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SOME INEQUALITIES INVOLVING THE GAMMA FUNCTION

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Abstract

In this short paper, as a complement of the double inequality on the Euler gamma function, obtained by József Sándor in the paper [A note on certain inequalities for the gamma function, *J. Ineq. Pure Appl. Math.*, **6**(3) (2005), Art. 61], several inequalities involving the Euler gamma function are established by using the same method of J. Sándor that is used in [2].

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1. Introduction and Lemma

In [1], C. Alsina and M.S. Tomas studied a very interesting inequality involving the gamma function and proved the following double inequality

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1, \qquad x \in [0,1],\ n \in \aleph, \tag{1.1}$$

by using a geometrical method [1]. In view of the interest in this type of inequalities, J. Sándor [2] extended this result to a more general case, and obtained the following inequality

$$\frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1, \qquad x \in [0,1],\ a \geq 1. \tag{1.2}$$

The method used in [2] to obtain these results is based on the following lemma.

Lemma 1.1. *For all $x > 0$, and all $a \geq 1$ one has*

$$\psi(1+ax) \geq \psi(1+x), \tag{1.3}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, $x > 0$ is the digamma function and has the following series representation

$$\psi(x) = -\gamma + (x-1)\sum_{k=0}^{\infty} \frac{1}{(k+1)(x+k)}. \tag{1.4}$$

Proof. See [2]. $\square$

In [3], A.McD. Mercer continued to create new inequalities on this subject and other special functions and obtained the following inequalities

$$\frac{\Gamma(1+x)^a}{\Gamma(1+ax)} < \frac{\Gamma(1+y)^a}{\Gamma(1+ay)}, \qquad 0 < a < 1, \tag{1.5}$$

$$\frac{\Gamma(1+x)^a}{\Gamma(1+ax)} > \frac{\Gamma(1+y)^a}{\Gamma(1+ay)}, \qquad a < 0 \text{ or } a > 1, \tag{1.6}$$

where $y > x > 0$, $1 + ax > 0$, and $1 + bx > 0$.

This paper is a continuation of the above papers. As in [2], our goal is to prove several inequalities involving the gamma function, using the same method of J. Sándor that is used in [2]. Here, the essential lemma is the following

Lemma 1.2. *For all $x > 0$, and all $a \geq b$ we have*

$$\psi(1+ax) \geq \psi(1+bx), \tag{1.7}$$

in which $1 + ax > 0$ and $1 + bx > 0$.

Proof. By the above series representation of ψ, observe that:

$$\psi(1+ax) - \psi(1+bx) = \sum_{k=0}^{\infty}\left[\frac{ax}{(k+1)(ax+k+1)} - \frac{bx}{(k+1)(bx+k+1)}\right],$$

$$\psi(1+ax) - \psi(1+bx) = (a-b)x\sum_{k=0}^{\infty}\frac{1}{(ax+k+1)(bx+k+1)} \geq 0,$$

by $a \geq b$, $1 + ax > 0$, $1 + bx > 0$, $x > 0$ and $k > 0$. Thus the inequality (1.7) is proved. The equality in (1.7) holds only if $a = b$. □

2. Main Results

Now we are in a position to give the following theorem.

Theorem 2.1. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b}, \qquad \forall x \geq 0,$$

in which $1 + ax > 0$ and $1 + bx > 0$, then for all $a \geq b > 0$ or $0 > a \geq b$ ($a > 0$ and $b < 0$), f is decreasing (increasing) respectively on $[0, \infty)$.

Proof. Let g be a function defined by

$$g(x) = \log f(x) = a \log \Gamma(1 + bx) - b \log \Gamma(1 + ax),$$

then

$$g'(x) = ab[\psi(1 + bx) - \psi(1 + ax)].$$

By Lemma 1.2, we get $g'(x) \leq 0$ if $a \geq b > 0$ or $0 > a \geq b$ ($g'(x) \geq 0$ if $0 > a \geq b$), i.e., g is decreasing on $[0, \infty)$ (increasing on $[0, \infty)$) respectively. Hence f is decreasing on $[0, \infty)$ if $a \geq b > 0$ or $0 > a \geq b$ (increasing if $a > 0$ and $b < 0$) respectively.
The proof is complete. □

Corollary 2.2. *For all $x \in [0, 1]$, and all $a \geq b > 0$ or $0 > a \geq b$, we have*

$$\frac{\Gamma(1+b)^a}{\Gamma(1+a)^b} \leq \frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b} \leq 1. \tag{2.1}$$

Proof. To prove (2.1), applying Theorem 2.1, and taking account of $\Gamma(1) = 1$ we get $f(1) \leq f(x) \leq f(0)$ for all $x \in [0,1]$, and we omit (2.1). □

Corollary 2.3. *For all $x \in [0,1]$, and all $a > 0$ and $b < 0$, we have*

$$1 \leq \frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b} \leq \frac{\Gamma(1+b)^a}{\Gamma(1+a)^b}. \tag{2.2}$$

Proof. Applying Theorem 2.1, we get $f(0) \leq f(x) \leq f(1)$ for all $x \in [0,1]$, and we omit (2.2). □

Now we consider the simplest cases of Corollary 2.2 to obtain the known results of C. Alsina and M.S. Tomas [1] and J. Sándor [2].

Remark 1. *Taking $a = n$ and $b = 1$ ($a \geq 1$ and $b = 1$), in Corollary 2.2, we obtain (1.1) ((1.2)) respectively.*

Also we conclude different generalizations of (1.5)–(1.6) which are obtained by A.McD. Mercer [3].

Corollary 2.4. *For all $x \in [0,1]$, and all $a \geq b > 0$ or $0 > a \geq b$, we have*

$$\frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b} < \frac{\Gamma(1+by)^a}{\Gamma(1+ay)^b}, \tag{2.3}$$

where $0 < y < x \leq 1$.

Corollary 2.5. *For all $x \in [0,1]$, and all $a > 0$ and $b < 0$, we have*

$$\frac{\Gamma(1+by)^a}{\Gamma(1+ay)^b} < \frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b}, \tag{2.4}$$

where $0 < y < x \leq 1$.

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