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AN APPLICATION OF THE GENERALIZED MALIGRANDA-ORLICZ'S LEMMA

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ABSTRACT. Using the generalized Maligranda-Orlicz's Lemma we will show that $BV_{(2,\alpha)}([a,b])$ is a Banach algebra.

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1. INTRODUCTION

Two centuries ago, around 1880, C. Jordan (see [2]) introduced the notion of a function of bounded variation and established the relation between these functions and monotonic ones. Later, the concept of bounded variation was generalized in various directions. In his 1908 paper de la Vallée Poussin (see [4]) generalized the Jordan bounded variation concept. De la Vallée Poussin, defined the bounded second variation of a function f on an interval $[a,b]$ by

$$V^2(f) = V^2(f,[a,b]) = \sup_{\Pi} \sum_{j=1}^{n-1} \left| \frac{f(t_{j+1}) - f(t_j)}{t_{j+1} - t_j} - \frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right|$$

where the supremum is taken over all partitions $\Pi : a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$. If $V^2(f,[a,b]) < \infty$, the function f is said to be of bounded second variation on $[a,b]$. The class of all functions which are of bounded second variation is denoted by $BV^2([a,b])$.

In 1970 the above class of functions was generalized with respect to a strictly increasing continuous function α (see [3]):

Let f be a real function defined on $[a,b]$. For a given partition of the form: $\Pi : a = t_1 < \cdots < t_n = b$, we set

$$\sigma_{(2,\alpha)}(f,\Pi) = \sum_{j=1}^{n-2} |f_\alpha[t_j,t_{j+1}] - f_\alpha[t_{j+1},t_{j+2}]|,$$

where

$$f_\alpha[p,q] = \frac{f(q)-f(p)}{\alpha(q)-\alpha(p)},$$

and

$$V_{(2,\alpha)}(f,[a,b]) = V_{(2,\alpha)}(f) = \sup_{\Pi} \sigma_{(2,\alpha)}(f,\Pi),$$

where the supremum is taken over all partitions Π of $[a,b]$.

If $V_{(2,\alpha)}(f) < \infty$, then f is said to be of $(2,\alpha)$-bounded variation.

The set of all these functions will be denoted by $BV_{(2,\alpha)}([a,b])$.

A function f is α-derivable at t_0 if

$$\lim_{t\to t_0} \frac{f(t)-f(t_0)}{\alpha(t)-\alpha(t_0)} \text{ exists.}$$

If this limit exists, we denote its value by $f'_\alpha(t_0)$, which we call the α-derivative of f at t_0.

The class $BV_{(2,\alpha)}([a,b])$ is a Banach space equipped with the norm

$$\|f\|_{BV_{(2,\alpha)([a,b])}} = |f(a)| + |f'_\alpha(a)| + V_{(2,\alpha)}(f).$$

Using the generalized Maligranda-Orlicz's Lemma (see Theorem 3.1 of the present paper) we will show that $BV_{(2,\alpha)}([a,b])$ is a Banach algebra.

2. DEFINITION AND NOTATION

We begin this section by giving a definition and several simple lemmas that will be used throughout the paper.

Definition 2.1. A function $f : [a,b] \to \mathbb{R}$ is said to be α-Lipschitz if there exists a constant $M > 0$ such that

$$|f(x)-f(y)| \le M|\alpha(x)-\alpha(y)|,$$

for all $x,y \in [a,b]$, $x \ne y$. By α-$\mathrm{Lip}[a,b]$ we will denote the space of functions which are α-Lipschitz. If $f \in \alpha$-$\mathrm{Lip}[a,b]$ we define

$$\mathrm{Lip}_\alpha(f) = \inf\{M > 0 : |f(x)-f(y)| \le M|\alpha(x)-\alpha(y)|, x \ne y \in [a,b]\}$$

and

$$\mathrm{Lip}'_\alpha(f) = \sup\left\{ \frac{|f(x)-f(y)|}{|\alpha(x)-\alpha(y)|} : x \ne y \in [a,b] \right\}.$$

It is not hard to prove that

$$\mathrm{Lip}_\alpha(f) = \mathrm{Lip}'_\alpha(f).$$

α-$\mathrm{Lip}[a,b]$ equipped with the norm

$$\|f\|_{\alpha\text{-}\mathrm{Lip}[a,b]} = |f(a)| + \mathrm{Lip}_\alpha(f)$$

is a Banach space.

Lemma 2.1. *If $f \in BV_{(2,\alpha)}([a,b])$, then there exists a constant $M > 0$ such that*

$$\left| \frac{f(x_2)-f(x_1)}{\alpha(x_2)-\alpha(x_1)} \right| = |f_\alpha[x_1,x_2]| \le M$$

for all $x_1, x_2 \in [a,b]$.

Lemma 2.2.

$$\|f\|_{\alpha\text{-Lip}[a,b]} \leq \|f\|_{BV_{(2,\alpha)}([a,b])},\ f \in BV_{(2,\alpha)}([a,b])$$

and

$$BV_{(2,\alpha)} \hookrightarrow \alpha\text{-Lip}[a,b].$$

Lemma 2.3. $\alpha\text{-Lip}[a,b] \hookrightarrow BV[a,b] \hookrightarrow B[a,b]$.

3. GENERALIZED MALIGRANDA-ORLICZ'S LEMMA

The following result generalizes the Maligranda-Orlicz Lemma which is due to the authors (see [1]).

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space whose elements are bounded functions, which is closed under pointwise multiplication of functions. Let us assume that $f \cdot g \in X$ such that*

$$\|fg\| \leq \|f\|_\infty \|g\| + \|f\| \|g\|_\infty + K\|f\|\|g\|,\ K > 0.$$

Then $(X, \|\cdot\|_1)$ equipped with the norm

$$\|f\|_1 = \|f\|_\infty + K\|f\|,\ f \in X,$$

is a Banach algebra. If $X \hookrightarrow B[a,b]$, then $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent.

4. $\mathbf{BV_{(2,\alpha)}([a,b])}$ AS A BANACH ALGEBRA

The following result shows us that $BV_{(2,\alpha)}([a,b])$ is closed under pointwise multiplication of functions.

Theorem 4.1. *If $f, g \in BV_{(2,\alpha)}([a,b])$, then $f \cdot g \in BV_{(2,\alpha)}([a,b])$.*

Proof. Let $\Pi : a = x_1 < x_2 < \cdot < x_n = b$ be a partition of $[a,b]$. Then

$$\begin{aligned}
\sigma_{(2,\alpha)}(f \cdot g, \Pi) &= \sum_{j=1}^{n-2} \big| (fg)_\alpha[x_j, x_{j+1}] - (fg)_\alpha[x_{j+1}, x_{j+2}] \big| \\
&= \sum_{j=1}^{n-2} \big| f(x_j) \cdot g_\alpha[x_j, x_{j+1}] + g(x_{j+1}) \cdot f_\alpha[x_j, x_{j+1}] \\
&\qquad - f(x_{j+1}) \cdot g_\alpha[x_{j+1}, x_{j+2}] - g(x_{j+2}) \cdot f_\alpha[x_{j+1}, x_{j+2}] \big| \\
&\leq \sum_{j=1}^{n-2} \big| f(x_j) \cdot g_\alpha[x_j, x_{j+1}] - f(x_j) \cdot g_\alpha[x_{j+1}, x_{j+2}] \\
&\qquad + f(x_j) \cdot g_\alpha[x_{j+1}, x_{j+2}] - f(x_{j+1}) \cdot g_\alpha[x_{j+1}, x_{j+2}] \big| \\
&\qquad + \sum_{j=1}^{n-2} \big| g(x_{j+1}) \cdot f_\alpha[x_j, x_{j+1}] - g(x_{j+1}) \cdot f_\alpha[x_{j+1}, x_{j+2}] \\
&\qquad + g(x_{j+1}) \cdot f_\alpha[x_{j+1}, x_{j+2}] - g(x_{j+2}) \cdot f_\alpha[x_{j+1}, x_{j+2}] \big|.
\end{aligned}$$

Since f and g are bounded, we have $|f(x_j)| \leq \|f\|_\infty$ and $|g(x_{j+1})| \leq \|g\|_\infty$.

Hence

$$\begin{aligned}\sigma_{(2,\alpha)}(f\cdot g,\Pi) \leq \|f\|_\infty \sum_{j=1}^{n-2}\big|g_\alpha[x_j,x_{j+1}]-g_\alpha[x_{j+1},x_{j+2}]\big| \\ +\sum_{j=1}^{n-2}\big|f(x_j)-f(x_{j+1})\big|\cdot\big|g_\alpha[x_{j+1},x_{j+2}]\big| \\ +\|g\|_\infty\sum_{j=1}^{n-2}\big|f_\alpha[x_j,x_{j+1}]-f_\alpha[x_{j+1},x_{j+2}]\big| \\ +\sum_{j=1}^{n-2}\big|g(x_{j+1})-g(x_{j+2})\big|\cdot\big|f_\alpha[x_{j+1},x_{j+2}]\big| \\ =\|f\|_\infty\cdot\sigma_{(2,\alpha)}(g,\Pi)+\|g\|_\infty\cdot\sigma_{(2,\alpha)}(f,\Pi) \\ +\sum_{j=1}^{n-2}\frac{|f(x_j)-f(x_{j+1})|}{|\alpha(x_j)-\alpha(x_{j+1})|}\cdot\frac{|g(x_{j+1})-g(x_{j+2})|}{|\alpha(x_{j+1})-\alpha(x_{j+2})|}|\alpha(x_j)-\alpha(x_{j+1})| \\ +\sum_{j=1}^{n-2}\frac{|g(x_{j+1})-g(x_{j+2})|}{|\alpha(x_{j+1})-\alpha(x_{j+2})|}\cdot\frac{|f(x_{j+1})-f(x_{j+2})|}{|\alpha(x_{j+1})-\alpha(x_{j+2})|}|\alpha(x_{j+1})-\alpha(x_{j+2})|.\end{aligned}$$

By Definition 2.1 and Lemma 2.1 we obtain

$$\frac{|f(x_j)-f(x_{j+1})|}{|\alpha(x_j)-\alpha(x_{j+1})|}\leq \mathrm{Lip}_\alpha(f)\ j=1,2,\cdots,n-1$$

and

$$\frac{|g(x_j)-g(x_{j+1})|}{|\alpha(x_j)-\alpha(x_{j+1})|}\leq \mathrm{Lip}_\alpha(g)\ j=1,2,\cdots,n-1.$$

Thus

$$\begin{aligned}\sigma_{(2,\alpha)}(f\cdot g,\Pi)\leq \|f\|_\infty\cdot\sigma_{(2,\alpha)}(g,\Pi)+\|g\|_\infty\cdot\sigma_{(2,\alpha)}(f,\Pi) \\ +(\mathrm{Lip}_\alpha(f))(\mathrm{Lip}_\alpha(g))\sum_{j=1}^{n-2}(\alpha(x_{j+1})-\alpha(x_j)+\alpha(x_{j+2})-\alpha(x_{j+1})).\end{aligned}$$

By Lemma 2.2 we have $\mathrm{Lip}_\alpha(f)<+\infty$ and $\mathrm{Lip}_\alpha(g)<+\infty$. Moreover

$$\begin{aligned}\sum_{j=1}^{n-2}(\alpha(x_{j+2})-\alpha(x_j)) &= \alpha(b)+\sum_{j=2}^{n-2}(\alpha(x_{j+1})-\alpha(x_j))-\alpha(a) \\ &\leq 2(\alpha(b)-\alpha(a)).\end{aligned}$$

Then

$$\begin{aligned}\sigma_{(2,\alpha)}(f\cdot g,\Pi)\leq \|f\|_\infty\cdot\sigma_{(2,\alpha)}(g,\Pi)+\|g\|_\infty\cdot\sigma_{(2,\alpha)}(f,\Pi) \\ +2(\alpha(b)-\alpha(a))(\mathrm{Lip}_\alpha(f))(\mathrm{Lip}_\alpha(g))\end{aligned}$$

for all partitions Π of $[a,b]$.

Hence

$$\begin{aligned}V_{(2,\alpha)}(f\cdot g)\leq \|f\|_\infty V_{(2,\alpha)}(g)+\|g\|_\infty V_{(2,\alpha)}(f) \\ +2(\alpha(b)-\alpha(a))(\mathrm{Lip}_\alpha(f))(\mathrm{Lip}_\alpha(g))<+\infty.\end{aligned}$$

Therefore $f \cdot g \in BV_{(2,\alpha)}([a,b])$.

This completes the proof of Theorem 4.1. □

Corollary 4.2. *If* $f, g \in BV_{(2,\alpha)}([a,b])$, *then*

$$\|f \cdot g\|_{BV_{(2,\alpha)}([a,b])} \leq \|f\|_\infty \|g\|_{BV_{(2,\alpha)}([a,b])} + \|g\|_\infty \|f\|_{BV_{(2,\alpha)}([a,b])} + 2(\alpha(b) - \alpha(a)) \|f\|_{BV_{(2,\alpha)}([a,b])} \|g\|_{BV_{(2,\alpha)}([a,b])}.$$

Proof. Note that

$$\mathrm{Lip}_\alpha(f) \leq \|f\|_{\alpha\text{-Lip}([a,b])} \leq \|f\|_{BV_{(2,\alpha)}([a,b])}$$

and

$$\mathrm{Lip}_\alpha(g) \leq \|g\|_{\alpha\text{-Lip}([a,b])} \leq \|g\|_{BV_{(2,\alpha)}([a,b])}$$

by Lemma 2.2.

From Theorem 4.1 we have

$$V_{(2,\alpha)}(f \cdot g) \leq \|f\|_\infty V_{(2,\alpha)}(g) + \|g\|_\infty V_{(2,\alpha)}(f) + 2(\alpha(b) - \alpha(a)) \|f\|_{BV_{(2,\alpha)}([a,b])} \|g\|_{BV_{(2,\alpha)}([a,b])}.$$

On the other hand,

$$\begin{aligned} |(fg)(a)| &\leq 2|f(a)| \cdot |g(a)| \leq \|f\|_\infty |g(a)| + \|g\|_\infty |f(a)| \\ |(fg)'_\alpha(a)| &\leq |f(a)| \cdot |g'_\alpha(a)| + |g(a)| \cdot |f'_\alpha(a)| \\ &\leq \|f\|_\infty |g'_\alpha(a)| + \|g\|_\infty |f'_\alpha(a)|. \end{aligned}$$

Adding we obtain

$$\begin{aligned} &|(fg)(a)| + |(fg)'_\alpha(a)| + V_{(2,\alpha)}(f \cdot g) \\ &\quad \leq \|f\|_\infty (|g(a)| + |g'_\alpha(a)| + V_{(2,\alpha)}(g)) + \|g\|_\infty (|f(a)| + |f'_\alpha(a)| + V_{(2,\alpha)}(f)) \\ &\qquad + 2(\alpha(b) - \alpha(a)) \|f\|_{BV_{(2,\alpha)}([a,b])} \|g\|_{BV_{(2,\alpha)}([a,b])}. \end{aligned}$$

Therefore

$$\|f \cdot g\|_{BV_{(2,\alpha)}([a,b])} \leq \|f\|_\infty \|g\|_{BV_{(2,\alpha)}([a,b])} + \|g\|_\infty \|f\|_{BV_{(2,\alpha)}([a,b])} + 2(\alpha(b) - \alpha(a)) \|f\|_{BV_{(2,\alpha)}([a,b])} \|g\|_{BV_{(2,\alpha)}([a,b])}.$$

This completes the proof of Corollary 4.2. □

5. MAIN RESULT

Theorem 5.1. $BV_{(2,\alpha)}([a,b])$ *equipped with the norm*

$$\|f\|^1_{BV_{(2,\alpha)}([a,b])} = \|f\|_\infty + 2(\alpha(b) - \alpha(a)) \|f\|_{BV_{(2,\alpha)}([a,b])}, \; f \in BV_{(2,\alpha)}([a,b])$$

is a Banach algebra and the norms $\|\cdot\|_{BV_{(2,\alpha)}([a,b])}$ *and* $\|\cdot\|^1_{BV_{(2,\alpha)}([a,b])}$ *are equivalent.*

Proof. First of all, we need to check the hypotheses from Theorem 3.1. Since $\alpha\text{-Lip}[a,b] \hookrightarrow B[a,b]$, by Lemma 2.2 we have $BV_{(2,\alpha)}([a,b]) \subset B[a,b]$. Next, from Theorem 4.1 we see that $BV_{(2,\alpha)}([a,b])$ is closed under pointwise multiplication of functions. Now observe that if we take $K = 2(\alpha(b) - \alpha(a))$, the inequality given in Corollary 4.2 coincides with the one given in Theorem 3.1. Also note that

$$BV_{(2,\alpha)}([a,b]) \hookrightarrow \alpha\text{-Lip}[a,b] \hookrightarrow B[a,b]$$

and

$$\|f\|_\infty \leq \max\{1, (\alpha(b) - \alpha(a))\}\|f\|_{BV_{(2,\alpha)}([a,b])}, \; f \in BV_{(2,\alpha)}([a,b]).$$

Therefore, invoking Theorem 3.1 we have that $(\mathbb{R}, BV_{(2,\alpha)}([a,b]), +, \cdot, \|\cdot\|^1_{BV_{(2,\alpha)}([a,b])})$ is a Banach algebra and the norms $\|\cdot\|_{BV_{(2,\alpha)}([a,b])}$ and $\|\cdot\|^1_{BV_{(2,\alpha)}([a,b])}$ are equivalent.

This completes the proof of Theorem 5.1. □

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