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PRICE AND HAAR TYPE FUNCTIONS AND UNIFORM DISTRIBUTION OF SEQUENCES

V. GROZDANOV AND S. STOILOVA

DEPARTMENT OF MATHEMATICS,
SOUTH WEST UNIVERSITY
66 IVAN MIHAILOV STR.
2700 BLAGOEVGRAD, BULGARIA.
vassgrozdanov@yahoo.com

stanislavast@yahoo.com

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ABSTRACT. The Weyl criterion is shown in the terms of Price functions and Haar type functions. We define the so-called modified integrals of Price and Haar type functions and obtain the analogues of the criterion of Weyl, the inequalities of LeVeque and Erdös-Turan and the formula of Koksma in the terms of the modified integrals of Price and Haar type functions.

Key words and phrases: Uniform distribution of sequences; Price and Haar type functions; Weyl criterion; Inequalities of LeVeque and Erdös-Turan; Formula of Koksma.

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1. INTRODUCTION

Let $\xi = (x_i)_{i\geq 0}$ be a sequence in the unit interval $[0,1)$. We define $A(\xi;J;N) = \{i : 0 \leq i \leq N-1,\ \ x_i \in J\}$ for an arbitrary integer $N \geq 1$ and an arbitrary subinterval $J \subseteq [0,1)$. The sequence ξ is called uniformly distributed (abbreviated u. d.) if for every subinterval J of $[0,1)$ the equality $\lim_{N\to\infty} \frac{A(\xi;J;N)}{N} = \mu(J)$ holds, and where $\mu(J)$ is the length of J.

Let $\xi_N = \{x_0, \ldots, x_{N-1}\}$ be an arbitrary net of real numbers in $[0,1)$. The extreme and quadratical discrepancies $D(\xi_N)$ and $T(\xi_N)$ of the net ξ_N are defined respectively as

$$D(\xi_N) = \sup_{J\subseteq[0,1)} |N^{-1}A(\xi_N;J;N) - \mu(J)|,$$

$$T(\xi_N) = \left(\int_0^1 |N^{-1}A(\xi_N;[0,x);N) - x|^2 dx\right)^{\frac{1}{2}}.$$

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The discrepancy $D_N(\xi)$ of the sequence ξ is defined as $D_N(\xi) = D(\xi_N)$, for each integer $N \geq 1$, and ξ_N is the net, composed of the first N elements of the sequence ξ. It is well-known that the sequence ξ is u. d. if and only if $\lim_{N\to\infty} D_N(\xi) = 0$.

According to Kuipers and Niederreiter [8, Corollaries 1.1 and 1.2], the sequence ξ is u. d. if and only if for each complex-valued and integrable in the sense of Riemann function f, defined on $\mathbb{R}$ and periodical with period 1, the following equality

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(x_i) = \int_0^1 f(x)dx \tag{1.1}$$

holds.

The theory of uniformly distributed sequences is divided into quantitative and qualitative parts. Quantitative theory considers measures, showing the deviation of the distribution of a concrete sequence from an ideal distribution. Qualitative theory main idea of uniformly distributed sequences is to find necessary and sufficient conditions for uniformity of the distribution of sequences.

Weyl [18] obtains such a condition (the so-called Weyl criterion) which is based on the use of the trigonometric functional system $\mathcal{T} = \{e_k(x) = \exp(2\pi \mathbf{i} kx), k \in \mathbb{Z}, x \in \mathbb{R}\}$. The criterion of Weyl is: The sequence $\xi = (x_i)_{i\geq 0}$ is uniformly distributed if and only if the equality $\lim_{N\to\infty} \frac{1}{N}\sum_{i=0}^{N-1} e_k(x_i) = 0$ holds for each integer $k \neq 0$.

The Walsh functional system has been recently used as an appropriate means of studying the uniformity of the distribution of sequences. Sloss and Blyth [13] use this system to obtain future necessary and sufficient conditions for a sequence to be u. d.

The link, which is realized for studying sequences in $[0, 1)$, constructed in a generalized number system and some orthonormal functional systems on $[0, 1)$, constructed in the same system, is quite natural. The purpose of our paper is to reveal the possibility some other classes of orthonormal functional system, as the Price functional system and two systems of Haar type functions to be used as a means of obtaining new necessary and sufficient conditions for uniform distribution of sequences.

In Section 2 we obtain new necessary and sufficient conditions for uniform distribution of sequences, which are analogues of the classical criterion of Weyl. These conditions are based on the functions of Price and Haar type functions.

In Section 3 we introduce the so-called modified integrals of the Price functions and Haar type functions. Integral analogues of the Weyl criterion are obtained in terms of these integrals. Analogues of the classical inequalities of LeVeque [9] and Erdös-Turan (see Kuipers and Niederreiter [8]), and the formula of Koksma, (see Kuipers [7]) are obtained.

In Section 4 we prove some preliminary statements, which are used to prove the main results. The proofs of the main results are given in Section 5. In Section 6 we give a conclusion, where we announce some open problems, having to do with the problems, solved in our paper.

The results of this paper were announced in Grozdanov and Stoilova [3] and [4]. Here we explain the full proofs of them. The results which are based on the Price functions generalize the ones of Sloss and Blyth [13]. The results which are based on the Haar type functions are new.

2. PRICE FUNCTIONAL SYSTEM, HAAR TYPE FUNCTIONAL SYSTEM AND ANALOGUES OF THE CRITERION OF WEYL

Let $\mathcal{B} = \{b_1, b_2, \ldots, b_j, \ldots : b_j \geq 2, j \geq 1\}$ be an arbitrary fixed sequence of integer numbers. We define $\omega_j = \exp\left(\frac{2\pi \mathbf{i}}{b_j}\right)$ for each integer $j \geq 1$. We define the set of the generalized powers $\{B_j\}_{j=0}^{\infty}$ as: $B_0 = 1$ and for each integer $j \geq 1$, $B_j = \prod_{s=1}^{j} b_s$.

Definition 2.1.

(i) For real $x \in [0,1)$ in the $\mathcal{B}-$adic form $x = \sum_{i=1}^{\infty} x_i B_i^{-1}$, where for $i \geq 1$ $x_i \in \{0,1,\dots,b_i-1\}$ and each integer $j \geq 0$, Price [10] defines the functions $\chi_{B_j}(x) = \omega_{j+1}^{x_{j+1}}$.

(ii) For each integer $k \geq 0$ in the $\mathcal{B}-$adic form $k = \sum_{j=0}^{n} k_{j+1} B_j$, where for $1 \leq j \leq n+1$, $k_j \in \{0,1,\dots,b_j-1\}$, $k_{n+1} \neq 0$ and real $x \in [0,1)$, the k-th function of Price $\chi_k(x)$ is defined as $\chi_k(x) = \prod_{j=0}^{n} (\chi_{B_j}(x))^{k_{j+1}}$.

The system $\chi(\mathcal{B}) = \{\chi_k\}_{k=0}^{\infty}$ is called the Price functional system. This system is a complete orthonormal system in $L_2[0,1)$.

Let $b_j = b$ in the sequence $\mathcal{B}$ for each $j \geq 1$. Then, the system $\{\chi_0\} \cup \{\chi_{b^k}\}_{k=0}^{\infty}$ is the Rademacher [11] system $\{\phi_k^{(b)}\}_{k=0}^{\infty}$ of order b. The system of Chrestenson [2] $\{\psi_k^{(b)}\}_{k=0}^{\infty}$ of order b is obtained from the system $\chi(\mathcal{B})$. If for each $j \geq 1$ $b_j = 2$, then the original system of Walsh [17] is obtained.

In 1947 Vilenkin [15] introduced the system $\chi(\mathcal{B})$ and Price [10] defined it independently of him in 1957. Some names are used about the system $\chi(\mathcal{B})$ in special literature: both Price system (see Agaev, Vilenkin, Dzafarly, Rubinstein [1]) and Vilenkin system (see Schipp, Wade, Simon [12]). We use the name Price functional system in this paper.

We will consider two kinds of the so-called Haar type functions. Starting from the original Haar [5] system, Vilenkin [16] proposes a new system of functions, which is called a Haar type system, (see Schipp, Wide, Simon [12]). This definition is:

Definition 2.2. For $x \in [0,1)$ the k^{th} Haar type function $h'_k(x)$, $k \geq 0$ to the base $\mathcal{B}$ is defined as follows: If $k = 0$, then $h'_0(x) = 1$, $\forall x \in [0,1)$. If $k \geq 1$ is an arbitrary integer and

$$k = B_n + p(b_{n+1} - 1) + s - 1, \tag{2.1}$$

where for some integer $n \geq 0$, $0 \leq p \leq B_n - 1$ and $s \in \{1,\dots,b_{n+1}-1\}$, then

$$h'_k(x) = \begin{cases} \sqrt{B_n}\,\omega_{n+1}^{sa}, & \text{if } \frac{pb_{n+1}+a}{B_{n+1}} \leq x < \frac{pb_{n+1}+a+1}{B_{n+1}} \text{ and } a = 0,1,\dots,b_{n+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

We will consider another one:

Definition 2.3. For $x \in [0,1)$ the k^{th} Haar type function $h''_k(x)$, $k \geq 0$ to the base $\mathcal{B}$ is defined as follows: If $k = 0$, then $h''_0(x) = 1$, $\forall x \in [0,1)$. If $k \geq 1$ is an arbitrary integer and

$$k = k_n B_n + p, \tag{2.2}$$

where for some integer $n \geq 0$, $0 \leq p \leq B_n - 1$ and $k_n \in \{1,\dots,b_{n+1}-1\}$, then

$$h''_k(x) = \begin{cases} \sqrt{B_n}\,\omega_{n+1}^{k_n a}, & \text{if } \frac{pb_{n+1}+a}{B_{n+1}} \leq x < \frac{pb_{n+1}+a+1}{B_{n+1}} \text{ and } a = 0,1,\dots,b_{n+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

It can be easily seen that the systems $\{h'_k\}_{k=0}^{\infty}$ and $\{h''_k\}_{k=0}^{\infty}$ are complete orthonormal systems in $L_2[0,1)$. In the case when for each $j \geq 1$ $b_j = 2$ the original system of Haar is obtained from the systems $\{h'_k\}_{k=0}^{\infty}$ and $\{h''_k\}_{k=0}^{\infty}$.

Theorem 2.1 (Analogues of the criterion of Weyl)**.** *The sequence $(x_i)_{i\geq 0}$ of $[0,1)$ is u. d. if and only if:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \chi_k(x_i) = 0, \quad \textit{for each } k \geq 1,$$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}h'_k(x_i)=0,\ \ \textit{for each}\ \ k\geq 1,$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}h''_k(x_i)=0,\ \ \textit{for each}\ \ k\geq 1.$$

The proof of this theorem is based on the equality (1.1) and the properties of Price and Haar type functional systems.

3. Price and Haar Type Integrals and u. d. of Sequences

We consider the integrals of Price and Haar type functions $J_k(x)=\int_0^x\chi_k(t)dt$, $\Psi'_k(x)=\int_0^x h'_k(t)dt$ and $\Psi''_k(x)=\int_0^x h''_k(t)dt$ for each integer $k\geq 1$ and $x\in[0,1)$.

For an arbitrary integer $k\geq 1$ we define the integer $n\geq 0$ by the condition $B_n\leq k<B_{n+1}$. We define the *modified integrals of Price function* as

$$J_{n,q,k}(x)=J_k(x)+\frac{1}{B_{n+1}}\cdot\frac{1}{\omega_{n+1}^q-1}\delta_{q.B_n,k}, \tag{3.1}$$

for all $x\in[0,1)$ and each $q=1,2,\dots,b_{n+1}-1$, and for arbitrary integers $i,j\geq 0$, $\delta_{i,j}$ is the Kronecker's symbol.

If k is an integer of the kind (2.1), we define

$$\Psi'_{n,s,k}(x)=\Psi'_k(x)+\frac{1}{b_{n+1}}B_n^{-\frac{3}{2}}\frac{1}{\omega_{n+1}^s-1}, \tag{3.2}$$

for all $x\in[0,1)$ and each $s=1,2,\dots,b_{n+1}-1$.

If k is an integer of the kind (2.2), we define

$$\Psi''_{n,k_n,k}(x)=\Psi''_k(x)+\frac{1}{b_{n+1}}B_n^{-\frac{3}{2}}\frac{1}{\omega_{n+1}^{k_n}-1}, \tag{3.3}$$

for all $x\in[0,1)$ and each $k_n=1,2,\dots,b_{n+1}-1$.

We will call the integrals $\Psi'_{n,s,k}(x)$ and $\Psi''_{n,k_n,k}(x)$ *modified integrals of Haar type functions*. The next theorems hold:

Theorem 3.1 (Analogues of the inequality of LeVeque)**.** *Let $\xi_N=\{x_0,x_1\dots,x_{N-1}\}$ be an arbitrary net, composed of $N\geq 1$ points of $[0,1)$. The discrepancy $D(\xi_N)$ of the net ξ_N satisfies the inequalities:*

$$D(\xi_N)\leq\left(\frac{12}{N^2}\sum_{n=0}^{\infty}\sum_{q=1}^{b_{n+1}-1}\sum_{k=qB_n}^{(q+1)B_n-1}\left|\sum_{m=0}^{N-1}J_{n,q,k}(x_m)\right|^2\right)^{\frac{1}{3}},$$

$$D(\xi_N)\leq\left(\frac{12}{N^2}\sum_{n=0}^{\infty}\sum_{s=1}^{b_{n+1}-1}\sum_{k=sB_n}^{(s+1)B_n-1}\left|\sum_{m=0}^{N-1}\Psi'_{n,s,k}(x_m)\right|^2\right)^{\frac{1}{3}},$$

$$D(\xi_N)\leq\left(\frac{12}{N^2}\sum_{n=0}^{\infty}\sum_{k_n=1}^{b_{n+1}-1}\sum_{k=k_nB_n}^{(k_n+1)B_n-1}\left|\sum_{m=0}^{N-1}\Psi''_{n,k_n,k}(x_m)\right|^2\right)^{\frac{1}{3}}.$$

Theorem 3.2 (Integral analogues of the criterion of Weyl)**.** *Let an absolute constant B exist, such as for each $j\geq 1$ $b_j\leq B$. Let $k\geq 1$ be an arbitrary integer and $B_n\leq k<B_{n+1}$. The sequence $\xi=(x_i)_{i\geq 0}$ of $[0,1)$ is u. d. if and only if:*

(i)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1} J_{n,q,k}(x_i)=0 \ \textit{ for each } \ q=1,2,\dots,b_{n+1}-1,$$

(ii) *If $k\geq 1$ is of the kind (2.1) then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1} \Psi'_{n,s,k}(x_i)=0 \ \textit{ for each } \ s=1,2,\dots,b_{n+1}-1,$$

(iii) *If $k\geq 1$ is of the kind (2.2) then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1} \Psi''_{n,k_n,k}(x_i)=0 \ \textit{ for each } \ k_n=1,2,\dots,b_{n+1}-1.$$

Theorem 3.3 (An analogue of the inequality of Erdös-Turan)**.** *Let an absolute constant B exist, such that $b_j\leq B$ for each $j\geq 1$ and we signify $b=\min\{b_j : j\geq 1\}$. Let $\xi_N=\{x_0,\dots,x_{N-1}\}$ be an arbitrary net, composed of $N\geq 1$ points of $[0,1)$. For an arbitrary integer $H>0$ we define the integers $M\geq 0$ and $q\in\{1,2,\dots,b_{M+1}-1\}$ as $qB_M\leq H<(q+1)B_M$. Then the following inequality holds*

$$\begin{aligned} D(\xi_N) \leq \Bigg(& 12\sum_{n=0}^{M-1}\sum_{s=1}^{b_{n+1}-1}\sum_{k=sB_n}^{(s+1)B_n-1}\left|\frac{1}{N}\sum_{m=0}^{N-1} J_{n,s,k}(x_m)\right|^2 \\ & + 12\sum_{s=1}^{q-1}\sum_{k=sB_M}^{(s+1)B_M-1}\left|\frac{1}{N}\sum_{m=0}^{N-1} J_{M,s,k}(x_m)\right|^2 \\ & + 12\sum_{k=qB_M}^{H}\left|\frac{1}{N}\sum_{m=0}^{N-1} J_{M,q,k}(x_m)\right|^2 + \frac{3B(1+2b\sin\frac{\pi}{B})^2}{(b-1)b\sin^2\frac{\pi}{B}}\frac{1}{B_M}\Bigg)^{\frac{1}{3}}. \end{aligned}$$

Theorem 3.4 (Analogues of the formula of Koksma)**.** *Let $\xi_N=\{x_0,x_1\dots,x_{N-1}\}$ be an arbitrary net, composed of $N\geq 1$ points of $[0,1)$. The quadratical discrepancy $T(\xi_N)$ of the net ξ_N satisfies the equalities*

$$(NT(\xi_N))^2=\left(\sum_{m=0}^{N-1}\left(x_m-\frac{1}{2}\right)\right)^2+\sum_{n=0}^{\infty}\sum_{q=1}^{b_{n+1}-1}\sum_{k=q.B_n}^{(q+1).B_n-1}\left|\sum_{m=0}^{N-1} J_{n,q,k}(x_m)\right|^2,$$

$$(NT(\xi_N))^2=\left(\sum_{m=0}^{N-1}\left(x_m-\frac{1}{2}\right)\right)^2+\sum_{n=0}^{\infty}\sum_{s=1}^{b_{n+1}-1}\sum_{k=sB_n}^{(s+1)B_n-1}\left|\sum_{m=0}^{N-1} \Psi'_{n,s,k}(x_m)\right|^2,$$

and

$$(NT(\xi_N))^2=\left(\sum_{m=0}^{N-1}\left(x_m-\frac{1}{2}\right)\right)^2+\sum_{n=0}^{\infty}\sum_{k_n=1}^{b_{n+1}-1}\sum_{k=k_nB_n}^{(k_n+1)B_n-1}\left|\sum_{m=0}^{N-1} \Psi''_{n,k_n,k}(x_m)\right|^2.$$

4. PRELIMINARY STATEMENTS

Let $x\in[0,1)$ have the $\mathcal{B}-$adic representation $x=\sum_{j=0}^{\infty}x_{j+1}B_{j+1}^{-1}$, where for $j\geq 0$, $x_{j+1}\in\{0,1,\dots,b_{j+1}-1\}$. For each integer $j\geq 0$ we have that $x_{j+1}=\frac{b_{j+1}}{2\pi}\arg\chi_{B_j}(x)$.

Hence, we obtain the representation

$$x = \frac{1}{2\pi} \sum_{j=0}^{\infty} \frac{1}{B_j} \arg \chi_{B_j}(x). \tag{4.1}$$

Lemma 4.1. *Let $k \geq 1$ be an arbitrary integer and $k = \beta_{n+1} B_n + k'$, where $\beta_{n+1} \in \{1, \dots, b_{n+1} - 1\}$ and $0 \leq k' < B_n$. For $x \in [0, 1)$ the k^{th} Price integral satisfies the following equality*

$$J_k(x) = \frac{1}{B_{n+1}} \frac{1 - \omega_{n+1}^{\frac{\beta_{n+1} b_{n+1}}{2\pi} \arg \chi_{B_n}(x)}}{1 - \omega_{n+1}^{\beta_{n+1}}} \chi_{k'}(x) + \frac{1}{2\pi B_n} \sum_{r=1}^{\infty} b_{n+1}^{-r} \arg \chi_{B_n}\left(b_{n+1}^r x\right) \chi_k(x). \tag{4.2}$$

Proof. Let $b \geq 2$ be a fixed integer and $\omega = \exp\left(\frac{2\pi \mathbf{i}}{b}\right)$. For an arbitrary integer β, $1 \leq \beta \leq b-1$ and real $x \in [0, 1)$ let

$$J_\beta^{(b)}(x) = \int_0^x \psi_\beta^{(b)}(t) dt.$$

We will prove the following equality

$$J_\beta^{(b)}(x) = \frac{1}{b} \frac{1 - \omega^{\frac{\beta b}{2\pi} \arg \phi_0^{(b)}(x)}}{1 - \omega^\beta} + \frac{1}{2\pi} \sum_{r=1}^{\infty} b^{-r} \arg \psi_{b^r}^{(b)}(x) \psi_\beta^{(b)}(x). \tag{4.3}$$

We put $s = [bx]$, where $[bx]$ denotes the integer part of bx and we have that

$$\begin{aligned} J_\beta^{(b)}(x) &= \int_0^x [\phi_0^{(b)}(t)]^\beta dt \\ &= \sum_{h=0}^{s-1} \int_{h/b}^{(h+1)/b} \omega^{h\beta} dt + \int_{s/b}^x [\phi_0^{(b)}(t)]^\beta dt \\ &= \frac{1}{b} \frac{1 - \omega^{\beta[bx]}}{1 - \omega^\beta} + \psi_\beta^{(b)}(x) \left(x - \frac{[bx]}{b}\right). \end{aligned} \tag{4.4}$$

From (4.1) and (4.4) we obtain (4.3).

If

$$J_{\beta_{n+1} \cdot B_n}^{(b_{n+1})}(x) = \int_0^x \chi_{\beta_{n+1} B_n}(t) dt,$$

then

$$J_k(x) = \chi_{k'}(x) J_{\beta_{n+1} B_n}^{(b_{n+1})}(x). \tag{4.5}$$

We have the equalities

$$\begin{aligned} J_{\beta_{n+1} \cdot B_n}^{(b_{n+1})}(x) &= \int_0^x \chi_{B_n}^{\beta_{n+1}}(t) dt \\ &= \int_0^x \left[\phi_0^{(b_{n+1})}(B_n t)\right]^{\beta_{n+1}} dt \\ &= \frac{1}{B_n} \int_0^{B_n x} \psi_{\beta_{n+1}}^{(b_{n+1})}(t) dt = \frac{1}{B_n} J_{\beta_{n+1}}^{(b_{n+1})}(B_n x), \end{aligned}$$

so that

$$J_{\beta_{n+1} \cdot B_n}^{(b_{n+1})}(x) = \frac{1}{B_n} J_{\beta_{n+1}}^{(b_{n+1})}(B_n x).$$

From the last equality and (4.5) we obtain that

$$J_k(x) = \frac{1}{B_n} \chi_{k'}(x) \cdot J_{\beta_{n+1}}^{(b_{n+1})}(B_n x). \tag{4.6}$$

From (4.3) we obtain

$$J^{(b_{n+1})}_{\beta_{n+1}}(B_n x) = \frac{1}{b_{n+1}} \frac{1-\omega^{\frac{\beta_{n+1}b_{n+1}}{2\pi}\arg\chi_{B_n}(x)}}{1-\omega^{\beta_{n+1}}_{n+1}} + \frac{1}{2\pi}\sum_{r=1}^{\infty} b^{-r}_{n+1}\arg\chi_{B_n}(b^r_{n+1}x)\chi_{\beta_{n+1}B_n}(x). \tag{4.7}$$

From (4.6) and (4.7) we obtain (4.2). □

For every integer $n \geq 1$ we consider the set $B(n) = \{b_1, b_2, \dots, b_n\}$. We define the "reverse" set $\widetilde{B}(n) = \{b_n, b_{n-1}, \dots, b_1\}$, so that $\widetilde{B}_1 = b_n$, $\widetilde{B}_2 = b_n b_{n-1}, \dots, \widetilde{B}_n = b_n \cdots b_1$. For an arbitrary integer p, $0 \leq p < B_n$ and for a $\mathcal{B}$-adic rational $\frac{p}{B_n}$ let $(p)_{B(n)}$, $(p)_{\widetilde{B}(n)}$, $\left(\frac{p}{B_n}\right)_{B(n)}$ and $\left(\frac{p}{B_n}\right)_{\widetilde{B}(n)}$ be the corresponding representations of p and $\frac{p}{B_n}$ to the systems $B(n)$ and $\widetilde{B}(n)$.

Lemma 4.2. *(Relationships between the Price and the Haar type functions)*

(i) *Let $k \geq 1$ be an arbitrary integer of the kind (2.1). Then for all $x \in [0,1)$*

$$h'_k(x) = \frac{1}{b_{n+1}\sqrt{B_n}} \sum_{\alpha=0}^{B_{n+1}-1}\sum_{a=0}^{b_{n+1}-1} \omega^{a.s}_{n+1}\overline{\chi}_{(pb_{n+1}+a)_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right)\chi_\alpha(x);$$

$$\Psi'_{n,s,k}(x) = \frac{1}{b_{n+1}\sqrt{B_n}} \times \sum_{j=0}^{n}\sum_{t=0}^{b_{j+1}-1}\sum_{\alpha=tB_j}^{(t+1)B_j-1}\sum_{a=0}^{b_{n+1}-1} \omega^{a.s}_{n+1}\overline{\chi}_{(pb_{n+1}+a)_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right)J_{j,t,\alpha}(x).$$

(ii) *Let $k \geq 1$ be an arbitrary integer of the kind (2.2). Then for all $x \in [0,1)$*

$$h''_k(x) = \frac{1}{b_{n+1}\sqrt{B_n}} \sum_{\alpha=0}^{B_{n+1}-1}\sum_{a=0}^{b_{n+1}-1} \omega^{a.k_n}_{n+1}\overline{\chi}_{(pb_{n+1}+a)_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right)\chi_\alpha(x); \tag{4.8}$$

$$\Psi''_{n,k_n,k}(x) = \frac{1}{b_{n+1}\sqrt{B_n}} \times \sum_{j=0}^{n}\sum_{t=0}^{b_{j+1}-1}\sum_{\alpha=tB_j}^{(t+1)B_j-1}\sum_{a=0}^{b_{n+1}-1} \omega^{a.k_n}_{n+1}\overline{\chi}_{(pb_{n+1}+a)_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right)J_{j,t,\alpha}(x). \tag{4.9}$$

Proof. For an arbitrary integer p, $0 \leq p < B_n$ and $x \in [0,1)$, following Kremer [6] we define the function

$$q(B_n;(p)_{\widetilde{B}(n)};x) = \begin{cases} 1, & \text{if } x \in \left[\left(\frac{p}{B_n}\right)_{B(n)}, \left(\frac{p+1}{B_n}\right)_{B(n)}\right) \\ 0 & \text{if } x \notin \left[\left(\frac{p}{B_n}\right)_{B(n)}, \left(\frac{p+1}{B_n}\right)_{B(n)}\right). \end{cases}$$

The equality

$$q(B_n;(p)_{\widetilde{B}(n)};x) = \frac{1}{B_n}\sum_{\alpha=0}^{B_n-1}\overline{\chi}_{(p)_{\widetilde{B}(n)}}\left(\left(\frac{\alpha}{B_n}\right)_{\widetilde{B}(n)}\right)\chi_\alpha(x)$$

holds. Let us use the significations: For an arbitrary integer p, $0 \leq p < B_n$, $(p)_{\widetilde{B}(n)} = (\tilde{p}_1\tilde{p}_2 \ldots \tilde{p}_n)_{\widetilde{B}(n)}$, for an arbitrary integer α, $0 \leq \alpha < B_n$, $\left(\frac{\alpha}{B_n}\right)_{\widetilde{B}(n)} = (0 \cdot \alpha_n\alpha_{n-1} \ldots \alpha_1)_{\widetilde{B}(n)}$, for real $x \in [0,1)$, $x = (0 \cdot x_1 \ldots x_{n+1} \ldots)_{\mathcal{B}}$. Then, we obtain the equalities

$$\chi_{(p)_{\widetilde{B}(n)}}\left(\left(\frac{\alpha}{B_n}\right)_{\widetilde{B}(n)}\right) = \omega_n^{\alpha_n\tilde{p}_n}\omega_{n-1}^{\alpha_{n-1}\tilde{p}_{n-1}} \ldots \omega_1^{\alpha_1\tilde{p}_1} \tag{4.10}$$

and

$$\chi_\alpha(x) = \omega_1^{\alpha_1 x_1}\omega_2^{\alpha_2 x_2} \ldots \omega_n^{\alpha_n x_n}. \tag{4.11}$$

If $x \in \left[\left(\frac{p}{B_n}\right)_{B(n)}, \left(\frac{p+1}{B_n}\right)_{B(n)}\right)$, then for all $j = 1, \ldots, n$, $x_j = \tilde{p}_j$. From (4.10) and (4.11) we obtain

$$\overline{\chi}_{(p)_{\widetilde{B}(n)}}\left(\left(\frac{\alpha}{B_n}\right)_{\widetilde{B}(n)}\right)\chi_\alpha(x) = 1.$$

If $x \notin \left[\left(\frac{p}{B_n}\right)_{B(n)}, \left(\frac{p+1}{B_n}\right)_{B(n)}\right)$, then, some δ, $1 \leq \delta \leq n$ exists, so that $x_\delta \neq \tilde{p}_\delta$. Then, we have that

$$\sum_{\alpha_\delta=0}^{b_\delta-1} \omega_\delta^{\alpha_\delta(x_\delta-\tilde{p}_\delta)} = 0.$$

From (4.10) and (4.11), we obtain

$$\sum_{\alpha=0}^{B_n-1} \overline{\chi}_{(p)_{\widetilde{B}(n)}}\left(\left(\frac{\alpha}{B_n}\right)_{\widetilde{B}(n)}\right)\chi_\alpha(x) = \prod_{j=1}^{n}\sum_{\alpha_j=0}^{b_j-1} \omega_j^{\alpha_j(x_j-\tilde{p}_j)} = 0.$$

Now let $k \geq 1$ be an integer of the kind (2.2). In order to prove (4.8) we note that for all $x \in [0,1)$

$$h''_k(x) = \sqrt{B_n}\sum_{a=0}^{b_{n+1}-1} \omega_{n+1}^{a.k_n} q\left(B_{n+1}; (pb_{n+1}+a)_{\widetilde{B}(n+1)}; x\right).$$

We will prove (4.9). Using the proved formula for $h''_k(x)$, we have that

$$\begin{aligned}
\Psi''_k(x) &= \int_0^x h''_k(t)dt \\
&= \frac{1}{b_{n+1}\sqrt{B_n}}\sum_{\alpha=0}^{B_{n+1}-1}\sum_{a=0}^{b_{n+1}-1} \omega_{n+1}^{a.k_n}\overline{\chi}_{(pb_{n+1}+a)_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right)\int_0^x \chi_\alpha(t)dt \\
&= \frac{1}{b_{n+1}\sqrt{B_n}}\sum_{j=0}^{n}\sum_{t=0}^{b_{j+1}-1}\sum_{\alpha=tB_j}^{(t+1)B_j-1}\sum_{a=0}^{b_{n+1}-1} \omega_{n+1}^{a.k_n}\overline{\chi}_{(pb_{n+1}+a)_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right) \\
&\qquad \times\left(J_{j,t,\alpha}(x) - \frac{1}{B_{j+1}}\frac{1}{\omega_{j+1}^t - 1}\delta_{tB_j,\alpha}\right).
\end{aligned} \tag{4.12}$$

It is not difficult to prove that

$$\sum_{j=0}^{n} \frac{1}{B_{j+1}} \sum_{t=0}^{b_{j+1}-1} \frac{1}{\omega_{j+1}^{t}-1} \sum_{\alpha=tB_j}^{(t+1)B_j-1} \sum_{a=0}^{b_{n+1}-1} \omega_{n+1}^{a.k_n} \times \overline{\chi}_{(pb_{n+1}+a)_{\tilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\tilde{B}(n+1)}\right) \delta_{tB_j,\alpha} = \frac{B_n^{-1}}{\omega_{n+1}^{k_n}-1}. \tag{4.13}$$

From (4.12) and (4.13), we obtain (4.9). □

In the following lemma the relationships in the opposite direction are proved.

Lemma 4.3. *Let $k \geq 1$ be an arbitrary integer and $k = sB_n + p$, where $s \in \{1, \dots, b_{n+1}-1\}$ and $0 \leq p \leq B_n - 1$. Then, for all $x \in [0,1)$ the equalities*

$$\chi_k(x) = \frac{1}{\sqrt{B_n}} \sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)} h''_{sB_n+\tilde{j}}(x); \tag{4.14}$$

$$J_{n,s,k}(x) = \frac{1}{\sqrt{B_n}} \sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)} \Psi''_{n,s,sB_n+\tilde{j}}(x); \tag{4.15}$$

$$\chi_k(x) = \frac{1}{\sqrt{B_n}} \sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)} h'_{B_n+\tilde{j}(b_{n+1}-1)+s-1}(x)$$

and

$$J_{n,s,k}(x) = \frac{1}{\sqrt{B_n}} \sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)} \Psi'_{n,s,B_n+\tilde{j}(b_{n+1}-1)+s-1}(x),$$

hold, where $\alpha_{p,\tilde{j}}^{(n)}$ are complex mumbers, so that $\sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)} = B_n \cdot \delta_{sB_n,k}$.

Proof. Let $x \in [0,1)$ be fixed. We define $\tilde{t} = (t)_{\tilde{B}(n)}$, $0 \leq \tilde{t} < B_n$ as $\left(\frac{\tilde{t}}{B_n}\right)_{B(n)} \leq x < \left(\frac{\tilde{t}+1}{B_n}\right)_{B(n)}$. We denote $\Delta_{\tilde{t}}^{(n)} = \left[\frac{\tilde{t}}{B_n}, \frac{\tilde{t}+1}{B_n}\right)$. It is obvious that $\Delta_{\tilde{t}}^{(n)} = \bigcup_{a=0}^{b_{n+1}-1} \Delta_{\tilde{t}b_{n+1}+a}^{(n+1)}$. There is some a, $0 \leq a \leq b_{n+1}-1$, so that $x \in \Delta_{\tilde{t}b_{n+1}+a}^{(n+1)}$. We have the equalities

$$\chi_k(x) = \chi_p\left(\left(\frac{\tilde{t}}{B_n}\right)_{B(n)}\right)\omega_{n+1}^{a.s} \quad \text{and} \quad h''_{s\cdot B_n+\tilde{t}}(x) = \sqrt{B_n}\omega_{n+1}^{a.s}.$$

Hence, we obtain

$$\chi_k(x) = \frac{1}{\sqrt{B_n}}\chi_p\left(\left(\frac{\tilde{t}}{B_n}\right)_{B(n)}\right) h''_{s\cdot B_n+\tilde{t}}(x).$$

Let $\tilde{t}'$ be an arbitrary integer, so that $0 \leq \tilde{t}' < B_n$, and $\tilde{t}' \neq \tilde{t}$. For $x \in \Delta_{\tilde{t}}^{(n)}$ we have that $h''_{s\cdot B_n+\tilde{t}'}(x) = 0$. Hence, we obtain the equality

$$\chi_k(x) = \frac{1}{\sqrt{B_n}} \sum_{j=0}^{B_n-1} \chi_p\left(\left(\frac{\tilde{j}}{B_n}\right)_{B(n)}\right) h''_{s\cdot B_n+\tilde{j}}(x).$$

Let for integers $0 \le p < B_n$ and $0 \le j < B_n$ we signify $\alpha_{p,\tilde{j}}^{(n)} = \chi_p\left(\left(\frac{\tilde{j}}{B_n}\right)_{B(n)}\right)$. We use the representations $p = (p_n p_{n-1} \dots p_1)_{B(n)}$ and $\left(\frac{\tilde{j}}{B_n}\right)_{B(n)} = (0 \cdot j_1 j_2 \dots j_n)_{B(n)}$, where for $1 \le \tau \le n \ \ p_\tau, j_\tau \in \{0, 1, \dots, b_\tau - 1\}$. Then, we obtain the equality

$$\sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)} = \prod_{\tau=1}^{n} \sum_{j_\tau=0}^{b_\tau-1} \omega_\tau^{p_\tau j_\tau}. \tag{4.16}$$

If $p = 0$, then, for $1 \le \tau \le n, \ \ p_\tau = 0$ and from (4.16) we obtain $\sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)} = B_n$. If $p \ne 0$ then, some $\delta, \ \ 1 \le \delta \le n$ exists, so that $p_\delta \ne 0$. From (4.16), we obtain $\sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)} = 0$.

We will prove (4.15). From (4.14) for all $x \in [0, 1)$ we have

$$J_k(x) = \frac{1}{\sqrt{B_n}} \sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)} \left[\Psi''_{sB_n+\tilde{j}}(x) + \frac{1}{b_{n+1}} B_n^{-\frac{3}{2}} \frac{1}{\omega_{n+1}^s - 1}\right] - \frac{1}{b_{n+1}} B_n^{-2} \frac{1}{\omega_{n+1}^s - 1} \sum_{j=0}^{B_n-1} \alpha_{p,\tilde{j}}^{(n)}.$$

From the last equality and (3.3) we obtain (4.15). □

Sobol [14] proved a similar result, giving the relationship between the original Haar and the Walsh functions.

For an arbitrary net $\xi_N = \{x_0, \dots, x_{N-1}\}$, composed of $N \ge 1$ points of $[0, 1)$ and $x \in [0, 1)$ we signify $R(\xi_N; x) = A(\xi_N; [0, x); N) - Nx$. Then, the next lemma holds:

Lemma 4.4.

(i) *The Fourier-Price coefficiens of* $R(\xi_N; x)$ *satisfy the equalities:*

$$a_0^{(\chi)} = -\sum_{m=0}^{N-1} \left(x_m - \frac{1}{2}\right);$$

and for each integer $k \ge 1$, $k = k_n B_n + p$, $k_n \in \{1, 2, \dots, b_{n+1} - 1\}$, $0 \le p < B_n$

$$\overline{a}_k^{(\chi)} = -\sum_{m=0}^{N-1} J_{n,k_n,k}(x_m).$$

(ii) *The Fourier-Haar type coefficiens of* $R(\xi_N; x)$ *satisfy the equalities:*

$$a_0^{(h')} = -\sum_{m=0}^{N-1} \left(x_m - \frac{1}{2}\right), \ \ a_0^{(h'')} = -\sum_{m=0}^{N-1} \left(x_m - \frac{1}{2}\right);$$

Let $k \ge 1$ *be an arbitrary integer of kind (2.1). Then,*

$$\overline{a}_k^{(h')} = -\sum_{m=0}^{N-1} \Psi'_{n,s,k}(x_m).$$

Let $k \ge 1$ *be an arbitrary integer of kind (2.2). Then,*

$$\overline{a}_k^{(h'')} = -\sum_{m=0}^{N-1} \Psi''_{n,k_n,k}(x_m). \tag{4.17}$$

Proof. Let for $0 \leq m \leq N-1$, $c_m(x)$ be the characteristic function of the interval $(x_m, 1)$. Then,

$$R(\xi_N; x) = \sum_{m=0}^{N-1} c_m(x) - Nx.$$

We will prove only (4.17). The proof of the remaining equalities of the lemma is similar. For an arbitrary integer $k \geq 1$ of the kind (2.2) we have:

$$\overline{a}_k^{(h'')} = \int_0^1 R(\xi_N; x) h_k''(x)dx = \sum_{m=0}^{N-1} \int_0^1 c_m(x) h_k''(x)dx - N\int_0^1 x h_k''(x)dx. \tag{4.18}$$

The equalities

$$\sum_{m=0}^{N-1} \int_0^1 c_m(x) h_k''(x)dx = \sum_{m=0}^{N-1} \left[\int_0^1 h_k''(x)dx - \int_0^{x_m} h_k''(x)dx\right] = -\sum_{m=0}^{N-1} \Psi_k''(x_m) \tag{4.19}$$

hold. From (4.1) and (4.8) we obtain

$$\begin{aligned}
&\int_0^1 x h_k''(x)dx \\
&= \frac{1}{2\pi b_{n+1}\sqrt{B_n}} \sum_{\alpha=0}^{B_{n+1}-1} \sum_{a=0}^{b_{n+1}-1} \omega_{n+1}^{a\cdot k_n} \overline{\chi}_{(pb_{n+1}+a)_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right) \\
&\quad \times \sum_{r=0}^{\infty} \frac{1}{B_r} \int_0^1 \arg \chi_{B_r}(x)\chi_\alpha(x)dx \\
&= \frac{1}{2\pi b_{n+1}\sqrt{B_n}} \sum_{\alpha=0}^{B_n-1} \overline{\chi}_{(pb_{n+1})_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right) \left[\sum_{a=0}^{b_{n+1}-1} \omega_{n+1}^{a\cdot k_n}\right] \\
&\quad \times \sum_{r=0}^{\infty} \frac{1}{B_r} \int_0^1 \arg \chi_{B_r}(x)\chi_\alpha(x)dx \\
&\quad + \frac{1}{2\pi b_{n+1}\sqrt{B_n}} \sum_{t=1}^{b_{n+1}-1} \sum_{\alpha=tB_n}^{(t+1)B_n-1} \overline{\chi}_{(pb_{n+1})_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right) \\
&\quad \times \left[\sum_{a=0}^{b_{n+1}-1} \omega_{n+1}^{(t-k_n)a}\right] \sum_{r=0}^{\infty} \frac{1}{B_r} \int_0^1 \arg \chi_{B_r}(x)\chi_\alpha(x)dx \\
&= \frac{1}{2\pi\sqrt{B_n}} \sum_{\alpha=k_nB_n}^{(k_n+1)B_n-1} \overline{\chi}_{(pb_{n+1})_{\widetilde{B}(n+1)}}\left(\left(\frac{\alpha}{B_{n+1}}\right)_{\widetilde{B}(n+1)}\right) \\
&\quad \times \sum_{r=0}^{\infty} \frac{1}{B_r} \int_0^1 \arg \chi_{B_r}(x)\chi_\alpha(x)dx.
\end{aligned} \tag{4.20}$$

The following equality can be proved: Let $n \geq 0$ and $q \in \{1, 2, \ldots, b_{n+1}-1\}$ be fixed integers. Then, for each integer $\alpha \geq 0$

$$\frac{1}{2\pi} \int_0^1 \arg \chi_{B_n}(x)\chi_\alpha(x)dx = \frac{1}{b_{n+1}} \cdot \frac{1}{\omega_{n+1}^q - 1} \delta_{q\cdot B_n, \alpha}. \tag{4.21}$$

From (4.20) and (4.21) we obtain

$$\int_0^1 x h_k''(x)dx = \frac{1}{b_{n+1}} B_n^{-\frac{3}{2}} \frac{1}{\omega_{n+1}^{k_n} - 1}. \tag{4.22}$$

From (4.18), (4.19), (4.22) and (3.3) we obtain (4.17). □

5. PROOFS OF THE MAIN RESULTS

Proof of Theorem 3.1. For an arbitrary net $\xi_N = \{x_0, \ldots, x_{N-1}\}$, composed of $N \geq 1$ points of $[0,1)$, following Kuipers and Niederreiter [8] we denote $S(\xi_N) = \sum_{m=0}^{N-1}\left(x_m - \frac{1}{2}\right)$ and for $x \in [0,1)$, $Q(\xi_N; x) = \frac{1}{N}(R(\xi_N; x) + S(\xi_N))$. The inequality

$$D^3(\xi_N) \leq 12 \int_0^1 Q^2(\xi_N; x)dx.$$

is proved. Hence, we obtain

$$D^3(\xi_N) \leq \frac{12}{N^2}\left(\int_0^1 R^2(\xi_N; x)dx - S^2(\xi_N)\right). \tag{5.1}$$

We obtain

$$D(\xi_N) \leq \left(\frac{12}{N^2}\sum_{n=0}^{\infty}\sum_{k_n=1}^{b_{n+1}-1}\sum_{k=k_nB_n}^{(k_n+1)B_n-1}\left|\sum_{m=0}^{N-1} J_{n,k_n,k}(x_m)\right|^2\right)^{\frac{1}{3}}$$

from Lemma 4.4 (i) and Parseval's equality.

The other inequalities of Theorem 3.1 follow from (5.1) and Lemma 4.4 (ii). □

Proof of Theorem 3.2.
Necessity of (i): We assume that the sequence is u. d. We will prove the equality

$$\lim_{N\to\infty}\sum_{i=0}^{N-1} J_{n,q,k}(x_i) = 0. \tag{5.2}$$

We use the representation $k = q \cdot B_n + p$, where $n \geq 0$, $q \in \{1, 2, \ldots, b_{n+1}-1\}$ and $0 \leq p < B_n$. From (3.1) and Lemma 4.1, we obtain

$$\begin{aligned} J_{n,q,k}(x) = &-\frac{1}{B_{n+1}} \cdot \frac{\chi_p(x)}{\omega_{n+1}^q - 1} + \frac{1}{B_{n+1}} \cdot \frac{1}{\omega_{n+1}^q - 1}\delta_{q.B_n,k} \\ &+ \frac{1}{B_{n+1}} \cdot \frac{\omega_{n+1}^{\frac{qb_{n+1}}{2\pi}\arg\chi_{B_n}(x)}}{\omega_{n+1}^q - 1}\chi_p(x) + \frac{1}{2\pi B_n}\sum_{r=1}^{\infty} b_{n+1}^{-r}\arg\chi_{B_n}\left(b_{n+1}^r x\right)\chi_k(x). \end{aligned} \tag{5.3}$$

Firstly, we assume that $k = q \cdot B_n$, hence, $p = 0$ and $\delta_{q \cdot B_n,k} = 1$. From (5.3) for $J_{n,q,k}(x)$, we have

$$J_{n,q,k}(x) = \frac{1}{B_{n+1}} \cdot \frac{\omega_{n+1}^{\frac{qb_{n+1}}{2\pi}\arg\chi_{B_n}(x)}}{\omega_{n+1}^q - 1} + \frac{1}{2\pi B_n}\sum_{r=1}^{\infty} b_{n+1}^{-r}\arg\chi_{B_n}\left(b_{n+1}^r x\right)\chi_k(x).$$

We will prove the equalities

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\omega_{n+1}^{\frac{qb_{n+1}}{2\pi}\arg\chi_{B_n}(x_i)} = 0 \quad\text{and}\quad \lim_{N\to\infty}\frac{1}{N}\sum_{i=0}^{N-1}\sum_{r=1}^{\infty} b_{n+1}^{-r}\arg\chi_{B_n}(b_{n+1}^r x_i)\chi_k(x_i) = 0.$$

From (1.1), it is sufficient to prove that

$$\int_0^1 \omega_{n+1}^{\frac{qb_{n+1}}{2\pi}\arg\chi_{B_n}(x)}dx = 0$$

and

$$\int_0^1 \sum_{r=1}^{\infty} b_{n+1}^{-r}\arg\chi_{B_n}\left(b_{n+1}^r x\right)\chi_k(x)dx = 0.$$

We have the following equalities

$$\begin{aligned}\int_0^1 \omega_{n+1}^{\frac{qb_{n+1}}{2\pi}\arg\chi_{B_n}(x)}dx &= \sum_{j=0}^{B_n-1}\sum_{s=0}^{b_{n+1}-1}\int_{\frac{jb_{n+1}+s}{B_{n+1}}}^{\frac{jb_{n+1}+s+1}{B_{n+1}}} \omega_{n+1}^{\frac{qb_{n+1}}{2\pi}\arg\chi_{B_n}(x)}dx \\ &= \frac{1}{b_{n+1}}\sum_{s=0}^{b_{n+1}-1}\omega_{n+1}^{sq} = 0.\end{aligned}$$

Since $k = q{\cdot}B_n$ from (4.21), we have the equalities

$$\begin{aligned}&\int_0^1 \left[\sum_{r=1}^{\infty} b_{n+1}^{-r}\arg\chi_{B_n}\left(b_{n+1}^r x\right)\chi_k(x)\right]dx \\ &\qquad= \sum_{r=1}^{\infty} b_{n+1}^{-r}\int_0^1 \arg\chi_{B_n}\left(b_{n+1}^r x\right)\chi_k(x)dx \\ &\qquad= \sum_{r=1}^{\infty} b_{n+1}^{-r}\int_0^1 \arg\phi_0^{(b_{n+1})}(b_{n+1}^r B_n x)\left[\phi_0^{(b_{n+1})}(B_n x)\right]^q dx \\ &\qquad= \sum_{r=1}^{\infty} b_{n+1}^{-r}\int_0^{B_n} \arg\phi_r^{(b_{n+1})}(t)\left[\phi_0^{(b_{n+1})}(t)\right]^q dt \\ &\qquad= B_n\sum_{r=1}^{\infty} b_{n+1}^{-r}\int_0^1 \arg\psi_{b_{n+1}^r}^{(b_{n+1})}(t)\psi_q^{(b_{n+1})}(t)dt = 0.\end{aligned}$$

If $k \neq q{\cdot}B_n$, then, from (5.3), we will obtain a useful formula for $J_{n,q,k}(x)$ and by analogy, we can prove the equality (5.2).

Sufficiency of (i): We assume that the sequence $\xi = (x_i)_{i\geq 0}$ is not u. d. Then, $\lim_{N\to\infty} D_N(\xi) = D > 0$.

Let $M > 0$ be a fixed integer. From Theorem 3.1 we have

$$\begin{aligned}D_N^3(\xi) \leq 12\sum_{n=0}^{M-1}\sum_{q=1}^{b_{n+1}-1}\sum_{k=qB_n}^{(q+1)B_n-1}&\left|\frac{1}{N}\sum_{m=0}^{N-1}J_{n,q,k}(x_m)\right|^2 \\ &+ 12\sum_{n=M}^{\infty}\sum_{q=1}^{b_{n+1}-1}\sum_{k=qB_n}^{(q+1)B_n-1}\left|\frac{1}{N}\sum_{m=0}^{N-1}J_{n,q,k}(x_m)\right|^2.\end{aligned} \tag{5.4}$$

For arbitrary integers $n \geq 0$, $q \in \{1, 2, \dots, b_{n+1} - 1\}$ and $qB_n \leq k < (q + 1)B_n$ we can prove

$$\left|\frac{1}{N}\sum_{m=0}^{N-1}J_{n,q,k}(x_m)\right| \leq \left(B + \frac{1}{2\sin\frac{\pi}{B}}\right)\frac{1}{B_{n+1}}. \tag{5.5}$$

Let $b = \min\{b_n : n \geq 1\}$. Then, using the inequality (5.5), we have

$$\sum_{n=M}^{\infty} \sum_{q=1}^{b_{n+1}-1} \sum_{k=q.B_n}^{(q+1)B_n-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{n,q,k}(x_m) \right|^2 < \left(B + \frac{1}{2\sin\frac{\pi}{B}} \right)^2 \sum_{n=M}^{\infty} \frac{1}{B_{n+1}} \tag{5.6}$$
$$\leq \left(B + \frac{1}{2\sin\frac{\pi}{B}} \right)^2 \sum_{n=M}^{\infty} \frac{1}{b^{n+1}}$$
$$= \frac{1}{b-1} \left(B + \frac{1}{2\sin\frac{\pi}{B}} \right)^2 \frac{1}{b^M}.$$

Now we choose M, so that

$$\frac{12}{b-1} \left(B + \frac{1}{2\sin\frac{\pi}{B}} \right)^2 \frac{1}{b^M} < \frac{1}{2} D^3. \tag{5.7}$$

Let $\varepsilon > 0$ be arbitrary. Then, an integer N_0 exists, so that for each $N \geq N_0, \;\; D^3 - \varepsilon < D^3_N(\xi)$. From (5.4), (5.6), (5.7) and the last inequality, we obtain

$$\frac{1}{2} D^3 - \varepsilon < 12 \sum_{n=0}^{M-1} \sum_{q=1}^{b_{n+1}-1} \sum_{k=qB_n}^{(q+1)B_n-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{n,q,k}(x_m) \right|^2 .$$

We choose an integer $\nu > 0$, so that $\frac{1}{2} D^3 - \varepsilon > \frac{1}{\nu} D^3$ and we obtain

$$0 < \frac{1}{\nu} D^3 < 12 \sum_{n=0}^{M-1} \sum_{q=1}^{b_{n+1}-1} \sum_{k=qB_n}^{(q+1)B_n-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{n,q,k}(x_m) \right|^2 .$$

Finally, (n, q, k) exists, such that

$$\left| \lim_{N\to\infty} \frac{1}{N} \sum_{m=0}^{N-1} J_{n,q,k}(x_m) \right| > 0.$$

The demonstration of (ii) and (iii) of the theorem is a consequence of the formulae obtained in Lemma 4.2, Lemma 4.3 and (i) of Theorem 3.2. □

Proof of Theorem 3.3.

$$D^3(\xi_N) \leq 12 \sum_{n=0}^{M-1} \sum_{s=1}^{b_{n+1}-1} \sum_{k=sB_n}^{(s+1)B_n-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{n,s,k}(x_m) \right|^2 \tag{5.8}$$
$$+ 12 \sum_{s=1}^{q-1} \sum_{k=sB_M}^{(s+1)B_M-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{M,s,k}(x_m) \right|^2$$
$$+ 12 \left\{ \sum_{k=qB_M}^{H} + \sum_{k=H+1}^{(q+1)B_M-1} \right\} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{M,q,k}(x_m) \right|^2$$
$$+ 12 \sum_{s=q+1}^{b_{M+1}-1} \sum_{k=sB_M}^{(s+1)B_M-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{M,s,k}(x_m) \right|^2$$

$$+ 12 \sum_{n=M+1}^{\infty} \sum_{s=1}^{b_{n+1}-1} \sum_{k=sB_n}^{(s+1)B_n-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{n,s,k}(x_m) \right|^2 \Bigg)^{\frac{1}{3}}.$$

We have the following inequality

$$\Sigma = \sum_{k=H+1}^{(q+1)B_M-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{M,q,k}(x_m) \right|^2 + \sum_{s=q+1}^{b_{M+1}-1} \sum_{k=sB_M}^{(s+1)B_M-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{M,s,k}(x_m) \right|^2$$
$$+ \sum_{n=M+1}^{\infty} \sum_{s=1}^{b_{n+1}-1} \sum_{k=sB_n}^{(s+1)B_n-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{n,s,k}(x_m) \right|^2$$
$$\leq \sum_{n=M}^{\infty} \sum_{s=1}^{b_{n+1}-1} \sum_{k=sB_n}^{(s+1)B_n-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} J_{n,s,k}(x_m) \right|^2. \tag{5.9}$$

The following inequalities

$$\left| \frac{1}{N} \sum_{m=0}^{N-1} J_{n,s,k}(x_m) \right|^2 \leq \left(\sum_{m=0}^{N-1} \frac{1}{N} \left| J_{n,s,k}(x_m) \right| \right)^2 \tag{5.10}$$

and

$$|J_{n,s,k}(x)| \leq |J_k(x)| + \frac{1}{2b \sin \frac{\pi}{B}} \cdot \frac{1}{B_n}, \quad \forall x \in [0,1) \tag{5.11}$$

hold. It is not difficult to prove that for each k, $B_n \leq k < B_{n+1}$

$$|J_k(x)| < \frac{1}{B_n}, \quad \forall x \in [0,1). \tag{5.12}$$

From (5.11) and (5.12), we obtain

$$|J_{n,s,k}(x)| \leq \left(1 + \frac{1}{2b \sin \frac{\pi}{B}} \right) \frac{1}{B_n}, \quad \forall x \in [0,1). \tag{5.13}$$

From (5.9), (5.10) and (5.13) the following inequalities

$$\Sigma \leq \sum_{n=M}^{\infty} \sum_{s=1}^{b_{n+1}-1} \sum_{h=sB_n}^{(s+1)B_n-1} \left(1 + \frac{1}{2b \sin \frac{\pi}{B}} \right)^2 \frac{1}{B_n^2} < \frac{B(1 + 2b \sin \frac{\pi}{B})^2}{4(b-1)b \sin^2 \frac{\pi}{B}} \frac{1}{B_M}$$

hold. The statement of the theorem holds from the last inequality and (5.8). □

Theorem 3.4 is a direct consequence of Lemma 4.4 and Parseval's equality.

6. CONCLUSION

In conclusion, the authors will present possible variants to extend this study.

The obtained results show that Price functions and Haar type functions can be used as a means of examining uniformly distributed sequences. The proved results raise the issue of their generalization. The most natural generalization is to obtain results in the $s-$dimensional cube $[0,1)^s$, $s \geq 2$.

This is not difficult to do when Price functions and Haar type functions are used (see Kuipers and Niederreiter [8, Corollaries 1.1 and 1.2]).

The obtaining of results, in which the modified integrals from the corresponding $s-$dimensional functions are in use, is connected with great technical difficulties. In the one-dimensional case the proof of sufficiency of Theorem 3.2 is based on the analog of the LeVeque inequality, exposed in Theorem 3.1. A multidimensional variant of the inequality of Le Veque is not known in this form to the authors. In this case, the proof of sufficiency of the multidimensional variant of Theorem 3.2 is connected with proving the multidimensional variant of LeVeque's inequality.

Another direction to generalize the obtained results is the possibility to use arbitrary orthonormal bases in $L_2[0,1)$ as a means of examining uniformity distributed sequences. The obtained results in such a study would have more general nature. The definition and the use of the modified integrals of the functions of an arbitrary system will be difficult in practice because these integrals depend on the concrete values of the corresponding functions.

Regarding the applications of uniformly distributed sequences, the inequality of Koksma-Hlawka gives an estimation of the error of $s-$dimensional quadrature formula in the terms of discrepancy of the used net. In this sense, the problem of obtaining quantitative estimations of discrepancy is interesting, as the functions presented in this paper can be used as a means of solving the above problem.

The shown generalizations are a part of the problems to solve in connection to the study of uniformity distributed sequences by orthonormal bases in $L_2[0,1)$.

The methods to prove the theorems in this paper, by suitable adaptation may be used to solve some of the exposed problems.

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