BOUNDS FOR SOME PERTURBED ČEBYŠEV FUNCTIONALS

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Abstract: Bounds for the perturbed Čebyšev functionals $C(f,g) - \mu C(e,g)$ and $C(f,g) - \mu C(e,g) - \nu C(f,e)$ when $\mu, \nu \in \mathbb{R}$ and e is the identity function on the interval $[a,b]$, are given. Applications for some Grüss' type inequalities are also provided.

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1. Introduction

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the *Čebyšev functional*:

$$C(f, g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt \int_a^b g(t)dt. \tag{1.1}$$

In 1934, Grüss [5] showed that

$$|C(f, g)| \le \frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a, b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [3], states that

$$|C(f, g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{1.4}$$

provided that f', g' exist and are continuous on $[a, b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ can be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f, g : [a, b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$ while $\|f'\|_\infty = ess\sup_{t\in[a,b]} |f'(t)|$.

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [9]:

$$|C(f, g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

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provided that f is Lebesgue integrable and satisfies (1.3) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (1.5).

The case of euclidean norms of the derivative was considered by A. Lupaş in [7] in which he proved that

$$|C(f,g)| \le \frac{1}{\pi^2}\|f'\|_2\|g'\|_2(b-a), \tag{1.6}$$

provided that f,g are absolutely continuous and $f',g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, P. Cerone and S.S. Dragomir [1] have proved the following results:

$$|C(f,g)| \le \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}}, \tag{1.7}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$ or $p=1$ and $q=\infty$, and

$$|C(f,g)| \le \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_1 \cdot \frac{1}{b-a} ess\sup_{t\in[a,b]}\left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|, \tag{1.8}$$

provided that $f \in L_p[a,b]$ and $g \in L_q[a,b]$ ($p>1$, $\frac{1}{p}+\frac{1}{q}=1$; $p=1$, $q=\infty$ or $p=\infty, q=1$).

Notice that for $q=\infty, p=1$ in (1.7) we obtain

$$\begin{aligned}|C(f,g)| &\le \inf_{\gamma\in\mathbb{R}}\|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b\left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \\ &\le \|g\|_\infty \cdot \frac{1}{b-a}\int_a^b\left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt\end{aligned} \tag{1.9}$$

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and if g satisfies (1.3), then

$$\begin{aligned} (1.10) \qquad |C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \\ &\leq \left\| g - \frac{n+N}{2} \right\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \\ &\leq \frac{1}{2}(N-n) \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \end{aligned}$$

The inequality between the first and the last term in (1.10) has been obtained by Cheng and Sun in [4]. However, the sharpness of the constant $\frac{1}{2}$, a generalisation for the abstract Lebesgue integral and the discrete version of it have been obtained in [2].

For other recent results on the Grüss inequality, see [6], [8] and [10] and the references therein.

The aim of the present paper is to establish Grüss type inequalities for some perturbed Čebyšev functionals. For this purpose, two integral representations of the functionals $C(f,g) - \mu C(e,g)$ and $C(f,g) - \mu C(e,g) - \nu C(f,e)$ when $\mu, \nu \in \mathbb{R}$ and $e(t) = t$, $t \in [a,b]$ are given.

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2. Representation Results

The following representation result can be stated.

Lemma 2.1. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and g is Lebesgue integrable on $[a,b]$, then*

$$C(f,g)=\frac{1}{(b-a)^2}\int_a^b\int_a^b Q(t,s)\,[g(s)-\lambda]\,f'(t)\,dsdt \tag{2.1}$$

for any $\lambda\in\mathbb{R}$, where the kernel $Q:[a,b]^2\to\mathbb{R}$ is given by

$$Q(t,s):=\begin{cases} t-b & \text{if } a\le s\le t\le b,\\ t-a & \text{if } a\le t<s\le b.\end{cases} \tag{2.2}$$

Proof. We observe that for $\lambda\in\mathbb{R}$ we have $C(f,\lambda)=0$ and thus it suffices to prove (2.1) for $\lambda=0$.

By Fubini's theorem, we have

$$\int_a^b\int_a^b Q(t,s)\,g(s)\,f'(t)\,dsdt=\int_a^b\left(\int_a^b Q(t,s)\,f'(t)\,dt\right)g(s)\,ds. \tag{2.3}$$

By the definition of $Q(t,s)$ and integrating by parts, we have successively,

$$\begin{aligned}\int_a^b Q(t,s)f'(t)\,dt &\\ &=\int_a^s Q(t,s)\,f'(t)\,dt+\int_s^b Q(t,s)\,f'(t)\,dt\\ &=\int_a^s (t-a)\,f'(t)\,dt+\int_s^b (t-b)\,f'(t)\,dt\end{aligned} \tag{2.4}$$

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$$= (s-a) f(s) - \int_a^s f(t)\,dt + (b-s) f(s) - \int_s^b f(t)\,dt$$
$$= (b-a) f(s) - \int_a^b f(t)\,dt,$$

for any $s \in [a,b]$.

Now, integrating (2.4) multiplied with $g(s)$ over $s \in [a,b]$, we deduce

$$\int_a^b \left(\int_a^b Q(t,s) f'(t)\,dt \right) g(s)\,ds = \int_a^b \left[(b-a) f(s) - \int_a^b f(t)\,dt \right] g(s)\,ds$$
$$= (b-a) \int_a^b f(s)g(s)ds - \int_a^b f(s)\,ds \cdot \int_a^b g(s)ds$$
$$= (b-a)^2 C(f,g)$$

and the identity is proved. □

Utilising the linearity property of $C(\cdot,\cdot)$ in each argument, we can state the following equality:

Theorem 2.2. *If $e : [a,b] \to \mathbb{R}$, $e(t) = t$, then under the assumptions of Lemma 2.1 we have:*

$$C(f,g) = \mu C(e,g) + \frac{1}{(b-a)^2} \int_a^b \int_a^b Q(t,s) [g(s) - \lambda] [f'(t) - \mu]\,dtds \tag{2.5}$$

for any $\lambda, \mu \in \mathbb{R}$, where

$$C(e,g) = \frac{1}{b-a} \int_a^b t g(t)\,dt - \frac{a+b}{2} \int_a^b g(t)\,dt. \tag{2.6}$$

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The second representation result is incorporated in

Lemma 2.3. *If $f, g : [a, b] \to \mathbb{R}$ are absolutely continuous on $[a, b]$, then*

$$C(f, g) = \frac{1}{(b-a)^2} \int_a^b \int_a^b K(t, s) f'(t) g'(s) \, dt ds, \tag{2.7}$$

where the kernel $K : [a, b] \to \mathbb{R}$ is defined by

$$K(t, s) := \begin{cases} (b-t)(s-a) & \text{if } a \le s \le t \le b, \\ (t-a)(b-s) & \text{if } a \le t < s \le b. \end{cases} \tag{2.8}$$

Proof. By Fubini's theorem we have

$$\int_a^b \int_a^b K(t, s) f'(t) g'(s) \, dt ds = \int_a^b \left(\int_a^b K(t, s) g'(s) \, ds \right) f'(t) \, dt. \tag{2.9}$$

By the definition of K and integrating by parts, we have successively:

$$\begin{aligned} \int_a^b K(t, s) g'(s) \, ds \quad & \\ &= \int_a^t K(t, s) g'(s) \, ds + \int_t^b K(t, s) g'(s) \, ds \\ &= (b-t) \int_a^t (s-a) g'(s) \, ds + (t-a) \int_t^b (b-s) g'(s) \, ds \\ &= (b-t) \left[(t-a) g(t) - \int_a^t g(s) \, ds \right] \\ &\quad + (t-a) \left[-(b-t) g(t) + \int_t^b g(s) \, ds \right] \end{aligned} \tag{2.10}$$

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$$= (t-a)\int_t^b g(s)\,ds - (b-t)\int_a^t g(s)\,ds,$$

for any $t \in [a,b]$.

Multiplying (2.10) by $f'(t)$ and integrating over $t \in [a,b]$, we have:

$$\begin{aligned}
(2.11)\quad & \int_a^b \left(\int_a^b K(t,s)\,g'(s)\,ds\right) f'(t)\,dt \\
&= \int_a^b \left[(t-a)\int_t^b g(s)\,ds - (b-t)\int_a^t g(s)\,ds\right] f'(t)\,dt \\
&= f(t)\left[(t-a)\int_t^b g(s)\,ds - (b-t)\int_a^t g(s)\,ds\right]\Bigg|_a^b \\
&\qquad - \int_a^b f(t)\left[(t-a)\int_t^b g(s)\,ds - (b-t)\int_a^t g(s)\,ds\right]' dt \\
&= \int_a^b f(t)\left[\int_t^b g(s)\,ds - (t-a)\,g(t) + \int_a^t g(s)\,ds - (b-t)\,g(t)\right] \\
&= -\int_a^b f(t)\left[\int_a^b g(s)\,ds - (b-a)\,g(t)\right] dt \\
&= (b-a)\int_a^b g(t)f(t)dt - \int_a^b f(t)\,dt \cdot \int_a^b g(t)dt \\
&= (b-a)^2\, C(f,g).
\end{aligned}$$

By (2.11) and (2.9) we deduce the desired result. □

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Theorem 2.4. *With the assumptions of Lemma 2.3, we have for any $\nu, \mu \in \mathbb{R}$ that:*

$$C(f,g) = \mu C(e,g) + \nu C(f,e) + \frac{1}{(b-a)^2} \int_a^b \int_a^b K(t,s)\, [f'(t) - \mu]\, [g'(s) - \nu]\, dt ds. \tag{2.12}$$

Proof. Follows by Lemma 2.3 on observing that $C(e,e) = 0$ and

$$C(f - \mu e, g - \nu e) = C(f,g) - \mu C(e,g) - \nu C(f,e)$$

for any $\mu, \nu \in \mathbb{R}$. □

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3. Bounds in Terms of Lebesgue Norms of g and f'

Utilising the representation (2.5) we can state the following result:

Theorem 3.1. *Assume that* $g : [a,b] \to \mathbb{R}$ *is Lebesgue integrable on* $[a,b]$ *and* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$ *, then*

$$
(3.1)\quad |C(f,g) - \mu C(e,g)| \le \begin{cases} \frac{1}{3}(b-a)\|f'-\mu\|_\infty \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty & \text{if } f', g \in L_\infty[a,b]; \\ \frac{2^{1/q}(b-a)^{\frac{p-q}{pq}}}{[(q+1)(q+2)]^{1/q}} \|f'-\mu\|_p \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_p & \text{if } f', g \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ (b-a)^{-1}\|f'-\mu\|_1 \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_1 \end{cases}
$$

for any $\mu \in \mathbb{R}$.

Proof. From (2.5), we have

$$
(3.2)\quad \begin{aligned} |C(f,g) - \mu C(e,g)| &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b |Q(t,s)|\,|g(s)-\lambda|\,|f'(t)-\mu|\,dtds \\ &\le \|g-\lambda\|_\infty \|f'-\mu\|_\infty \frac{1}{(b-a)^2}\int_a^b\int_a^b |Q(t,s)|\,dtds. \end{aligned}
$$

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However, by the definition of Q we have for $\alpha \geq 1$ that

$$\begin{aligned} I(\alpha) &:= \int_a^b \int_a^b |Q(t,s)|^\alpha \, dtds \\ &= \int_a^b \left(\int_a^t |t-b|^\alpha \, ds + \int_t^b |t-a|^\alpha \, ds \right) dt \\ &= \int_a^b \left[(t-a)(b-t)^\alpha + (b-t)(t-a)^\alpha \right] dt. \end{aligned}$$

Since

$$\int_a^b (t-a)(b-t)^\alpha \, dt = \frac{(b-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}$$

and

$$\int_a^b (b-t)(t-a)^\alpha \, dt = \frac{(b-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)},$$

hence

$$I(\alpha) = \frac{2(b-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}, \qquad \alpha \geq 1.$$

Then we have

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b |Q(t,s)| \, dtds = \frac{b-a}{3},$$

and taking the infimum over $\lambda \in \mathbb{R}$ in (3.2), we deduce the first part of (3.1).

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Utilising the Hölder inequality for double integrals we also have

$$\int_a^b \int_a^b |Q(t,s)| \, |g(s) - \lambda| \, |f'(t) - \mu| \, dtds$$
$$\leq \left(\int_a^b \int_a^b |Q(t,s)|^q \, dtds \right)^{\frac{1}{q}} \left(\int_a^b \int_a^b |g(s) - \lambda|^p \, |f'(t) - \mu|^p \, dtds \right)^{\frac{1}{p}}$$
$$= \frac{2^{1/q} (b-a)^{1+\frac{2}{q}}}{[(q+1)(q+2)]^{1/q}} \|g - \lambda\|_p \|f' - \mu\|_p ,$$

which provides, by the first inequality in (3.2), the second part of (3.1).

For the last part, we observe that $\sup_{(t,s)\in[a,b]^2} |Q(t,s)| = b - a$ and then

$$\int_a^b \int_a^b |Q(t,s)| \, |g(s) - \lambda| \, |f'(t) - \mu| \, dtds \leq (b-a) \|g - \lambda\|_1 \|f' - \mu\|_1 .$$

This completes the proof. □

Remark 1. The above inequality (3.1) is a source of various inequalities as will be shown in the following.

1. For instance, if $-\infty < m \leq g(t) \leq M < \infty$ for a.e. $t \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2}(M-m)$ and $\left\| g - \frac{m+M}{2} \right\|_p \leq \frac{1}{2}(M-m)(b-a)^{1/p}$, $p \geq 1$. Then for any $\mu \in \mathbb{R}$ we have

$$(3.3) \quad |C(f,g) - \mu C(e,g)|$$
$$\leq \begin{cases} \frac{1}{6}(b-a)(M-m)\|f' - \mu\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{2^{-1/p}(b-a)^{1/q}}{[(q+1)(q+2)]^{1/q}}(M-m)\|f' - \mu\|_p & \text{if } f' \in L_p[a,b], \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2}(M-m)\|f' - \mu\|_1 , \end{cases}$$

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which gives for $\mu = 0$ that

$$(3.4)\qquad |C(f,g)| \le \begin{cases} \frac{1}{6}(b-a)(M-m)\|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{2^{-1/p}(b-a)^{1/q}}{[(q+1)(q+2)]^{1/q}}(M-m)\|f'\|_p & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(M-m)\|f'\|_1. \end{cases}$$

2. If $-\infty < \gamma \le f'(t) \le \Gamma < \infty$ for a.e. $t \in [a,b]$, then $\left\|f' - \frac{\gamma+\Gamma}{2}\right\|_\infty \le \frac{1}{2}|\Gamma-\gamma|$ and $\left\|f' - \frac{\gamma+\Gamma}{2}\right\|_p \le \frac{1}{2}|\Gamma-\gamma|(b-a)^{1/p}$, $p \ge 1$. Then we have from (3.1) that

$$(3.5)\qquad \left|C(f,g) - \frac{\gamma+\Gamma}{2}C(e,g)\right| \le \begin{cases} \frac{1}{6}(b-a)(\Gamma-\gamma)\inf_{\xi\in\mathbb{R}}\|g-\xi\|_\infty & \text{if } g \in L_\infty[a,b]; \\ \frac{2^{-1/p}(b-a)^{1/q}}{[(q+1)(q+2)]^{1/q}}(\Gamma-\gamma)\inf_{\xi\in\mathbb{R}}\|g-\xi\|_p & \text{if } g \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(\Gamma-\gamma)\inf_{\xi\in\mathbb{R}}\|g-\xi\|_1. \end{cases}$$

Moreover, if we also assume that $-\infty < m \le g(t) \le M < \infty$ for a.e. $t \in [a,b]$, then by (3.5) we also deduce:

$$(3.6)\qquad \left|C(f,g) - \frac{\gamma+\Gamma}{2}C(e,g)\right|$$

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$$\leq \begin{cases} \frac{1}{12}(b-a)(\Gamma-\gamma)(M-m) \\ \frac{2^{1-1/p}(b-a)}{[(q+1)(q+2)]^{1/q}}(\Gamma-\gamma)(M-m) & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}(\Gamma-\gamma)(M-m)(b-a). \end{cases}$$

Observe that the first inequality in (3.6) is better than the others.

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4. Bounds in Terms of Lebesgue Norms of f' and g'

We have the following result:

Theorem 4.1. *Assume that $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$, then*

$$|C(f,g)-\mu C(e,g)-\nu C(f,e)| \tag{4.1}$$

$$\leq\begin{cases}\frac{1}{12}(b-a)^2\|f'-\mu\|_\infty\|g'-\nu\|_\infty & \text{if } f',g'\in L_\infty[a,b];\\ \left[\frac{B(q+1,q+1)}{q+1}\right]^{\frac{1}{q}}(b-a)^{2/q}\|f'-\mu\|_p\|g'-\nu\|_p & \text{if } f',g'\in L_p[a,b],\\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \frac{1}{4}\|f'-\mu\|_1\|g'-\nu\|_1;\end{cases}$$

for any $\mu,\nu\in\mathbb{R}$.

Proof. From (2.12), we have

$$|C(f,g)-\mu C(e,g)-\nu C(f,e)| \leq \frac{1}{(b-a)^2}\int_a^b\int_a^b|K(t,s)||f'(t)-\mu||g'(s)-\nu|\,dtds. \tag{4.2}$$

Define

$$\begin{aligned} J(\alpha) &:= \int_a^b\int_a^b|K(t,s)|^\alpha\,dtds \\ &= \int_a^b\left[\int_a^t(b-t)^\alpha(s-a)^\alpha\,ds+\int_t^b(t-a)^\alpha(b-s)^\alpha\,ds\right]dt \\ &= \frac{1}{\alpha+1}\left[\int_a^b(b-t)^\alpha(t-a)^{\alpha+1}\,dt+\int_a^b(t-a)^\alpha(b-t)^{\alpha+1}\,dt\right]. \end{aligned} \tag{4.3}$$

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Since

$$\int_a^b (t-a)^p (b-t)^q \, dt = (b-a)^{p+q+1} \int_0^1 s^p (1-s)^q \, ds$$
$$= (b-a)^{p+q+1} B(p+1, q+1),$$

hence, by (4.3),

$$J(\alpha) = \frac{2(b-a)^{2\alpha+2}}{\alpha+1} B(\alpha+1, \alpha+2), \qquad \alpha \geq 1.$$

As it is well known that

$$B(p, q+1) = \frac{q}{p+q} B(p,q),$$

then for $p = \alpha + 1$, $q = \alpha + 1$ we have $B(\alpha+1, \alpha+2) = \frac{1}{2} B(\alpha+1, \alpha+1)$.

Then we have

$$J(\alpha) = \frac{(b-a)^{2\alpha+2}}{\alpha+1} B(\alpha+1, \alpha+1), \qquad \alpha \geq 1.$$

Taking into account that

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b |K(t,s)| \, |f'(t) - \mu| \, |g'(s) - \nu| \, dtds$$
$$\leq \|f' - \mu\|_\infty \|g' - \nu\|_\infty \frac{1}{(b-a)^2} \int_a^b \int_a^b |K(t,s)| \, dtds$$
$$= \|f' - \mu\|_\infty \|g' - \nu\|_\infty (b-a)^2 B(2,3)$$
$$= \frac{1}{12} (b-a)^2 \|f' - \mu\|_\infty \|g' - \nu\|_\infty,$$

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we deduce from (4.2) the first part of (4.1).

By the Hölder integral inequality for double integrals, we have

$$\begin{aligned}(4.4)\qquad &\int_a^b\int_a^b |K(t,s)|\,|f'(t)-\mu|\,|g'(s)-\nu|\,dtds\\ &\le\left(\int_a^b\int_a^b |K(t,s)|^q\,dtds\right)^{\frac{1}{q}}\|f'-\mu\|_p\,\|g'-\nu\|_p\\ &=\left[\frac{(b-a)^{2q+2}}{q+1}B(q+1,q+2)\right]^{\frac{1}{q}}\|f'-\mu\|_p\,\|g'-\nu\|_p\\ &=(b-a)^{2+2/q}\left[\frac{B(q+1,q+1)}{q+1}\right]^{\frac{1}{q}}\|f'-\mu\|_p\,\|g'-\nu\|_p\,.\end{aligned}$$

Utilising (4.2) and (4.4) we deduce the second part of (4.1).

By the definition of $K(t,s)$ we have, for $a\le s\le t\le b$, that

$$K(t,s)=(b-t)(s-a)\le(b-t)(t-a)\le\frac{1}{4}(b-a)^2$$

and for $a\le t<s\le b$, that

$$K(t,s)=(t-a)(b-s)\le(t-a)(b-t)\le\frac{1}{4}(b-a)^2,$$

therefore

$$\sup_{(t,s)\in[a,b]}|K(t,s)|=\frac{1}{4}(b-a)^2.$$

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Due to the fact that

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b |K(t,s)|\,|f'(t)-\mu|\,|g'(s)-\nu|\,dtds$$
$$\leq \sup_{(t,s)\in[a,b]} |K(t,s)|\,\frac{1}{(b-a)^2}\int_a^b\int_a^b |f'(t)-\mu|\,|g'(s)-\nu|\,dtds$$
$$= \frac{1}{4}\,\|f'-\mu\|_1\,\|g'-\nu\|_1\,,$$

then from (4.2) we obtain the last part of (4.1). □

Remark 2. When $\mu=\nu=0$, we obtain from (4.1) the following Grüss type inequalities:

$$(4.5)\quad |C(f,g)| \leq \begin{cases} \frac{1}{12}(b-a)^2\,\|f'\|_\infty\,\|g'\|_\infty & \text{if } f',g'\in L_\infty[a,b]\,; \\ \left[\frac{B(q+1,q+1)}{q+1}\right]^{\frac{1}{q}}(b-a)^{2/q}\,\|f'\|_p\,\|g'\|_p & \text{if } f',g'\in L_p[a,b]\,, \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{4}\,\|f'\|_1\,\|g'\|_1\,. \end{cases}$$

Notice that the first inequality in (4.5) is exactly the Čebyšev inequality for which $\frac{1}{12}$ is the best possible constant.

If we assume that there exists γ,Γ,ϕ,Φ such that $-\infty<\gamma\leq f'(t)\leq\Gamma<\infty$ and $-\infty<\phi\leq g'(t)\leq\Phi<\infty$ for a.e. $t\in[a,b]$, then we deduce from (4.1) the

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following inequality

$$\left|C(f,g)-\frac{\gamma+\Gamma}{2}\cdot C(e,g)-\frac{\phi+\Phi}{2}\cdot C(f,e)\right| \leq \frac{1}{48}(b-a)^2(\Gamma-\gamma)(\Phi-\phi)\,. \tag{4.6}$$

We also observe that the constant $\frac{1}{48}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

The sharpness of the constant follows by the fact that for $\Gamma=-\gamma$, $\Phi=-\phi$ we deduce from (4.6) the Čebyšev inequality which is sharp.

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