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## A NOTE ON CERTAIN INEQUALITIES FOR THE GAMMA FUNCTION

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Abstract

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## Abstract

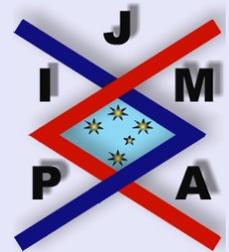
We obtain a new proof of a generalization of a double inequality on the Euler gamma function, obtained by C. Alsina and M. S. Tomás [1].

*2000 Mathematics Subject Classification:* 33B15.

*Key words:* Euler gamma function, Digamma function.

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# 1. Introduction

The Euler Gamma function  $\Gamma$  is defined for  $x > 0$  by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

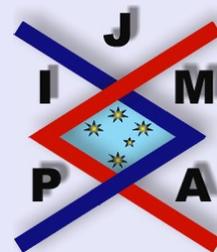
By using a geometrical method, recently C. Alsina and M. S. Tomás [1] have proved the following double inequality:

**Theorem 1.1.** *For all  $x \in [0, 1]$ , and all nonnegative integers  $n$  one has*

$$(1.1) \quad \frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$

While the interesting method of [1] is geometrical, we will show in what follows that, by certain simple analytical arguments it can be proved that (1.1) holds true for all real numbers  $n$ , and all  $x \in [0, 1]$ . In fact, this will be a consequence of a monotonicity property.

Let  $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$  ( $x > 0$ ) be the "digamma function". For properties of this function, as well as inequalities, or representation theorems, see e.g. [2], [4], [5], [7]. See also [3] and [6] for a survey of results on the gamma and related functions.



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## 2. Main Results

Our method is based on the following auxiliary result:

**Lemma 2.1.** For all  $x > 0$  one has the series representation

$$(2.1) \quad \psi(x) = -\gamma + (x-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(x+k)}.$$

This is well-known. For proofs, see e.g. [4], [7].

**Lemma 2.2.** For all  $x > 0$ , and all  $a \geq 1$  one has

$$(2.2) \quad \psi(1+ax) \geq \psi(1+x).$$

*Proof.* By (2.1) we can write  $\psi(1+ax) \geq \psi(1+x)$  iff

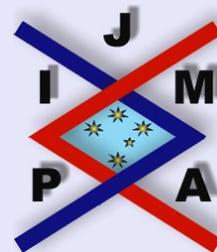
$$-\gamma + ax \sum_{k=0}^{\infty} \frac{1}{(k+1)(1+ax+k)} \geq -\gamma + x \sum_{k=0}^{\infty} \frac{1}{(k+1)(1+x+k)}.$$

Now, remark that

$$\frac{a}{(k+1)(1+ax+k)} - \frac{1}{(k+1)(1+x+k)} = \frac{a-1}{(1+x+k)(1+ax+k)} \geq 0$$

by  $a \geq 1$ ,  $x > 0$ ,  $k \geq 0$ . Thus inequality (2.2) is proved. There is equality only for  $a = 1$ .  $\square$

We notice that (2.2) trivially holds true for  $x = 0$  for all  $a$ .



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**Theorem 2.3.** For all  $a \geq 1$ , the function

$$f(x) = \frac{\Gamma(1+x)^a}{\Gamma(1+ax)}$$

is a decreasing function of  $x \geq 0$ .

*Proof.* Let

$$g(x) = \log f(x) = a \log \Gamma(1+x) - \log \Gamma(1+ax).$$

Since

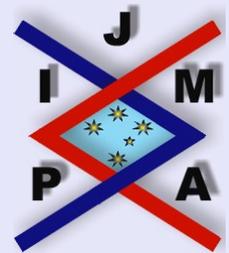
$$g'(x) = a[\psi(1+x) - \psi(1+ax)],$$

by Lemma 2.2 we get  $g'(x) \leq 0$ , so  $g$  is decreasing. This implies the required monotonicity of  $f$ .  $\square$

**Corollary 2.4.** For all  $a \geq 1$  and all  $x \in [0, 1]$  one has

$$(2.3) \quad \frac{1}{\Gamma(1+a)} \leq \frac{\Gamma(x+1)^a}{\Gamma(ax+1)} \leq 1.$$

*Proof.* For  $x \in (0, 1]$ , by Theorem 2.3,  $f(1) \leq f(x) \leq f(0)$ , which by  $\Gamma(1) = \Gamma(2) = 1$  implies (2.3). For  $a = n \geq 1$  integer, this yields relation (1.1).  $\square$



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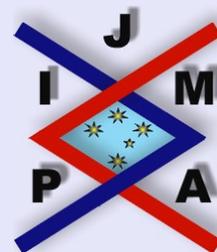
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