A STUDY ON STARLIKE AND CONVEX PROPERTIES FOR HYPERGEOMETRIC FUNCTIONS

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Abstract: The objective of the present paper is to give some characterizations for a (Gaussian) hypergeometric function to be in various subclasses of starlike and convex functions. We also consider an integral operator related to the hypergeometric function.

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Contents

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1. Introduction

Let T be the class consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqslant 0), \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $T(\lambda, \alpha)$ be the subclass of T consisting of functions which satisfy the condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{\lambda z f'(z) + (1-\lambda) f(z)}\right\} > \alpha, \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda < 1)$ and for all $z \in U$.

Also, let $C(\lambda, \alpha)$ denote the subclass of T consisting of functions which satisfy the condition:

$$\operatorname{Re}\left\{\frac{f'(z) + zf''(z)}{f'(z) + \lambda z f''(z)}\right\} > \alpha, \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda < 1)$ and for all $z \in U$.

From (1.2) and (1.3), we have

$$f(z) \in C(\lambda, \alpha) \Leftrightarrow zf'(z) \in T(\lambda, \alpha). \tag{1.4}$$

We note that $T(0, \alpha) = T^*(\alpha)$, the class of starlike functions of order α $(0 \leq \alpha < 1)$ and $C(0, \alpha) = C(\alpha)$, the class of convex functions of order α $(0 \leq \alpha < 1)$ (see Silverman [6]).

Let $F(a, b; c; z)$ be the (Gaussian) hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{1.5}$$

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Study on Starlike and Convex Properties
A. O. Mostafa
vol. 10, iss. 3, art. 87, 2009

where $c \neq 0, -1, -2, \ldots$, and $(\theta)_n$ is the Pochhammer symbol defined by

$$(\theta)_n = \begin{cases} 1, & n = 0 \\ \theta(\theta+1)\cdots(\theta+n-1) & n \in N = \{1, 2, \ldots\}. \end{cases}$$

We note that $F(a, b; c; 1)$ converges for $\mathrm{Re}(c - a - b) > 0$ and is related to the Gamma function by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.6}$$

Silverman [7] gave necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $T^*(\alpha)$ and $C(\alpha)$, also examining a linear operator acting on hypergeometric functions. For other interesting developments on $zF(a, b; c; z)$ in connection with various subclasses of univalent functions, the reader can refer the to works of Carlson and Shaffer [2], Merkes and Scott [4], Ruscheweyh and Singh [5] and Cho et al. [3].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Main Results

To establish our main results, we need the following lemma due to Altintas and Owa [1].

Lemma 2.1.

(i) *A function* $f(z)$ *defined by* (1.1) *is in the class* $T(\lambda, \alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}(n - \lambda\alpha n - \alpha + \lambda\alpha)a_n \leq 1 - \alpha. \tag{2.1}$$

(ii) *A function* $f(z)$ *defined by* (1.1) *is in the class* $C(\lambda, \alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} n(n - \lambda\alpha n - \alpha + \lambda\alpha)a_n \leq 1 - \alpha. \tag{2.2}$$

Theorem A.

(i) *If* $a, b > -1, c > 0$ *and* $ab < 0$, *then* $zF(a, b; c; z)$ *is in* $T(\lambda, \alpha)$ *if and only if*

$$c > a + b + 1 - \frac{(1 - \lambda\alpha)ab}{1 - \alpha}. \tag{2.3}$$

(ii) *If* $a, b > 0$ *and* $c > a + b + 1$, *then* $F_1(a, b; c; z) = z[2 - F(a, b; c; z)]$ *is in* $T(\lambda, \alpha)$ *if and only if*

$$\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\left[1 + \frac{(1 - \lambda\alpha)ab}{(1 - \alpha)(c - a - b - 1)}\right] \leq 2. \tag{2.4}$$

Study on Starlike and Convex Properties
A. O. Mostafa
vol. 10, iss. 3, art. 87, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Proof. (i) Since

$$zF(a,b;c;z) = z + \frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n \tag{2.5}$$

$$= z - \left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n,$$

according to (i) of Lemma 2.1, we must show that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \le \left|\frac{c}{ab}\right|(1-\alpha)\,. \tag{2.6}$$

Note that the left side of (2.6) diverges if $c < a+b+1$. Now

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$= (1-\lambda\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= (1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \frac{(1-\alpha)c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= (1-\lambda\alpha)\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{(1-\alpha)c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right].$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Hence, (2.6) is equivalent to

$$\text{(2.7)}\quad \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[(1-\lambda\alpha)+\frac{(1-\alpha)(c-a-b-1)}{ab}\right] \le (1-\alpha)\left[\left|\frac{c}{ab}\right|+\frac{c}{ab}\right]=0.$$

Thus, (2.7) is valid if and only if

$$(1-\lambda\alpha)+\frac{(1-\alpha)(c-a-b-1)}{ab}\le 0,$$

or equivalently,

$$c \geqslant a+b+1-\frac{(1-\lambda\alpha)ab}{1-\alpha}.$$

(ii) Since

$$F_1(a,b;c;z)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\le 1-\alpha.$$

Now,

$$\text{(2.8)}\quad \sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \sum_{n=2}^{\infty}[(n-1)(1-\lambda\alpha)+(1-\alpha)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$= (1-\lambda\alpha)\sum_{n=1}^{\infty} n\frac{(a)_n(b)_n}{(c)_n(1)_n} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$= (1-\lambda\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}.$$

Noting that $(\theta)_n = \theta((\theta+1)_{n-1}$ then, (2.8) may be expressed as

$$(1-\lambda\alpha)\frac{ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + (1-\alpha)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$= (1-\lambda\alpha)\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1-\alpha)\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right]$$
$$= (1-\lambda\alpha)\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right]$$
$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[(1-\alpha) + \frac{ab(1-\lambda\alpha)}{(c-a-b-1)}\right] - (1-\alpha).$$

But this last expression is bounded above by $1-\alpha$ if and only if (2.4) holds. □

Theorem B.

(*i*) *If* $a, b > -1$, $ab < 0$, *and* $c > a+b+2$, *then* $zF(a,b;c;z)$ *is in* $C(\lambda,\alpha)$ *if and only if*

$$(1-\lambda\alpha)(a)_2(b)_2 + (3-2\lambda\alpha-\alpha)ab(c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \geqslant 0. \tag{2.9}$$

journal of inequalities in pure and applied mathematics
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(ii) *If* $a, b > 0$ *and* $c > a + b + 2$, *then* $F_1(a, b; c; z) = z[2 - F(a, b; c; z)]$ *is in* $C(\lambda, \alpha)$ *if and only if*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left\{1 + \frac{(1-\lambda\alpha)(a)_2(b)_2}{(1-\alpha)(c-a-b-2)_2} + \left(\frac{3-2\lambda\alpha-\alpha}{1-\alpha}\right)\left(\frac{ab}{c-a-b-1}\right)\right\} \leq 2. \tag{2.10}$$

Proof. (i) Since $zF(a, b; c; z)$ has the form (2.5), we see from (ii) of Lemma 2.1, that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} n(n - \lambda\alpha n - \alpha + \lambda\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left|\frac{c}{ab}\right|(1-\alpha). \tag{2.11}$$

Note that for $c > a + b + 2$, the left side of (2.11) converges. Writing

$$(n+2)[(n+2)(1-\lambda\alpha) - \alpha(1-\lambda)] = (n+1)^2(1-\lambda\alpha) + (n+1)(2-\alpha-\lambda\alpha) + (1-\alpha),$$

we see that

$$\sum_{n=0}^{\infty}(n+2)[(n+2)(1-\lambda\alpha) - \alpha(1-\lambda)]\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$= (1-\lambda\alpha)\sum_{n=0}^{\infty}(n+1)^2\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$
$$+ (2-\alpha-\lambda\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$= (1-\lambda\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$

$$+ (2-\alpha-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1-\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= (1-\lambda\alpha)\sum_{n=0}^{\infty}n\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (3-\alpha-2\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$

$$+ (1-\alpha)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n}$$

$$= \frac{(1-\lambda\alpha)(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}$$

$$+ (3-\alpha-2\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1-\alpha)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Big[-(1-\lambda\alpha)(a+1)(b+1)$$

$$+(3-\alpha-2\lambda\alpha)(c-a-b-2) + \frac{(1-\alpha)}{ab}(c-a-b-2)_2\Big] - \frac{(1-\alpha)c}{ab}.$$

This last expression is bounded above by $\left|\frac{c}{ab}\right|(1-\alpha)$ if and only if

$$(1-\lambda\alpha)(a+1)(b+1)+(3-\alpha-2\lambda\alpha)(c-a-b-2)+\frac{(1-\alpha)}{ab}(c-a-b-2)_2 \le 0,$$

which is equivalent to (2.9).

Study on Starlike and Convex Properties
A. O. Mostafa
vol. 10, iss. 3, art. 87, 2009

journal of inequalities in pure and applied mathematics
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(ii) In view of (ii) of Lemma 2.1, we need to show that

$$\sum_{n=2}^{\infty} n(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \le (1-\alpha)\,.$$

Now

$$\sum_{n=2}^{\infty} n(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \tag{2.12}$$
$$=\sum_{n=0}^{\infty}(n+2)[(n+2)(1-\lambda\alpha)-\alpha(1-\lambda)]\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$=(1-\lambda\alpha)\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}-\alpha(1-\lambda)\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}.$$

Writing $(n+2)=(n+1)+1$, we have

$$\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}=\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \tag{2.13}$$
$$=\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

and

$$\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \tag{2.14}$$
$$=\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+2\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$$= \sum_{n=1}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}} + 3\sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + \sum_{n=1}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.$$

Substituting (2.13) and (2.14) into the right side of (2.12), yields

$$(2.15)\quad (1-\lambda\alpha)\sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_{n}} + (3-2\lambda\alpha-\alpha)\sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}} + (1-\alpha)\sum_{n=1}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.$$

Since $(a)_{n+k} = (a)_k(a+k)_n$, we may write (2.15) as

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(1-\lambda\alpha)(a)_2(b)_2}{(c-a-b-2)_2} + \frac{(3-2\lambda\alpha-\alpha)ab}{(c-a-b-1)} + (1-\alpha)\right] - (1-\alpha).$$

By a simplification, we see that the last expression is bounded above by $(1-\alpha)$ if and only if (2.10) holds. □

Putting $\lambda = 0$ in (i) of Theorem B, we have:

Corollary 2.2. *If* $a, b > -1$, $ab < 0$, *and* $c > a+b+2$, *then* $zF(a,b;c;z)$ *is in* $C(\alpha)$ *if and only if*

$$(a)_2(b)_2 + (3-\alpha)ab(c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \geqslant 0.$$

Remark 1. Corollary 2.2, corrects the result obtained by Silverman [7, Theorem 4].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. An Integral Operator

In the theorems below, we obtain similar results in connection with a particular integral operator $G(a,b;c;z)$ acting on $F(a,b;c;z)$ as follows:

$$G(a,b;c;z)=\int_0^z F(a,b;c;t)dt. \tag{3.1}$$

Theorem C. *Let* $a,b>-1, ab<0$ *and* $c>\max\{0,a+b\}$. *Then* $G(a,b;c;z)$ *defined by* (3.1) *is in* $T(\lambda,\alpha)$ *if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(1-\lambda\alpha)}{ab}-\frac{\alpha(1-\lambda)(c-a-b)}{(a-1)_2(b-1)_2}\right] + \frac{\alpha(1-\lambda)(c-1)_2}{(a-1)_2(b-1)_2}\leq 0. \tag{3.2}$$

Proof. Since

$$G(a,b;c;z)=z-\left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}z^n,$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty}(n-\lambda\alpha n-\alpha+\lambda\alpha)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}\leq\left|\frac{c}{ab}\right|(1-\alpha).$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Now

$$\sum_{n=2}^{\infty}[n(1-\lambda\alpha)-\alpha(1-\lambda)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}$$
$$=(1-\lambda\alpha)\sum_{n=0}^{\infty}(n+2)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}-\alpha(1-\lambda)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}$$
$$=(1-\lambda\alpha)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}-\alpha(1-\lambda)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n+1}}$$
$$=(1-\lambda\alpha)\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_n}-\alpha(1-\lambda)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}$$
$$=(1-\lambda\alpha)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-\alpha(1-\lambda)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}$$
$$=(1-\lambda\alpha)\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]$$
$$-\alpha(1-\lambda)\frac{(c-1)_2}{(a-1)_2(b-1)_2}\sum_{n=2}^{\infty}\frac{(a-1)_n(b-1)_n}{(c-1)_n(1)_n}$$
$$=\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(1-\lambda\alpha)}{ab}-\frac{\alpha(1-\lambda)(c-a-b)}{(a-1)_2(b-1)_2}\right]$$
$$+\frac{\alpha(1-\lambda)(c-1)_2}{(a-1)_2(b-1)_2}-\frac{(1-\alpha)c}{ab},$$

which is bounded above by $(1-\alpha)\left|\frac{c}{ab}\right|$ if and only if (3.2) holds. $\square$

Now, we observe that $G(a,b;c;z)\in C(\lambda,\alpha)$ if and only if $zF(a,b;c;z)\in$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$T(\lambda, \alpha)$. Thus any result of functions belonging to the class $T(\lambda, \alpha)$ about $zF(a, b; c; z)$ leads to that of functions belonging to the class $C(\lambda, \alpha)$. Hence we obtain the following analogous result to Theorem A.

Theorem 3.1. *Let $a, b > -1, ab < 0$ and $c > a + b + 2$. Then $G(a, b; c; z)$ defined by (3.1) is in $C(\lambda, \alpha)$ if and only if*

$$c > a + b + 1 - \frac{(1 - \lambda\alpha)ab}{1 - \alpha}.$$

Remark 2. Putting $\lambda = 0$ in the above results, we obtain the results of Silverman [7].

journal of inequalities in pure and applied mathematics
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journal of inequalities in pure and applied mathematics

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