SOME INEQUALITIES FOR THE GAMMA FUNCTION

ARMEND SH. SHABANI
Department of Mathematics
University of Prishtina
Avenue "Mother Theresa", 5 Prishtine
10000, Kosova-UNMIK
EMail: armend_shabani@hotmail.com

Received: 04 May, 2007

Accepted: 15 May, 2007

Communicated by: A. Laforgia

2000 AMS Sub. Class.: 33B15.

Key words: Euler gamma function, Inequalities.

Abstract: In this paper are established some inequalities involving the Euler gamma function. We use the ideas and methods that were used by J. Sándor in his paper [2].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

vol. 8, iss. 2, art. 49, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

The Psi or digamma function, the logarithmic derivative of the gamma function is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0.$$

C. Alsina and M.S. Tomás in [1] proved the following double inequality:

Theorem 1.1. *For all $x \in [0,1]$ and all nonnegative integers n, the following double inequality is true:*

$$\frac{1}{n!} \le \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \le 1. \tag{1.1}$$

Using the series representation of $\psi(x)$, J. Sándor in [2] proved the following generalized result of (1.1):

Theorem 1.2. *For all $a \ge 1$ and all $x \in [0,1]$, one has:*

$$\frac{1}{\Gamma(1+a)} \le \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \le 1. \tag{1.2}$$

In this paper, using the series representation of $\psi(x)$ and ideas used in [2] we will establish some double inequalities involving the gamma function, "similar" to (1.2).

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Main Results

In order to establish the proof of the theorems, we need the following lemmas:

Lemma 2.1. *If* $x > 0$, *then the digamma function* $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ *has the following series representation*

$$\psi(x) = -\gamma + (x-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(k+x)}, \tag{2.1}$$

where γ *is the Euler's constant.*

Proof. See [3]. □

Lemma 2.2. *Let* $x \in [0,1]$ *and* a, b *be two positive real numbers such that* $a \geq b$. *Then*

$$\psi(a+bx) \geq \psi(b+ax). \tag{2.2}$$

Proof. It is easy to verify that $a+bx > 0$, $b+ax > 0$. Then by (2.1) we obtain:

$$\begin{aligned}\psi(a+bx) - \psi(b+ax) &= (a+bx-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(a+bx+k)} \\ &\quad - (b+ax-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(b+ax+k)} \\ &= \sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{a+bx-1}{a+bx+k} - \frac{b+ax-1}{b+ax+k}\right) \\ &= \sum_{k=0}^{\infty}\frac{(a-b)(1-x)}{(a+bx+k)(b+ax+k)} \geq 0.\end{aligned}$$

□

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Alternative proof of Lemma 2.2. Let $x > 0, y > 0$ and $x \geq y$. Then

$$\psi(x) - \psi(y) = (x-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(x+k)} - (y-1)\sum_{k=0}^{\infty}\frac{1}{(k+1)(y+k)}$$
$$= \sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{x-1}{x+k} - \frac{y-1}{y+k}\right)$$
$$= \sum_{k=0}^{\infty}\frac{(x-y)}{(x+k)(y+k)} \geq 0.$$

So $\psi(x) \geq \psi(y)$.

In our case: since $a+bx > 0, b+ax > 0$ it is easy to verify that for $x \in [0,1], a \geq b > 0$ we have $a+bx \geq b+ax$, so $\psi(a+bx) \geq \psi(b+ax)$. □

Lemma 2.3. *Let $x \in [0,1]$, a, b $(a \geq b)$ be two positive real numbers such that $\psi(b+ax) > 0$. Let c, d be two given positive real numbers such that $bc \geq ad > 0$. Then*

$$bc\psi(a+bx) - ad\psi(b+ax) \geq 0. \tag{2.3}$$

Proof. Since $\psi(b+ax) > 0$, by (2.2) it is clear that $\psi(a+bx) > 0$. Now, since $bc \geq ad$, using Lemma 2.2, we have:

$$bc\psi(a+bx) \geq ad\psi(a+bx) \geq ad\psi(b+ax).$$

So $bc\psi(a+bx) - ad\psi(b+ax) \geq 0$. □

Theorem 2.4. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d},$$

Inequalities for the Gamma Function
Armend Sh. Shabani
vol. 8, iss. 2, art. 49, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

where $x \in [0,1]$, $a \geq b > 0$, c, d are positive real numbers such that: $bc \geq ad > 0$ and $\psi(b+ax) > 0$. Then f is an increasing function on $[0,1]$, and the following double inequality holds:

$$\frac{\Gamma(a)^c}{\Gamma(b)^d} \leq \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d} \leq \frac{\Gamma(a+b)^c}{\Gamma(a+b)^d}.$$

Proof. Let $g(x)$ be a function defined by $g(x) = \log f(x)$. Then:

$$g(x) = c\log\Gamma(a+bx) - d\log\Gamma(b+ax).$$

So

$$g^{'}(x) = bc\frac{\Gamma^{'}(a+bx)}{\Gamma(a+bx)} - ad\frac{\Gamma^{'}(b+ax)}{\Gamma(b+ax)} = bc\psi(a+bx) - ad\psi(b+ax).$$

Using (2.3), we have $g^{'}(x) \geq 0$. It means that $g(x)$ is increasing on $[0,1]$. This implies that $f(x)$ is increasing on $[0,1]$.

So for $x \in [0,1]$ we have $f(0) \leq f(x) \leq f(1)$ or

$$\frac{\Gamma(a)^c}{\Gamma(b)^d} \leq \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d} \leq \frac{\Gamma(a+b)^c}{\Gamma(a+b)^d}.$$

This concludes the proof of Theorem 2.4. □

In a similar way, it is easy to prove the following lemmas and theorems.

Lemma 2.5. *Let $x \geq 1$ and a, b be two positive real numbers such that $b \geq a$. Then*

$$\psi(a+bx) \geq \psi(b+ax).$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Lemma 2.6. *Let $x \geq 1, a, b$ $(b \geq a)$ be two positive real numbers such that $\psi(b + ax) > 0$ and c, d be any two given real numbers such that $bc \geq ad > 0$. Then*

$$bc\psi(a + bx) - ad\psi(b + ax) \geq 0.$$

Theorem 2.7. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(a + bx)^c}{\Gamma(b + ax)^d},$$

where $x \geq 1, b \geq a > 0$, c, d are positive real numbers such that $bc \geq ad > 0$ and $\psi(b + ax) > 0$. Then f is an increasing function on $[1, +\infty)$.

Lemma 2.8. *Let $x \in [0, 1]$, a, b $(a \geq b)$ be two positive real numbers such that $\psi(a + bx) < 0$ and c, d be any two given real numbers such that $ad \geq bc > 0$. Then*

$$bc\psi(a + bx) - ad\psi(b + ax) \geq 0.$$

Using Lemmas 2.2 and 2.8, and the methods we used in Theorem 2.4, the following theorem can be proved:

Theorem 2.9. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(a + bx)^c}{\Gamma(b + ax)^d},$$

where $x \in [0, 1]$, $a \geq b > 0$, c, d are positive real numbers such that $ad \geq bc > 0$ and $\psi(a + bx) < 0$. Then f is an increasing function on $[0, 1]$.

Lemma 2.10. *Let $x \geq 1, a, b$ $(b \geq a)$ be two positive real numbers such that $\psi(a + bx) < 0$ and c, d be any two given real numbers such that $ad \geq bc > 0$. Then*

$$bc\psi(a + bx) - ad\psi(b + ax) \geq 0.$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Using Lemmas 2.5 and 2.10, and the methods we used in Theorem 2.4, the following theorem can be proved:

Theorem 2.11. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(a+bx)^c}{\Gamma(b+ax)^d},$$

where $x > 1, b \geq a > 0$, c, d are positive real numbers such that $ad \geq bc > 0$ and $\psi(a+bx) < 0$. Then f is an increasing function on $[1, +\infty)$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] C. ALSINA AND M.S. TOMÁS, A geometrical proof of a new inequality for the gamma function, *J. Ineq. Pure Appl. Math.*, **6**(2) (2005), Art. 48. [ONLINE: http://jipam.vu.edu.au/article.php?sid=517].

[2] J. SÁNDOR, A note on certain inequalities for the gamma function, *J. Ineq. Pure Appl. Math.*, **6**(3) (2005), Art. 61. [ONLINE: http://jipam.vu.edu.au/article.php?sid=534].

[3] E.T. WHITTAKER AND G.N. WATSON, *A Course of Modern Analysis*, Camb. Univ. Press, 1996.

[4] W. RUDIN, *Principles of Mathematical Analysis*, New York, McGraw-Hill, 1976.

vol. 8, iss. 2, art. 49, 2007

journal of inequalities in pure and applied mathematics

issn: 1443-5756