence f is univalent in $\mathbb{E}$.

In view of Remark 1 and Corollary 3.4, we obtain the following result.

Corollary 3.5. *Let $0<\alpha\leq 2$ be a real number. Suppose that $f\in\mathcal{A}$, $f'(z)\neq 0,\ z\in\mathbb{E}$, satisfies the condition*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec F(z).$$

Then f is close-to-convex and hence univalent in $\mathbb{E}$, where F is the conformal mapping of the unit disk $\mathbb{E}$ with $F(0)=1$ and

$$F(\mathbb{E})=\mathbb{C}\setminus\left\{w\in\mathbb{C}:\Re\,w=\alpha,\ |\Im\,w|\geq\sqrt{\alpha(2-\alpha)}\right\}.$$

From Corollary 3.4, we obtain the following result of Singh, Gupta and Singh [7].

Corollary 3.6. *Let $0<\alpha<1$ be a real number. If $f\in\mathcal{A}$, $f'(z)\neq 0,\ z\in\mathbb{E}$, satisfies the differential inequality*

$$\Re\left[(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>\alpha,$$

then $\Re\,f'(z)>0,\ z\in\mathbb{E}$. Therefore, f is close-to-convex and hence f is univalent in $\mathbb{E}$.

From Corollary 3.4, we obtain the following result.

Corollary 3.7. *Let* $1 < \alpha \leq 2$, *be a real number. If* $f \in \mathcal{A}$, $f'(z) \neq 0,\ z \in \mathbb{E}$, *satisfies the differential inequality*

$$\Re\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < \alpha,$$

then $\Re f'(z) > 0,\ z \in \mathbb{E}$. *Therefore,* f *is close-to-convex and hence* f *is univalent in* $\mathbb{E}$.

When we select $q(z) = \frac{1+z}{1-z}$ in Theorem 3.2, we obtain the following result.

Corollary 3.8. *If* $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies the differential subordination*

$$(1-2\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec (1-\alpha)\frac{1+z}{1-z} + 2\alpha\frac{z}{1-z^2} = F_1(z),$$

where $0 < \alpha \leq 2$ *is a real number, then* $f \in \mathcal{S}^*$.

In view of Corollary 3.8, we have the following result.

Corollary 3.9. *Let* $0 < \alpha \leq 2$ *be a real number. Suppose that* $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies the condition*

$$(1-2\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec F_1(z).$$

Then $f \in \mathcal{S}^*$, *where* F_1 *is the conformal mapping of the unit disk* $\mathbb{E}$ *with* $F_1(0) = 1-\alpha$ *and*

$$F_1(\mathbb{E}) = \mathbb{C} \setminus \left\{w \in \mathbb{C} : \Re w = 0,\ |\Im w| \geq \sqrt{\alpha(2-\alpha)}\right\}.$$

In view of Corollary 3.8, we have the following result.

Corollary 3.10. *Let* $0 < \alpha < 1$ *be a real number. If* $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies the differential inequality*

$$\Re\left[(1-2\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0,$$

then $f \in \mathcal{S}^*$.

In view of Corollary 3.8, we also have the following result.

Corollary 3.11. *Let* $1 < \alpha \leq 2$, *be a real number. If* $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies the differential inequality*

$$\Re\left[(1-2\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] < 0,$$

then $f \in \mathcal{S}^*$.

When we select $q(z) = \frac{1+z}{1-z}$ in Theorem 3.3, we obtain the following result.

Corollary 3.12. *Let* $0 < \alpha \leq 2$ *be a real number. Let* $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, *satisfy the differential subordination*

$$(1-\alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}\right) \prec (1-\alpha)\frac{1+z}{1-z} + 2\alpha\frac{z}{1-z^2} = F_1(z).$$

Then $\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+z}{1-z}$, *where* ϕ *is analytic in a domain containing* $f(\mathbb{E})$, $\phi(0) = 0, \phi'(0) = 1$ *and* $\phi(w) \neq 0$ *for* $w \in f(\mathbb{E}) \setminus \{0\}$.

In view of Corollary 3.12, we obtain the following result.

Corollary 3.13. *Let* $0 < \alpha \leq 2$ *be a real number. Let* $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0,\ z \in \mathbb{E}$, *satisfy the condition*

$$(1-\alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z[\phi(f(z))]'}{\phi(f(z))}\right) \prec F_1(z).$$

Then f *is* ϕ*-like in* $\mathbb{E}$, *where* ϕ *is analytic in a domain containing* $f(\mathbb{E})$, $\phi(0) = 0, \phi'(0) = 1$ *and* $\phi(w) \neq 0$ *for* $w \in f(\mathbb{E}) \setminus \{0\}$ *and* F_1 *is the conformal mapping of the unit disk* $\mathbb{E}$ *with* $F_1(0) = 1 - \alpha$ *and*

$$F_1(\mathbb{E}) = \mathbb{C} \setminus \left\{w \in \mathbb{C} : \Re\, w = 0,\ |\Im\, w| \geq \sqrt{\alpha(2-\alpha)}\right\}.$$

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