

# SOME INEQUALITIES FOR THE GAMMA FUNCTION

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*Abstract:* In this paper are established some inequalities involving the Euler gamma function. We use the ideas and methods that were used by J. Sándor in his paper [2].



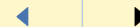
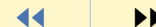
**Inequalities for the  
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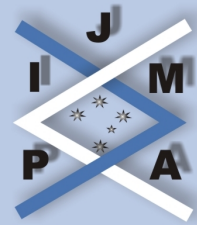
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## 1. Introduction

The Euler gamma function  $\Gamma(x)$  is defined for  $x > 0$  by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

The Psi or digamma function, the logarithmic derivative of the gamma function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

C. Alsina and M.S. Tomás in [1] proved the following double inequality:

**Theorem 1.1.** *For all  $x \in [0, 1]$  and all nonnegative integers  $n$ , the following double inequality is true:*

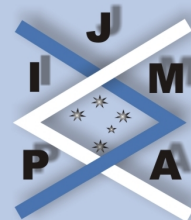
$$(1.1) \quad \frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$

Using the series representation of  $\psi(x)$ , J. Sándor in [2] proved the following generalized result of (1.1):

**Theorem 1.2.** *For all  $a \geq 1$  and all  $x \in [0, 1]$ , one has:*

$$(1.2) \quad \frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \leq 1.$$

In this paper, using the series representation of  $\psi(x)$  and ideas used in [2] we will establish some double inequalities involving the gamma function, "similar" to (1.2).



## 2. Main Results

In order to establish the proof of the theorems, we need the following lemmas:

**Lemma 2.1.** *If  $x > 0$ , then the digamma function  $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$  has the following series representation*

$$(2.1) \quad \psi(x) = -\gamma + (x-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+x)},$$

where  $\gamma$  is the Euler's constant.

*Proof.* See [3]. □

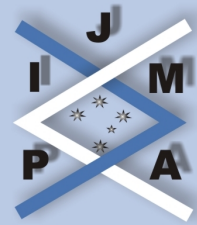
**Lemma 2.2.** *Let  $x \in [0, 1]$  and  $a, b$  be two positive real numbers such that  $a \geq b$ . Then*

$$(2.2) \quad \psi(a+bx) \geq \psi(b+ax).$$

*Proof.* It is easy to verify that  $a+bx > 0$ ,  $b+ax > 0$ . Then by (2.1) we obtain:

$$\begin{aligned} \psi(a+bx) - \psi(b+ax) &= (a+bx-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(a+bx+k)} \\ &\quad - (b+ax-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(b+ax+k)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1} \left( \frac{a+bx-1}{a+bx+k} - \frac{b+ax-1}{b+ax+k} \right) \\ &= \sum_{k=0}^{\infty} \frac{(a-b)(1-x)}{(a+bx+k)(b+ax+k)} \geq 0. \end{aligned}$$

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*Alternative proof of Lemma 2.2.* Let  $x > 0, y > 0$  and  $x \geq y$ . Then

$$\begin{aligned}\psi(x) - \psi(y) &= (x-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(x+k)} - (y-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(y+k)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k+1} \left( \frac{x-1}{x+k} - \frac{y-1}{y+k} \right) \\ &= \sum_{k=0}^{\infty} \frac{(x-y)}{(x+k)(y+k)} \geq 0.\end{aligned}$$

So  $\psi(x) \geq \psi(y)$ .

In our case: since  $a+bx > 0, b+ax > 0$  it is easy to verify that for  $x \in [0, 1], a \geq b > 0$  we have  $a+bx \geq b+ax$ , so  $\psi(a+bx) \geq \psi(b+ax)$ .  $\square$

**Lemma 2.3.** Let  $x \in [0, 1], a, b (a \geq b)$  be two positive real numbers such that  $\psi(b+ax) > 0$ . Let  $c, d$  be two given positive real numbers such that  $bc \geq ad > 0$ . Then

$$(2.3) \quad bc\psi(a+bx) - ad\psi(b+ax) \geq 0.$$

*Proof.* Since  $\psi(b+ax) > 0$ , by (2.2) it is clear that  $\psi(a+bx) > 0$ . Now, since  $bc \geq ad$ , using Lemma 2.2, we have:

$$bc\psi(a+bx) \geq ad\psi(a+bx) \geq ad\psi(b+ax).$$

So  $bc\psi(a+bx) - ad\psi(b+ax) \geq 0$ .  $\square$

**Theorem 2.4.** Let  $f$  be a function defined by

$$f(x) = \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d},$$



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where  $x \in [0, 1]$ ,  $a \geq b > 0$ ,  $c, d$  are positive real numbers such that:  $bc \geq ad > 0$  and  $\psi(b + ax) > 0$ . Then  $f$  is an increasing function on  $[0, 1]$ , and the following double inequality holds:

$$\frac{\Gamma(a)^c}{\Gamma(b)^d} \leq \frac{\Gamma(a + bx)^c}{\Gamma(b + ax)^d} \leq \frac{\Gamma(a + b)^c}{\Gamma(a + b)^d}.$$

*Proof.* Let  $g(x)$  be a function defined by  $g(x) = \log f(x)$ . Then:

$$g(x) = c \log \Gamma(a + bx) - d \log \Gamma(b + ax).$$

So

$$g'(x) = bc \frac{\Gamma'(a + bx)}{\Gamma(a + bx)} - ad \frac{\Gamma'(b + ax)}{\Gamma(b + ax)} = bc\psi(a + bx) - ad\psi(b + ax).$$

Using (2.3), we have  $g'(x) \geq 0$ . It means that  $g(x)$  is increasing on  $[0, 1]$ . This implies that  $f(x)$  is increasing on  $[0, 1]$ .

So for  $x \in [0, 1]$  we have  $f(0) \leq f(x) \leq f(1)$  or

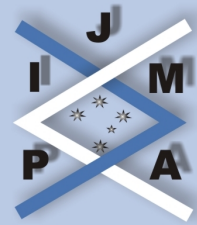
$$\frac{\Gamma(a)^c}{\Gamma(b)^d} \leq \frac{\Gamma(a + bx)^c}{\Gamma(b + ax)^d} \leq \frac{\Gamma(a + b)^c}{\Gamma(a + b)^d}.$$

This concludes the proof of Theorem 2.4. □

In a similar way, it is easy to prove the following lemmas and theorems.

**Lemma 2.5.** Let  $x \geq 1$  and  $a, b$  be two positive real numbers such that  $b \geq a$ . Then

$$\psi(a + bx) \geq \psi(b + ax).$$



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**Lemma 2.6.** Let  $x \geq 1$ ,  $a, b$  ( $b \geq a$ ) be two positive real numbers such that  $\psi(b + ax) > 0$  and  $c, d$  be any two given real numbers such that  $bc \geq ad > 0$ . Then

$$bc\psi(a + bx) - ad\psi(b + ax) \geq 0.$$

**Theorem 2.7.** Let  $f$  be a function defined by

$$f(x) = \frac{\Gamma(a + bx)^c}{\Gamma(b + ax)^d},$$

where  $x \geq 1$ ,  $b \geq a > 0$ ,  $c, d$  are positive real numbers such that  $bc \geq ad > 0$  and  $\psi(b + ax) > 0$ . Then  $f$  is an increasing function on  $[1, +\infty)$ .

**Lemma 2.8.** Let  $x \in [0, 1]$ ,  $a, b$  ( $a \geq b$ ) be two positive real numbers such that  $\psi(a + bx) < 0$  and  $c, d$  be any two given real numbers such that  $ad \geq bc > 0$ . Then

$$bc\psi(a + bx) - ad\psi(b + ax) \geq 0.$$

Using Lemmas 2.2 and 2.8, and the methods we used in Theorem 2.4, the following theorem can be proved:

**Theorem 2.9.** Let  $f$  be a function defined by

$$f(x) = \frac{\Gamma(a + bx)^c}{\Gamma(b + ax)^d},$$

where  $x \in [0, 1]$ ,  $a \geq b > 0$ ,  $c, d$  are positive real numbers such that  $ad \geq bc > 0$  and  $\psi(a + bx) < 0$ . Then  $f$  is an increasing function on  $[0, 1]$ .

**Lemma 2.10.** Let  $x \geq 1$ ,  $a, b$  ( $b \geq a$ ) be two positive real numbers such that  $\psi(a + bx) < 0$  and  $c, d$  be any two given real numbers such that  $ad \geq bc > 0$ . Then

$$bc\psi(a + bx) - ad\psi(b + ax) \geq 0.$$



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Using Lemmas 2.5 and 2.10, and the methods we used in Theorem 2.4, the following theorem can be proved:

**Theorem 2.11.** *Let  $f$  be a function defined by*

$$f(x) = \frac{\Gamma(a + bx)^c}{\Gamma(b + ax)^d},$$

*where  $x > 1, b \geq a > 0, c, d$  are positive real numbers such that  $ad \geq bc > 0$  and  $\psi(a + bx) < 0$ . Then  $f$  is an increasing function on  $[1, +\infty)$ .*



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