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# ZERO AND COEFFICIENT INEQUALITIES FOR HYPERBOLIC POLYNOMIALS

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ABSTRACT. In this paper using classical inequalities and Cardan-Viète formulae some inequalities involving zeroes and coefficients of hyperbolic polynomials are given. Furthermore, considering real polynomials whose zeros lie in $\mathrm{Re}(z) > 0$, the previous results have been extended and new inequalities are obtained.

*Key words and phrases:* Zeroes and coefficients, Inequalities in the complex plane, Inequalities for polynomials with real zeros, Hyperbolic polynomials.

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## 1. INTRODUCTION

The problem of finding relations between the zeroes and coefficients of a polynomial occupies a central role in the theory of equations. The most well known of such relations are Cardan-Viète's formulae [1]. Many papers devoted to obtaining inequalities between the zeros and coefficient have been written giving new bounds or improving the classical known ones ([2], [3], [4]). Furthermore, inequalities for polynomials with all zeros real also called hyperbolic polynomials, have been fully documented in [5]. In this paper, using some classical inequalities, several inequalities involving zeros and coefficients of polynomials with real zeros have been obtained and the main result has been extended to polynomials whose zeros lie in the right half plane.

## 2. THE INEQUALITIES

In what follows some zero and coefficient inequalities involving polynomials whose zeros are strictly positive real numbers are obtained. We begin with

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**Theorem 2.1.** *Let $A(x) = \sum_{k=0}^{n} a_k x^k$, $a_n \neq 0$, be a hyperbolic polynomial with all its zeroes $x_1, x_2, \ldots, x_n$ strictly positive. If $\alpha, p$ and $b$ are strictly positive real numbers such that $\alpha < p$, then*

$$\sum_{k=1}^{n} \frac{1}{[x_k^p + b]^{\frac{1}{\alpha}}} \leq \frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}} \left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}} \left|\frac{a_1}{a_0}\right|. \tag{2.1}$$

*Equality holds when* $A(x) = a_n \left(x - \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}\right)^n$.

*Proof.* Let $\beta$ and $a$ be strictly positive real numbers defined by $\beta = 1 - \frac{\alpha}{p} > 0$ and $a = \frac{b\alpha}{p\beta} > 0$. Taking into account that $\frac{\alpha}{p} + \beta = 1$ and applying the powered AM-GM inequality, we have for all $k$, $1 \leq k \leq n$,

$$(x_k^p)^{\frac{\alpha}{p}} a^{\beta} \leq \frac{\alpha}{p} x_k^p + \beta a. \tag{2.2}$$

Inverting the terms in (2.2) yields

$$\frac{1}{\frac{\alpha}{p} x_k^p + \beta a} \leq \frac{1}{x_k^{\alpha} a^{\beta}}, \quad 1 \leq k \leq n,$$

or equivalently

$$\frac{1}{x_k^p + \frac{p}{\alpha}\beta a} \leq \frac{\alpha}{p} \cdot \frac{1}{x_k^{\alpha} a^{\beta}}.$$

Taking into account that $\frac{p}{\alpha}\beta a = b$ and $\beta = \frac{p-\alpha}{p}$, we have

$$\begin{aligned}
\frac{1}{x_k^p + b} &\leq \frac{\alpha}{p} \cdot \frac{1}{\left(\frac{b\alpha}{p\beta}\right)^{\beta}} \frac{1}{x_k^{\alpha}} \\
&= \frac{\alpha}{p} \cdot \frac{p^{\beta}\beta^{\beta}}{b^{\beta}\alpha^{\beta}} \frac{1}{x_k^{\alpha}} \\
&= \frac{\alpha}{p} \cdot \frac{p^{1-\frac{\alpha}{p}} \left(\frac{p-\alpha}{p}\right)^{1-\frac{\alpha}{p}}}{b^{1-\frac{\alpha}{p}} \alpha^{1-\frac{\alpha}{p}}} \cdot \frac{1}{x_k^{\alpha}} \\
&= \frac{\alpha^{\frac{\alpha}{p}}}{p} \left(\frac{p-\alpha}{b}\right)^{1-\frac{\alpha}{p}} \frac{1}{x_k^{\alpha}}.
\end{aligned}$$

Raising to $\frac{1}{\alpha}$ both sides of the preceding inequality, yields

$$\frac{1}{[x_k^p + b]^{\frac{1}{\alpha}}} \leq \frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}} \left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}} \frac{1}{x_k}, \qquad 1 \leq k \leq n.$$

Finally, adding up the preceding inequalities, we obtain

$$\sum_{k=1}^{n} \frac{1}{[x_k^p + b]^{\frac{1}{\alpha}}} \leq \frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}} \left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}} \sum_{k=1}^{n} \frac{1}{x_k} = \frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}} \left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}} \left|\frac{a_1}{a_0}\right|$$

and (2.1) is proved.

Notice that equality holds in (2.1) if and only if equality holds in (2.2) for $1 \leq k \leq n$. Namely, equality holds when $x_k^p = a$, $1 \leq k \leq n$ or $x_k = a^{\frac{1}{p}} = \left(\frac{b\alpha}{p\beta}\right)^{\frac{1}{p}} = \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}$. That is, when $A(x) = a_n \left(x - \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}\right)^n$, $a_n \neq 0$. □

When $\alpha > p$ changing $\alpha$ by $\frac{1}{\alpha}$ and $p$ by $\frac{1}{p}$ into (2.1), we have the following:

**Corollary 2.2.** *If $\alpha, p$ and $b$ are strictly positive real numbers such that $\alpha > p$, then*

$$\sum_{k=1}^{n} \frac{1}{[x_k^{\frac{1}{p}} + b]^{\alpha}} \leq \frac{p^p}{\alpha^\alpha} \left(\frac{\alpha - p}{b}\right)^{\alpha - p} \left|\frac{a_1}{a_0}\right|.$$

Multiplying both sides of (2.2) by $\frac{p}{\alpha}$ and raising to $\frac{1}{\alpha}$, we obtain for $1 \leq k \leq n$,

$$\left(x_k^p + \frac{p}{\alpha}\beta a\right)^{\frac{1}{\alpha}} \geq \left(\frac{p}{\alpha}\right)^{\frac{1}{\alpha}} a^{\frac{\beta}{\alpha}} x_k.$$

Setting $\beta = 1 - \frac{\alpha}{p}$, $a = \frac{b\alpha}{p\beta}$ into the preceding expression and, after adding up the resulting inequalities, we get

**Corollary 2.3.** *If $\alpha, p$ and $b$ are strictly positive real numbers such that $\alpha < p$, then*

$$\sum_{k=1}^{n} (x_k^p + b)^{\frac{1}{\alpha}} \geq \frac{p^{\frac{1}{\alpha}}}{\alpha^{\frac{1}{p}}} \left(\frac{b}{p - \alpha}\right)^{\frac{1}{\alpha} - \frac{1}{p}} \left|\frac{a_{n-1}}{a_n}\right|$$

*holds.*

Another, immediate consequence of (2.1) is the following.

**Corollary 2.4.** *Let $A(x) = \sum_{k=0}^n a_k x^k$, $a_n \neq 0$, be a hyperbolic polynomial with all its zeroes $x_1, x_2, \dots, x_n$ strictly positive. Then,*

$$\sum_{k=1}^{n} \frac{1}{[x_k^n + 2n - 1]^2} \leq \frac{1}{4n^2} \left|\frac{a_1}{a_0}\right|$$

*holds.*

*Proof.* Setting $\alpha = \frac{1}{2}$, $p = n$ and $b = 2n - 1$ into (2.1), we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{[x_k^n + 2n - 1]^2} &\leq \frac{\left(\frac{1}{2}\right)^{\frac{1}{n}}}{n^2} \left(\frac{n - \frac{1}{2}}{2n - 1}\right)^{2 - \frac{1}{n}} \left|\frac{a_1}{a_0}\right| \\
&= \frac{\left(\frac{1}{2}\right)^{\frac{1}{n}}}{n^2} \left(\frac{1}{2}\right)^{2 - \frac{1}{n}} \left|\frac{a_1}{a_0}\right| \\
&= \frac{1}{4n^2} \left|\frac{a_1}{a_0}\right|.
\end{aligned}$$

Note that equality holds when $x_k = 1,\ 1 \leq k \leq n$. That is, when $A(x) = a_n(x-1)^n$. This completes the proof. $\square$

Considering the reverse polynomial $A^*(x) = x^n \overline{A(1/\overline{x})} = \sum_{k=0}^n a_{n-k} x^k$, we have the following

**Theorem 2.5.** *If $\alpha, p$ and $b$ are strictly positive real numbers such that $\alpha < p$, then*

$$\sum_{k=1}^{n} \left(\frac{x_k^p}{x_k^p + b}\right)^{\frac{1}{\alpha}} \leq \frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}} b^{\frac{1}{p}}} \cdot (p - \alpha)^{\frac{1}{\alpha} - \frac{1}{p}} \left|\frac{a_{n-1}}{a_n}\right|. \tag{2.3}$$

*Equality holds when $A(x) = a_n \left(x - \left(\frac{b(p-\alpha)}{\alpha}\right)^{\frac{1}{p}}\right)^n,\ a_n \neq 0.$*

*Proof.* Since $A^*(x)$ has zeros $\frac{1}{x_1},\dots,\frac{1}{x_n}$, then applying (2.1) to it, we get

$$\sum_{k=1}^{n}\frac{1}{\left[\left(\frac{1}{x_k}\right)^p+b\right]^{\frac{1}{\alpha}}}\le\frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}}\cdot\left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}}\left|\frac{a_{n-1}}{a_n}\right|.$$

Developing the LHS of the preceding inequality, we have

$$\frac{1}{b^{\frac{1}{\alpha}}}\sum_{k=1}^{n}\left(\frac{x_k^p}{\frac{1}{b}+x_k^p}\right)^{\frac{1}{\alpha}}\le\frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}}\cdot\left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}}\left|\frac{a_{n-1}}{a_n}\right|,$$

and rearranging terms, yields

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{x_k^p}{\frac{1}{b}+x_k^p}\right)^{\frac{1}{\alpha}}&\le b^{\frac{1}{\alpha}}\cdot\frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}}\cdot\left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}}\left|\frac{a_{n-1}}{a_n}\right|\\&=\frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}}\cdot b^{\frac{1}{p}}\cdot(p-\alpha)^{\frac{1}{\alpha}-\frac{1}{p}}\left|\frac{a_{n-1}}{a_n}\right|.\end{aligned}$$

Finally, replacing $b$ by $1/b$ in the preceding inequality we get (2.3) as claimed.

Applying Theorem 2.1, equality in (2.3) holds when $A^*(x)=a_n\left(x-\left(\frac{\alpha}{b(p-\alpha)}\right)^{\frac{1}{p}}\right)^n$. Taking into account that we have changed $b$ by $1/b$, equality will hold if and only if $A(x)=a_n\left(x-\left(\frac{b(p-\alpha)}{\alpha}\right)^{\frac{1}{p}}\right)^n$, $a_n\neq 0$ and the proof is completed. ☐

Next, we state and prove the following:

**Theorem 2.6.** *Let $A(x)$ be a hyperbolic polynomial with zeros $x_1,x_2,\dots,x_n$ such that $x_1\le x_2\le\dots\le x_n$. Let $\alpha,p$ and $b$ be strictly positive real numbers such that $\alpha<p$. If $a<x_1$ or $a>x_n$, then*

$$\sum_{k=1}^{n}\frac{1}{\left[|x_k-a|^p+b\right]^{\frac{1}{\alpha}}}\le\frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}}\left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}}\left|\frac{P'(a)}{P(a)}\right|.\tag{2.4}$$

*Equality holds when*

$$A(x)=a_n\left(x-\left[a+\left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}\right]\right)^n\quad\textit{or}$$
$$A(x)=a_n\left(x-\left[a-\left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}\right]\right)^n.$$

*Proof.* First, we observe that (2.1) applied to polynomial $P(-t)$ where $P(t)$ has all its zeros $t_1,t_2,\dots,t_n$ negative, yields

$$\sum_{k=1}^{n}\frac{1}{\left[|t_k|^p+b\right]^{\frac{1}{\alpha}}}\le\frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}}\cdot\left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}}\left|\frac{a_1}{a_0}\right|,\tag{2.5}$$

where equality holds when $P(t)=a_n\left(t+\left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}\right)^n$, $a_n\neq 0$.

Now, we consider the hyperbolic polynomial of the statement and assume that (i) $a<x_1$ or (ii) $a>x_n$. Let $B(x)=A(x+a)$, the zeros of which are $x_1-a,x_2-a,\dots,x_n-a$. Observe that, they are positive for case (i), and negative for case (ii). On the other hand, coefficients $a_0$

and $a_1$ of $B(x)$ are $B(0) = A(a)$ and $B'(0) = A'(a)$ respectively. Applying (2.1) to $B(x)$ in case (i) or (2.5) in case (ii) we get (2.4).

Finally, we see that equality in (2.4) holds in the case (i) when $B(x) = a_n \left(x - \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}\right)^n$, or equivalently when $A(x) = a_n \left(x - \left[a + \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}\right]\right)^n$. In case (ii) we will get equality when $B(x) = a_n \left(x + \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}\right)^n$, $a_n \neq 0$. That is, when $A(x) = a_n \left(x - \left[a - \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}\right]\right)^n$ and the proof is completed. $\square$

Finally, in the sequel we will extend the result obtained in Theorem 2.1 to real polynomials whose zeros lie in the half plane $\mathrm{Re}(z) > 0$ and they have an imaginary part “sufficiently small”. This is stated and proved in the following.

**Theorem 2.7.** *Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial with real coefficients whose zeros $z_1, z_2, \dots, z_n$ lie in $\mathrm{Re}(z) > 0$ and suppose that $|\mathrm{Im}(z)| \leq r\,\mathrm{Re}(z_k)$, $1 \leq k \leq n$ for some real $r \geq 0$. Let $\alpha, p$ and $b$ be strictly positive real numbers such that $\alpha < p$, then*

$$\sum_{k=1}^{n} \frac{1}{[|z_k|^p + b]^{\frac{1}{\alpha}}} \leq \frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}} \cdot \left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}} \cdot \sqrt{1+r^2}\left|\frac{a_1}{a_0}\right|. \tag{2.6}$$

*For $r > 0$, equality holds when $n$ is even and*

$$A(z) = \left(z^2 - \frac{2}{\sqrt{1+r^2}} \cdot \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}} z + \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{2}{p}}\right)^{\frac{n}{2}}.$$

Note that when $r = 0$ the preceding result reduces to (2.1).

*Proof.* Setting $x_k = |z_k|$ and repeating the procedure followed in proving (2.1), we get

$$\sum_{k=1}^{n} \frac{1}{[|z_k|^p + b]^{\frac{1}{\alpha}}} \leq \frac{\alpha^{\frac{1}{p}}}{p^{\frac{1}{\alpha}}} \cdot \left(\frac{p-\alpha}{b}\right)^{\frac{1}{\alpha}-\frac{1}{p}} \sum_{k=1}^{n} \frac{1}{|z_k|}.$$

Next, we will find an upper bound for the sum $S = \sum_{k=1}^{n} \frac{1}{|z_k|}$. Reordering the zeros of $A(z)$ in the way $z_1, \overline{z_1}, z_2, \overline{z_2}, \dots, z_s, \overline{z_s}, x_1, \dots, x_t$,where $x_1, \dots, x_t$ are the real zeros (if any), then the preceding sum becomes

$$S = 2\sum_{k=1}^{s} \frac{1}{|z_k|} + \sum_{k=1}^{t} \frac{1}{|x_k|} = 2\sum_{k=1}^{s} \frac{|z_k|}{|z_k|^2} + \sum_{k=1}^{t} \frac{1}{|x_k|}.$$

On the other hand, by Cardan-Viète formulae, we have

$$-\frac{a_1}{a_0} = \sum_{k=1}^{s} \left[\frac{1}{z_k} + \frac{1}{\overline{z_k}}\right] + \sum_{k=1}^{t} \frac{1}{x_k} = 2\sum_{k=1}^{s} \frac{\mathrm{Re}\, z_k}{|z_k|^2} + \sum_{k=1}^{t} \frac{1}{x_k}.$$

Taking into account that $|z_k| = \sqrt{(\operatorname{Re} z_k)^2 + (\operatorname{Im} z_k)^2} \le \sqrt{1+r^2}\,|\operatorname{Re} z_k|$ and the fact that the zeros of $A(z)$ lie in $\operatorname{Re}(z) > 0$, yields

$$\begin{aligned} S &= 2\sum_{k=1}^{s} \frac{|z_k|}{|z_k|^2} + \sum_{k=1}^{t} \frac{1}{|x_k|} \\ &\le 2\sqrt{1+r^2} \sum_{k=1}^{s} \frac{|\operatorname{Re} z_k|}{|z_k|^2} + \sum_{k=1}^{t} \frac{1}{|x_k|} \\ &\le \sqrt{1+r^2}\left(2\sum_{k=1}^{s} \frac{|\operatorname{Re} z_k|}{|z_k|^2} + \sum_{k=1}^{t} \frac{1}{|x_k|}\right) \\ &= \sqrt{1+r^2}\left|\frac{a_1}{a_0}\right|, \end{aligned} \tag{2.7}$$

from which (2.6) immediately follows.

Next, we will see when equality holds in (2.6). If $r > 0$, to get equality in (2.6) we require that (i) all the zeros of $A(z)$ have modulus $|z_k| = \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}$, because when $x_k = |z_k|$ the powered GM-AM inequality (2.2) must become equality, (ii) $|\operatorname{Im} z_k| = r \operatorname{Re} z_k$, $1 \le k \le s$, due to the fact that the inequality in (2.7) must become equality, and (iii) all the zeros of $A(z)$ must be complex because the second inequality in (2.7) also must be an equality. Now it is easy to see that the previous conditions are equivalent to say that $n$ is even and

$$z_k = \frac{1}{\sqrt{1+r^2}}\left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}}[1+ri], \quad 1 \le k \le \frac{n}{2}.$$

Multiplying the preceding zeros we get that inequality in (2.6) holds when $n$ is even and

$$A(z) = \left(z^2 - \frac{2}{\sqrt{1+r^2}}\left(\frac{b\alpha}{p-\alpha}\right)^{\frac{1}{p}} z + \left(\frac{b\alpha}{p-\alpha}\right)^{\frac{2}{p}}\right)^{\frac{n}{2}}.$$

This completes the proof. □

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