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DIFFERENCE OF GENERAL INTEGRAL MEANS

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ABSTRACT. In this paper we present sharp estimates for the difference of general integral means with respect to even different finite measures. This is achieved by the use of the Ostrowski and Fink inequalities and the Geometric Moment Theory Method. The produced inequalities are with respect to the supnorm of a derivative of the involved function.

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1. INTRODUCTION

Here our work is motivated by the works of J. Duoandikoetxea [5] and P. Cerone [4]. We use Ostrowski's ([8]) and Fink's ([6]) inequalities along with the Geometric Moment Theory Method, see [7], [1], [3], to prove our results.

We compare general averages of functions with respect to various finite measures over different subintervals of a domain, even disjoint. Our estimates are sharp and the inequalities are attained. They are with respect to the supnorm of a derivative of the involved function f.

To the best of our knowledge this type of work is totally new.

2. RESULTS

Part A

As motivation we give the following proposition.

Proposition 2.1. *Let μ_1, μ_2 be finite Borel measures on $[a,b] \subseteq \mathbb{R}$, $[c,d]$, $[\tilde{e},g] \subseteq [a,b]$, $f \in C^1([a,b])$. Denote $\mu_1([c,d]) = m_1 > 0$, $\mu_2([\tilde{e},g] = m_2 > 0$. Then*

$$\left| \frac{1}{m_1} \int_c^d f(x)d\mu_1 - \frac{1}{m_2} \int_{\tilde{e}}^g f(x)d\mu_2 \right| \leq \|f'\|_\infty (b-a). \tag{2.1}$$

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Proof. From the mean value theorem we have

$$|f(x) - f(y)| \le \|f'\|_\infty (b-a) =: \gamma, \quad \forall x, y \in [a,b],$$

that is,

$$-\gamma \le f(x) - f(y) \le \gamma, \quad \forall x, y \in [a,b],$$

and by fixing y we get

$$-\gamma \le \frac{1}{m_1}\int_c^d f(x)d\mu_1 - f(y) \le \gamma.$$

The last statement holds $\forall y \in [\tilde{e}, g]$. Hence

$$-\gamma \le \frac{1}{m_1}\int_c^d f(x)d\mu_1 - \frac{1}{m_2}\int_{\tilde{e}}^g f(x)d\mu_2 \le \gamma,$$

proving the claim. □

As a related result we have

Corollary 2.2. *Let* $f \in C^1([a,b])$*,* $[c,d]$*,* $[\tilde{e}, g] \subseteq [a,b] \subseteq \mathbb{R}$*. Then we have*

$$\left|\frac{1}{d-c}\int_c^d f(x)dx - \frac{1}{g-\tilde{e}}\int_{\tilde{e}}^g f(x)dx\right| \le \|f'\|_\infty \cdot (b-a). \tag{2.2}$$

We use the following famous Ostrowski inequality, see [8], [2].

Theorem 2.3. *Let* $f \in C^1([a,b])$*,* $x \in [a,b]$*. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{\|f'\|_\infty}{2(b-a)}\left((x-a)^2 + (x-b)^2\right), \tag{2.3}$$

and inequality (2.3) is sharp, see [2].

We also have

Corollary 2.4. *Let* $f \in C^1([a,b])$*,* $x \in [c,d] \subseteq [a,b] \subseteq \mathbb{R}$*. Then*

$$\begin{aligned} &\left|f(x) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\qquad \le \frac{\|f'\|_\infty}{2(b-a)} \max\left\{((c-a)^2 + (c-b)^2), ((d-a)^2 + (d-b)^2)\right\}. \end{aligned} \tag{2.4}$$

Proof. Obvious. □

We denote by $\mathcal{P}([a,b])$ the power set of $[a,b]$. We give the following.

Theorem 2.5. *Let* $f \in C^1([a,b])$*,* μ *be a finite measure on* $([c,d], \mathcal{P}([c,d]))$*, where* $[c,d] \subseteq [a,b] \subseteq \mathbb{R}$ *and* $m := \mu([c,d]) > 0$*. Then*

(1)

$$\begin{aligned} &\left|\frac{1}{m}\int_{[c,d]} f(x)d\mu - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\qquad \le \frac{\|f'\|_\infty}{2(b-a)} \max\left\{((c-a)^2 + (c-b)^2), ((d-a)^2 + (d-b)^2)\right\}. \end{aligned} \tag{2.5}$$

(2) *Inequality (2.5) is attained when* $d = b$*.*

Proof. 1) By (2.4) integrating against μ/m.

2) Here (2.5) collapses to

$$\left|\frac{1}{m}\int_{[c,b]} f(x)d\mu - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{\|f'\|_\infty}{2}(b-a). \tag{2.6}$$

We prove that (2.6) is attained. Take

$$f^*(x) = \frac{2x-(a+b)}{b-a}, \quad a \le x \le b.$$

Then $f^{*\prime}(x) = \frac{2}{b-a}$ and $\|f^{*\prime}\|_\infty = \frac{2}{b-a}$, along with

$$\int_a^b f^*(x)dx = 0.$$

Therefore (2.6) becomes

$$\left|\frac{1}{m}\int_{[c,b]} f^*(x)d\mu\right| \le 1. \tag{2.7}$$

Finally pick $\frac{\mu}{m} = \delta_{\{b\}}$ the Dirac measure supported at $\{b\}$, then (2.7) turns to equality. □

We further have

Corollary 2.6. *Let $f \in C^1([a,b])$ and $[c,d] \subseteq [a,b] \subseteq \mathbb{R}$. Let $M(c,d) := \{\mu\colon \mu$ a measure on $([c,d], \mathcal{P}([c,d]))$ of finite positive mass$\}$, denoted $m := \mu([c,d])$. Then*

(1) *The following result holds*

$$\sup_{\mu\in M(c,d)}\left|\frac{1}{m}\int_{[c,d]} f(x)d\mu - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{\|f'\|_\infty}{2(b-a)}\max\left\{((c-a)^2+(c-b)^2),((d-a)^2+(d-b)^2)\right\} \tag{2.8}$$
$$= \frac{\|f'\|_\infty}{2(b-a)} \times \begin{Bmatrix} (d-a)^2+(d-b)^2, & \text{if } d+c \ge a+b \\ (c-a)^2+(c-b)^2, & \text{if } d+c \le a+b \end{Bmatrix}$$
$$\le \frac{\|f'\|_\infty}{2}(b-a). \tag{2.9}$$

Inequality (2.9) becomes equality if $d=b$ or $c=a$ or both.

(2) *The following result holds*

$$\sup_{\substack{\text{all } c,d \\ a\le c<d\le b}}\left(\sup_{\mu\in M(c,d)}\left|\frac{1}{m}\int_{[c,d]} f(x)d\mu - \frac{1}{b-a}\int_a^b f(x)dx\right|\right) \le \frac{\|f'\|_\infty}{2}(b-a). \tag{2.10}$$

Next we restrict ourselves to a subclass of $M(c,d)$ of finite measures μ with given first moment and by the use of the Geometric Moment Theory Method, see [7], [1], [3], we produce an inequality sharper than (2.8). For that we need

Lemma 2.7. *Let ν be a probability measure on $([a,b], \mathcal{P}([a,b]))$ such that*

$$\int_{[a,b]} x\, d\nu = d_1 \in [a,b] \tag{2.11}$$

is given. Then

i)

$$U_1 := \sup_{\nu \text{ as in } (2.11)} \int_{[a,b]} (x-a)^2 d\nu = (b-a)(d_1-a), \tag{2.12}$$

and

ii)

$$U_2 := \sup_{\nu \text{ as in } (2.11)} \int_{[a,b]} (x-b)^2 d\nu = (b-a)(b-d_1). \tag{2.13}$$

Proof. i) We observe the graph

$$G_1 = \{(x,(x-a)^2) \colon a \le x \le b\},$$

which is a convex arc above the x-axis. We form the closed convex hull of G_1 and we call it $\widehat{G}_1$ which has as an upper concave envelope the line segment ℓ_1 from $(a,0)$ to $(b,(b-a)^2)$. We consider the vertical line $x = d_1$ which cuts ℓ_1 at the point Q_1. Then U_1 is the distance from $(d_1,0)$ to Q_1. By using the equal ratios property of similar triangles related here we get

$$\frac{d_1-a}{b-a} = \frac{U_1}{(b-a)^2},$$

which proves the claim.

ii) We observe the graph

$$G_2 = \{(x,(x-b)^2) \colon a \le x \le b\},$$

which is a convex arc above the x-axis. We form the closed convex hull of G_2 and we call it $\widehat{G}_2$ which has as an upper concave envelope the line segment ℓ_2 from $(b,0)$ to $(a,(b-a)^2)$. We consider the vertical line $x = d_1$ which intersects ℓ_2 at the point Q_2.

Then U_2 is the distance from $(d_1,0)$ to Q_2. By using the equal ratios property of the related similar triangles we obtain

$$\frac{U_2}{(b-a)^2} = \frac{b-d_1}{b-a},$$

which proves the claim. □

Furthermore we need

Lemma 2.8. *Let $[c,d] \subseteq [a,b] \subseteq \mathbb{R}$ and let ν be a probability measure on $([c,d],\mathcal{P}([c,d]))$ such that*

$$\int_{[c,d]} x\, d\nu = d_1 \in [c,d] \tag{2.14}$$

is given. Then

(i)

$$U_1 := \sup_{\nu \text{ as in (2.14)}} \int_{[c,d]} (x-a)^2 d\nu = d_1(c+d-2a) - cd + a^2, \tag{2.15}$$

and

(ii)

$$U_2 := \sup_{\nu \text{ as in (2.14)}} \int_{[c,d]} (x-b)^2 d\nu = d_1(c+d-2b) - cd + b^2. \tag{2.16}$$

(iii) *The following also holds:*

$$\sup_{\nu \text{ as in (2.14)}} \int_{[c,d]} \left[(x-a)^2 + (x-b)^2\right] d\nu = U_1 + U_2. \tag{2.17}$$

Proof. (i) We see that

$$\int_c^d (x-a)^2 d\nu = (c-a)^2 + 2(c-a)(d_1-c) + \int_c^d (x-c)^2 d\nu.$$

Using (2.12) which is applied on $[c,d]$, we find

$$\begin{aligned}\sup_{\nu \text{ as in (2.14)}} \int_c^d (x-a)^2 d\nu &= (c-a)^2 + 2(c-a)(d_1-c) \\ &\quad + \sup_{\nu \text{ as in (2.14)}} \int_c^d (x-c)^2 d\nu \\ &= (c-a)^2 + 2(c-a)(d_1-c) + (d-c)(d_1-c) \\ &= d_1(c+d-2a) - cd + a^2,\end{aligned}$$

proving the claim.
(ii) We see that

$$\int_c^d (x-b)^2 d\nu = (b-d)^2 + 2(b-d)(d-d_1) + \int_c^d (x-d)^2 d\nu.$$

Using (2.13) which is applied on $[c,d]$, we obtain

$$\begin{aligned}\sup_{\nu \text{ as in (2.14)}} \int_c^d (x-b)^2 d\nu &= (b-d)^2 + 2(b-d)(d-d_1) \\ &\quad + \sup_{\nu \text{ as in (2.14)}} \int_c^d (x-d)^2 d\nu \\ &= (b-d)^2 + 2(b-d)(d-d_1) + (d-c)(d-d_1) \\ &= d_1(c+d-2b) - cd + b^2,\end{aligned}$$

proving the claim.
(iii) Similar to Lemma 2.7 and above and obvious on noting that $(x-a)^2 + (x-b)^2$ is convex, etc. □

Now we are ready to present

Theorem 2.9. *Let* $[c,d] \subseteq [a,b] \subseteq \mathbb{R}$, $f \in C^1([a,b])$, μ *a finite measure on* $([c,d], \mathcal{P}([c,d]))$ *of mass* $m := \mu([c,d]) > 0$. *Assume that*

$$\frac{1}{m}\int_c^d x\, d\mu = d_1, \quad c \le d_1 \le d, \tag{2.18}$$

is given.

Then

$$\begin{aligned}\sup_{\mu \text{ as above}} &\left| \frac{1}{m}\int_c^d f(x) d\mu - \frac{1}{b-a}\int_a^b f(x) dx \right| \\ &\le \frac{\|f'\|_\infty}{(b-a)} \left[d_1\big((c+d)-(a+b)\big) - cd + \frac{a^2+b^2}{2} \right].\end{aligned} \tag{2.19}$$

Proof. Denote

$$\beta(x) := \frac{\|f'\|_\infty}{2(b-a)}\big((x-a)^2 + (x-b)^2\big),$$

then by Theorem 2.3 we have

$$-\beta(x) \le f(x) - \frac{1}{b-a}\int_a^b f(x)dx \le \beta(x), \quad \forall x \in [c,d].$$

Thus

$$-\frac{1}{m}\int_c^d \beta(x)d\mu \le \frac{1}{m}\int_c^d f(x)d\mu - \frac{1}{b-a}\int_a^b f(x)dx \le \frac{1}{m}\int_c^d \beta(x)d\mu,$$

and

$$\left|\frac{1}{m}\int_c^d f(x)d\mu - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{m}\int_c^d \beta(x)d\mu =: \theta.$$

Here $\nu := \frac{\mu}{m}$ is a probability measure subject to (2.18) on $([c,d], \mathcal{P}([c,d]))$ and

$$\begin{aligned}\theta &= \frac{\|f'\|_\infty}{2(b-a)}\left(\int_c^d (x-a)^2\frac{d\mu}{m} + \int_c^d (x-b)^2\frac{d\mu}{m}\right)\\ &= \frac{\|f'\|_\infty}{2(b-a)}\left(\int_c^d (x-a)^2 d\nu + \int_c^d (x-b)^2 d\nu\right).\end{aligned}$$

Using (2.14), (2.15), (2.16) and (2.17) we get

$$\begin{aligned}\theta &\le \frac{\|f'\|_\infty}{2(b-a)}\big\{(d_1(c+d-2a) - cd + a^2) + (d_1(c+d-2b) - cd + b^2)\big\}\\ &= \frac{\|f'\|_\infty}{(b-a)}\left[d_1((c+d)-(a+b)) - cd + \frac{a^2+b^2}{2}\right],\end{aligned}$$

proving the claim. □

We make the following remark.

Remark 2.10 (Remark on Theorem 2.9)**.** (1) Case of $c+d \ge a+b$, using $d_1 \le d$ we obtain

$$d_1\big((c+d)-(a+b)\big) - cd + \frac{a^2+b^2}{2} \le \frac{(d-a)^2+(d-b)^2}{2}. \tag{2.20}$$

(2) Case of $c+d \le a+b$, using $d_1 \ge c$ we find that

$$d_1\big((c+d)-(a+b)\big) - cd + \frac{a^2+b^2}{2} \le \frac{(c-a)^2+(c-b)^2}{2}. \tag{2.21}$$

Hence under (2.18) inequality (2.19) is sharper than (2.8).

We also give

Corollary 2.11. *Let all the assumptions in Theorem 2.9 hold. Then*

$$\begin{aligned}&\left|\frac{1}{m}\int_c^d f(x)d\mu - \frac{1}{b-a}\int_a^b f(x)dx\right|\\ &\qquad\le \frac{\|f'\|_\infty}{(b-a)}\left[d_1\big((c+d)-(a+b)\big) - cd + \frac{a^2+b^2}{2}\right].\end{aligned} \tag{2.22}$$

By Remark 2.10, inequality (2.22) is sharper than (2.5).

Part B

Here we follow Fink's work [6]. We require the following theorem.

Theorem 2.12 ([6]). *Let $f\colon [a,b]\to\mathbb{R}$, $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $n\geq 1$. Then*

$$\begin{aligned} f(x) &= \frac{n}{b-a}\int_a^b f(t)dt \\ &\quad + \sum_{k=1}^{n-1}\left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k}{b-a}\right) \\ &\quad + \frac{1}{(n-1)!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)dt, \end{aligned} \tag{2.23}$$

where

$$k(t,x) := \begin{cases} t-a, & a\leq t\leq x\leq b, \\ t-b, & a\leq x< t\leq b. \end{cases} \tag{2.24}$$

For $n=1$ the sum in (2.23) is taken as zero.

We also need Fink's inequality

Theorem 2.13 ([6]). *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ and $f^{(n)}\in L_\infty(a,b)$, $n\geq 1$. Then*

$$\begin{aligned} &\left|\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\qquad \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)}\left[(b-x)^{n+1}+(x-a)^{n+1}\right], \quad \forall x\in[a,b], \end{aligned} \tag{2.25}$$

where

$$F_k(x) := \left(\frac{n-k}{k!}\right)\left(\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}\right). \tag{2.26}$$

Inequality (2.25) is sharp, in the sense that it is attained by an optimal f for any $x\in[a,b]$.

We give

Corollary 2.14. *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ and $f^{(n)}\in L_\infty(a,b)$, $n\geq 1$. Then $\forall x\in[c,d]\subseteq[a,b]$ we have*

$$\begin{aligned} &\left|\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k(x)\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\qquad \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)}\left[(b-x)^{n+1}+(x-a)^{n+1}\right] \\ &\qquad \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!}(b-a)^n. \end{aligned} \tag{2.27}$$

Also we have

Proposition 2.15. *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ and $f^{(n)}\in L_\infty(a,b)$, $n\geq 1$. Let μ be a finite measure of mass $m>0$ on*

$$\big([c,d],\mathcal{P}([c,d])\big), \quad [c,d]\subseteq[a,b]\subseteq\mathbb{R}.$$

Then

$$K := \left|\frac{1}{n}\left(\frac{1}{m}\int_{[c,d]} f(x)d\mu + \sum_{k=1}^{n-1}\frac{1}{m}\int_{[c,d]} F_k(x)d\mu\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)}\left[\frac{1}{m}\int_{[c,d]}(b-x)^{n+1}d\mu + \frac{1}{m}\int_{[c,d]}(x-a)^{n+1}d\mu\right]$$
$$\leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!}(b-a)^n. \tag{2.28}$$

Proof. By (2.27). ☐

Similarly, based on Theorem A of [6] we also conclude

Proposition 2.16. *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ and $f^{(n)} \in L_p(a,b)$, where $1 < p < \infty$, $n \geq 1$. Let μ be a finite measure of mass $m > 0$ on $([c,d], \mathcal{P}([c,d]))$, $[c,d] \subseteq [a,b] \subseteq \mathbb{R}$.*

Here $p' > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\left|\frac{1}{n}\left(\frac{1}{m}\int_{[c,d]} f(x)d\mu + \sum_{k=1}^{n-1}\frac{1}{m}\int_{[c,d]} F_k(x)d\mu\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \left(\frac{B\big((n-1)p'+1, p'+1\big)^{1/p'}\|f^{(n)}\|_p}{n!(b-a)}\right)$$
$$\cdot\left(\frac{1}{m}\int_{[c,d]}\big((x-a)^{np'+1} + (b-x)^{np'+1}\big)^{1/p'}d\mu\right)$$
$$\leq \left(\frac{B\big((n-1)p'+1, p'+1\big)^{1/p'}(b-a)^{n-1+\frac{1}{p'}}}{n!}\right)\|f^{(n)}\|_p. \tag{2.29}$$

We make the following remark.

Remark 2.17. Clearly we have the following for

$$g(x) := (b-x)^{n+1} + (x-a)^{n+1} \leq (b-a)^{n+1}, \quad a \leq x \leq b, \tag{2.30}$$

where $n \geq 1$. Here $x = \frac{a+b}{2}$ is the only critical number of g and

$$g''\left(\frac{a+b}{2}\right) = n(n+1)\frac{(b-a)^{n-1}}{2^{n-2}} > 0,$$

giving that $g\left(\frac{a+b}{2}\right) = \frac{(b-a)^{n+1}}{2^n} > 0$ is the global minimum of g over $[a,b]$. Also g is convex over $[a,b]$. Therefore for $[c,d] \subseteq [a,b]$ we have

$$M := \max_{c\leq x\leq d}\big\{(x-a)^{n+1} + (b-x)^{n+1}\big\}$$
$$= \max\big\{(c-a)^{n+1} + (b-c)^{n+1}, (d-a)^{n+1} + (b-d)^{n+1}\big\}. \tag{2.31}$$

We get further that

$$M = \begin{cases} (d-a)^{n+1} + (b-d)^{n+1}, & \text{if } c+d \geq a+b \\ (c-a)^{n+1} + (b-c)^{n+1}, & \text{if } c+d \leq a+b. \end{cases} \tag{2.32}$$

If $d = b$ or $c = a$ or both then

$$M = (b-a)^{n+1}. \tag{2.33}$$

Based on Remark 2.17 we give

Theorem 2.18. *Let all assumptions, terms and notations be as in Proposition 2.15. Then*

(1)

$$K \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)} \max\Big\{(c-a)^{n+1}+(b-c)^{n+1}, \tag{2.34}$$
$$(d-a)^{n+1}+(b-d)^{n+1}\Big\}$$
$$= \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)} \times \left\{ \begin{array}{ll} (d-a)^{n+1}+(b-d)^{n+1}, & \text{if } c+d \geq a+b, \\ (c-a)^{n+1}+(b-c)^{n+1}, & \text{if } c+d \leq a+b \end{array} \right\}$$

$$\leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!}(b-a)^n, \tag{2.35}$$

where K is as in (2.28). If $d=b$ or $c=a$ or both, then (2.35) becomes equality. When $d=b$, $\frac{\mu}{m}=\delta_{\{b\}}$ and $f(x)=\frac{(x-a)^n}{n!}$, $a \leq x \leq b$, then inequality (2.34) is attained, i.e. it becomes equality, proving that (2.34) is a sharp inequality.

(2) *We also have*

$$\sup_{\mu \in M(c,d)} K \leq \textit{R.H.S (2.34)} \tag{2.36}$$

and

$$\sup_{\substack{\text{all } c,d \\ a \leq c \leq d \leq b}} \left(\sup_{\mu \in M(c,d)} K \right) \leq \textit{R.H.S (2.35)} \tag{2.37}$$

Proof. It remains to prove only the sharpness, via attainability of (2.34) when $d=b$. In that case (2.34) collapses to

$$\left| \frac{1}{n}\left(\frac{1}{m}\int_{[c,d]} f(x)d\mu + \sum_{k=1}^{n-1} \frac{1}{m}\int_{[c,b]} F_k(x)d\mu \right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \tag{2.38}$$
$$\leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!}(b-a)^n.$$

The optimal measure here will be $\frac{\mu}{m}=\delta_{\{b\}}$ and then (2.38) becomes

$$\left| \frac{1}{n}\left(f(b) + \sum_{k=1}^{n-1} F_k(b) \right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!}(b-a)^n. \tag{2.39}$$

The optimal function here will be

$$f^*(x) = \frac{(x-a)^n}{n!}, \quad a \leq x \leq b.$$

Then we see that

$$f^{*(k-1)}(x) = \frac{(x-a)^{n-k+1}}{(n-k+1)!}, \quad k-1 = 0,1,\dots,n-2,$$

and $f^{*(k-1)}(a)=0$ for $k-1=0,1,\dots,n-2$. Clearly here $F_k(b)=0$, $k=1,\dots,n-1$. Also we have

$$\frac{1}{b-a}\int_a^b f^*(x)dx = \frac{(b-a)^n}{(n+1)!} \qquad \text{and} \qquad \|f^{*(n)}\|_\infty = 1.$$

Putting all these elements in (2.39) we have

$$\left|\frac{(b-a)^n}{nn!}-\frac{(b-a)^n}{(n+1)!}\right|=\frac{(b-a)^n}{n(n+1)!},$$

proving the claim. ☐

Next, we again restrict ourselves to the subclass of $M(c,d)$ of finite measures μ with given first moment and by the use of the Geometric Moment Theory Method, see [7], [1], [3], we produce an inequality sharper than (2.36). For that we need the follwing result.

Lemma 2.19. *Let* $[c,d]\subseteq[a,b]\subseteq\mathbb{R}$ *and* ν *be a probability measure on* $([c,d],\mathcal{P}([c,d]))$ *such that*

$$\int_{[c,d]} x\,d\nu = d_1 \in [c,d] \tag{2.40}$$

is given, $n\geq 1$. *Then*

$$W_1 := \sup_{\nu \text{ as in } (2.40)} \int_{[c,d]} (x-a)^{n+1} d\nu \tag{2.41}$$

$$= \left(\sum_{k=0}^{n}(d-a)^{n-k}(c-a)^k\right)(d_1-d)+(d-a)^{n+1}. \tag{2.42}$$

Proof. We observe the graph

$$G_1=\left\{(x,(x-a)^{n+1})\colon c\leq x\leq d\right\},$$

which is a convex arc above the x-axis. We form the closed convex hull of G_1 and we call it $\widehat{G}_1$, which has as an upper concave envelope the line segment $\overline{\ell}_1$ from $(c,(c-a)^{n+1})$ to $(d,(d-a)^{n+1})$. Call ℓ_1 the line through $\overline{\ell}_1$. The line ℓ_1 intersects the x-axis at $(t,0)$, where $a\leq t\leq c$. We need to determine t: the slope of ℓ_1 is

$$\tilde{m}=\frac{(d-a)^{n+1}-(c-a)^{n+1}}{d-c}=\sum_{k=0}^{n}(d-a)^{n-k}(c-a)^k.$$

The equation of line ℓ_1 is

$$y=\tilde{m}\cdot x+(d-a)^{n+1}-\tilde{m}d.$$

Hence $\tilde{m}t+(d-a)^{n+1}-\tilde{m}d=0$ and

$$t=d-\frac{(d-a)^{n+1}}{\tilde{m}}.$$

Next we consider the moment right triangle with vertices $(t,0)$, $(d,0)$ and $(d,(d-a)^{n+1})$. Clearly $(d_1,0)$ is between $(t,0)$ and $(d,0)$. Consider the vertical line $x=d_1$, it intersects $\overline{\ell}_1$ at Q. Clearly then $W_1=\text{length}\overline{((d_1,0),Q)}$, the line segment of which length we find by the formed two similar right triangles with vertices $\{(t,0),(d_1,0),Q\}$ and $\{(t,0),(d,0),(d,(d-a)^{n+1})\}$. We have the equal ratios

$$\frac{d_1-t}{d-t}=\frac{W_1}{(d-a)^{n+1}},$$

i.e.

$$W_1=(d-a)^{n+1}\left(\frac{d_1-t}{d-t}\right).$$

☐

We also need

Lemma 2.20. *Let* $[c,d] \subseteq [a,b] \subseteq \mathbb{R}$ *and* ν *be a probability measure on* $([c,d], \mathcal{P}([c,d]))$ *such that*

$$\int_{[c,d]} x\, d\nu = d_1 \in [c,d] \tag{2.43}$$

is given, $n \geq 1$*. Then*

(1)

$$W_2 := \sup_{\nu \text{ as in } (2.43)} \int_{[c,d]} (b-x)^{n+1} d\nu$$

$$= \left(\sum_{k=0}^{n} (b-c)^{n-k}(b-d)^k\right)(c-d_1) + (b-c)^{n+1}. \tag{2.44}$$

(2) *The following result holds*

$$\sup_{\nu \text{ as in } (2.43)} \int_{[c,d]} \left[(x-a)^{n+1} + (b-x)^{n+1}\right] d\nu = W_1 + W_2, \tag{2.45}$$

where W_1 *is as in (2.41).*

Proof. (1) We observe the graph

$$G_2 = \left\{(x,(b-x)^{n+1})\colon c \leq x \leq d\right\},$$

which is a convex arc above the x-axis. We form the closed convex hull of G_2 and we call it $\widehat{G}_2$, which has as an upper concave envelope the line segment $\overline{\ell}_2$ from $(c,(b-c)^{n+1})$ to $(d,(b-d)^{n+1})$. Call ℓ_2 the line through $\overline{\ell}_2$. The line ℓ_2 intersects the x-axis at $(t^*, 0)$, where $d \leq t^* \leq b$. We need to determine t^*: The slope of ℓ_2 is

$$\tilde{m}^* = \frac{(b-c)^{n+1} - (b-d)^{n+1}}{c-d} = -\left(\sum_{k=0}^{n}(b-c)^{n-k}(b-d)^k\right).$$

The equation of line ℓ_2 is

$$y = \tilde{m}^* x + (b-c)^{n+1} - \tilde{m}^* c.$$

Hence

$$\tilde{m}^* t^* + (b-c)^{n+1} - \tilde{m}^* c = 0$$

and

$$t^* = c - \frac{(b-c)^{n+1}}{\tilde{m}^*}.$$

Next we consider the moment right triangle with vertices $(c,(b-c)^{n+1})$, $(c,0)$, $(t^*,0)$. Clearly $(d_1,0)$ is between $(c,0)$ and $(t^*,0)$. Consider the vertical line $x = d_1$, it intersects $\overline{\ell}_2$ at Q^*. Clearly then

$$W_2 = \text{length}\overline{((d_1,0),Q^*)},$$

the line segment of which length we find by the formed two similar right triangles with vertices $\{Q^*, (d_1,0), (t^*,0)\}$ and $\{(c,(b-c)^{n+1}), (c,0), (t^*,0)\}$. We have the equal ratios

$$\frac{t^* - d_1}{t^* - c} = \frac{W_2}{(b-c)^{n+1}},$$

i.e.

$$W_2 = (b-c)^{n+1}\left(\frac{t^* - d_1}{t^* - c}\right).$$

(2) Similar to that above and obvious.

□

We make the following useful remark.

Remark 2.21. By Lemmas 2.19, 2.20 we obtain

$$\begin{aligned}\lambda &:= W_1 + W_2 \\ &= \left(\sum_{k=0}^{n}(d-a)^{n-k}(c-a)^k\right)(d_1-d) \\ &\quad + \left(\sum_{k=0}^{n}(b-c)^{n-k}(b-d)^k\right)(c-d_1) + (d-a)^{n+1} + (b-c)^{n+1} > 0,\end{aligned} \tag{2.46}$$

$n \geq 1$.

We present the following important result.

Theorem 2.22. *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ and $f^{(n)} \in L_\infty(a,b)$, $n \geq 1$. Let μ be a finite measure of mass $m > 0$ on $([c,d], \mathcal{P}([c,d]))$, $[c,d] \subseteq [a,b] \subseteq \mathbb{R}$. Furthermore we assume that*

$$\frac{1}{m}\int_{[c,d]} x\,d\mu = d_1 \in [c,d] \tag{2.47}$$

is given. Then

$$\sup_{\mu \text{ as above}} K \leq \frac{\|f^{(n)}\|_\infty}{n(n+1)!(b-a)}\lambda, \tag{2.48}$$

and

$$K \leq \text{R.H.S (2.48)}, \tag{2.49}$$

where K is as in (2.28) and λ is as in (2.46).

Proof. By Proposition 2.15 and Lemmas 2.19 and 2.20. □

We make the following remark.

Remark 2.23. We compare M as in (2.31) and (2.32) and λ as in (2.46). We easily obtain that

$$\lambda \leq M. \tag{2.50}$$

As a result we have that (2.49) is sharper than (2.34) and (2.48) is sharper than (2.36). That is reasonable since we restricted ourselves to a subclass of $M(c,d)$ of measures μ by assuming the moment condition (2.47).

We finish with the following comment.

Remark 2.24.

I) When $c = a$ and $d = b$ then d_1 plays no role in the best upper bounds we found with the Geometric Moment Theory Method. That is, the restriction on measures μ via the first moment d_1 has no effect in producing sharper estimates as it happens when $a < c < d < b$. More precisely we notice that:

(a)

$$\text{R.H.S.}(2.19) = \frac{\|f'\|_\infty}{2}(b-a) = \text{R.H.S.}(2.9), \tag{2.51}$$

(b) by (2.46) here $\lambda = (b-a)^{n+1}$ and

$$\text{R.H.S.}(2.48) = \frac{\|f^{(n)}\|_\infty}{n(n+1)!}(b-a)^n = \text{R.H.S.}(2.35). \tag{2.52}$$

II) Further differences of general means over any $[c_1, d_1]$ and $[c_2, d_2]$ subsets of $[a, b]$ (even disjoint) with respect to μ_1 and μ_2, respectively, can be found by straightforward application of the above results and the triangle inequality.

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