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A GENERALIZATION OF ANDERSSON'S INEQUALITY

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ABSTRACT. Andersson's Inequality is generalized by replacing the integration there with a positive linear functional which operates on a composition of two functions. These two functions have rather light restrictions and this leads to considerable generalizations of Andersson's result.

Key words and phrases: Andersson's inequality, Convex functions, Linear functional.

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1. INTRODUCTION

In all that follows we shall use the terms *increasing, decreasing, positive* and *negative* in the wide sense, meaning *non-decreasing, non-increasing,* etc. Andersson [1] or [2, p. 256] showed that if the functions f_k are convex and increasing in $[0,1]$ with $f_k(0) = 0$ then

$$\int_0^1 f_1(x)f_2(x)\cdots f_n(x)dx \geq \frac{2^n}{n+1}\int_0^1 f_1(x)dx\int_0^1 f_2(x)dx\cdots\int_0^1 f_n(x)dx. \tag{1.1}$$

Then in [3] Fink showed that these hypotheses can be lightened to

$$f_k(0) = 0,\ f_k \in C[0,1] \text{ and } x^{-1}f_k(x) \text{ is increasing} \tag{1.2}$$

Note 1. In (1.2) $x^{-1}f_k(x)$ is initially undefined at the origin but since its limit from the right at $x = 0$ exists, this can be taken as its definition there.

Note 2. That the hypotheses in (1.2) are lighter than those used by Andersson can be seen immediately from the convexity condition

$$f(x) \leq \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) \quad \text{if } 0 < a \leq x \leq b$$

by letting $a \to 0$.

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An interesting special case of (1.1) is obtained by taking all the f_k to be the same function f, when we get

$$\int_0^1 f^n(x)dx \geq \frac{2^n}{n+1}\left(\int_0^1 f(x)dx\right)^n, \quad n=1,2,\dots \tag{1.3}$$

and the obvious question arises here concerning the case of non-integral n.

It is the purpose of this paper to generalize Andersson's result in a way that involves positive linear functionals. With this aim in mind, in the next section we make some preparations and then state our theorems.

2. Definitions and Statement of Results

Let the functions f_k satisfy (1.2) and L denote a positive linear functional defined on $C[0,1]$. We introduce a second set of functions ϕ_k defined by

$$\phi_k = e_1 \frac{L(f_k)}{L(e_1)}, \quad \text{where } e_1(x) = x.$$

Finally we let F_k denote functions defined and differentiable on the ranges of f_k and ϕ_k. (If only one of each of the functions above is involved in certain places we shall omit the subscript). We now introduce our two theorems.

Theorem 2.1 will be a generalization of the special case (1.3) and Theorem 2.2 will be a generalization of (1.1). We choose to proceed in this order since, on the one hand, Theorem 2.1 is of interest in its own right and, secondly, once it is proved, it is a simple matter to prove Theorem 2.2.

Theorem 2.1. *With f satisfying (1.2) and ϕ, F and L being as introduced above we have:*

(a) *If F' and g are increasing then*

$$L[F(f)g] \geq L[F(\phi)g]. \tag{2.1}$$

(b) *If F' and g are decreasing then*

$$L[F(f)g] \leq L[F(\phi)g].$$

Theorem 2.2. *With f_k satisfying (1.2) and ϕ_k, F_k and L being as introduced above we have*

(a) *If all the F_k and F_k' are increasing then*

$$L\left[\prod_{k=1}^n F_k(f_k)\right] \geq L\left[\prod_{k=1}^n F_k(\phi_k)\right]$$

and

(b) *If all the F_k and F_k' are decreasing then*

$$L\left[\prod_{k=1}^n F_k(f_k)\right] \leq L\left[\prod_{k=1}^n F_k(\phi_k)\right]$$

Before proceeding we give an example of Theorem 2.1.

Example 2.1. In Theorem 2.1 take $F(u) = u^\alpha, g(u) = 1$ and let L be defined by

$$L(w) = \int_0^1 w(t)dt.$$

Then

(a)

$$\int_0^1 f^{\,\alpha}(x)dx \geq \frac{2^\alpha}{\alpha+1}\left(\int_0^1 f(x)dx\right)^\alpha \quad \text{for } -1<\alpha\leq 0 \text{ or } \alpha\geq 1 \tag{2.2}$$

and

(b)

$$\int_0^1 f^{\,\alpha}(x)dx \leq \frac{2^\alpha}{\alpha+1}\left(\int_0^1 f(x)dx\right)^\alpha \quad \text{for } 0\leq\alpha\leq 1. \tag{2.3}$$

The values of α are determined by the behaviour of F' (except that the condition $-1<\alpha$ is required to ensure the convergence of the integral on the left).

The above example answers the question which arose at (1.3).

3. PROOFS

First we need two lemmas.

Lemma 3.1. *Let $p, q \in C[0,1]$, and L be a positive linear functional. Suppose that $L(p)=0$ and that $p(x)$ changes sign once, from negative to positive, in the interval and suppose that $q(x)$ is increasing there. Then*

(a)

$$L(pq)\geq 0$$

(b) *If $q(x)$ is decreasing then the inequality is to be reversed.*

Proof of Lemma 3.1(a). If $q(x)$ is constant the result is trivial. Otherwise there is $\gamma\in(0,1)$ such that $p(\gamma)=0$.

Then, defining

$$p_1(x)=\min(0,p(x)) : p_2(x)=\max(0,p(x)) \quad \text{in } [0,1]$$

we have

$$p_1(x)q(x)\geq p_1(x)q(\gamma) \quad \text{in } [0,\gamma)$$

and

$$p_2(x)q(x)\geq p_2(x)q(\gamma) \quad \text{in } [\gamma,1]$$

So

$$\begin{aligned} L(pq) &= L(p_1q)+L(p_2q)\\ &\geq L(p_1q(\gamma))+L(p_2q(\gamma))\\ &= q(\gamma)L(p)=0 \end{aligned}$$

which completes the proof of (a). The proof of Lemma 3.1(b) is similar. □

The next lemma was proved in [3] for the case in which L is integration over $[0,1]$. Here we give a different proof which refers to a general positive linear functional.

Lemma 3.2. *With f and ϕ as above we have*

(a)

$$L[(f-\phi)g]\equiv L\left[\left(f-e_1\frac{L(f)}{L(e_1)}\right)g\right]\geq 0 \quad \textit{for all } f$$

if g is increasing. (If g is constant or if $f=\phi$ we will have equality)

(b) *The inequality is to be reversed if g is decreasing.*

Proof of Lemma 3.2(a). First we observe that the difference

$$f(x) - \phi(x) \equiv f(x) - x\frac{L(f)}{L(e_1)}$$

changes sign in $(0,1)$ because $x^{-1}f(x)$ is increasing and both

$$f(x) - x\frac{L(f)}{L(e_1)} > 0 \quad \text{and} \quad f(x) - x\frac{L(f)}{L(e_1)} < 0 \quad \text{in } 0 \leq x \leq 1$$

are seen to be impossible, on operating through with L. Clearly, this sign change is from minus to plus.

It is also clear that

$$L(f - \phi) = 0$$

and so the result follows on taking

$$f - \phi = p \text{ and } g = q$$

in Lemma 3.1(a). The proof of part (b) is similar. □

Proof of Theorem 2.1(a). F' and g are increasing functions. Then

$$[F(f(x)) - F(\phi(x))]g(x) = [f(x) - \phi(x)]Q(x)g(x), \tag{3.1}$$

where

$$Q(x) \equiv \frac{1}{[f(x) - \phi(x)]}\int_{\phi(x)}^{f(x)} F'(t)dt$$

It is a simple matter to see that this quotient is increasing with x.

In fact, it is obvious since $Q(x)$ is the average value of the increasing function F' over the interval $(f(x), \phi(x))$ [or $(\phi(x), f(x))$], each of whose end-points moves to the right with increasing x.

Since Qg is an increasing function, then from (3.1) we get

$$L[F(f)g - F(\phi)g] = L[(f - \phi)Qg] \geq 0$$

on applying Lemma 3.2 to the right hand side.

This concludes the proof of Theorem 2.1(a) and the proof of part (b) is similar. □

Note 3. The function g played no significant part in this proof but its presence is needed when we come to deduce Theorem 2.2 from Theorem 2.1.

Proof of Theorem 2.2(a). For the sake of brevity we shall take $n = 3$ because this will indicate the method of proof for any $n > 1$.

We have

$$L[F_1(f_1)F_2(f_2)F_3(f_3)] \geq L[F_1(f_1)F_2(f_2)F_3(\phi_3)]$$

on reading $F_1(f_1)F_2(f_2)$ as g in (2.1).

Then

$$L[F_1(f_1)F_2(f_2)F_3(\phi_3)] \geq L[F_1(f_1)F_2(\phi_2)F_3(\phi_3)]$$

on reading $F_1(f_1)F_3(\phi_3)$ as g in (2.1).

Finally

$$L[F_1(f_1)F_2(\phi_2)F_3(\phi_3)] \geq L[F_1(\phi_1)F_2(\phi_2)F_3(\phi_3)]$$

on reading $F_2(\phi_2)F_3(\phi_3)$ as g in (2.1).

The general case is proved in exactly the same way. This concludes the proof of Theorem 2.2(a) and that of part (b) is similar. □

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