

# Journal of Inequalities in Pure and Applied Mathematics

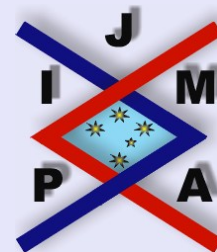
## A REFINEMENT OF AN INEQUALITY FROM INFORMATION THEORY

GARRY T. HALLIWELL AND PETER R. MERCER

Department of Mathematics,  
SUNY College at Buffalo,  
NY 14222, USA.

*EMail:* [hallgt31@mail.buffalostate.edu](mailto:hallgt31@mail.buffalostate.edu)

*EMail:* [mercerpr@math.buffalostate.edu](mailto:mercerpr@math.buffalostate.edu)



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## Abstract

We discuss a refinement of an inequality from Information Theory using other well known inequalities. Then we consider relationships between the logarithmic mean and inequalities of the geometric-arithmetic means.

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# 1. Results

The following inequality is well known in Information Theory [1], see also [4].

**Proposition 1.1.** *Let  $p_i, g_i > 0$ , where  $1 \leq i \leq n$  and  $\sum_{i=1}^n p_i = \sum_{i=1}^n g_i$ . Then  $0 \leq \sum_{i=1}^n p_i \ln(p_i/g_i)$  with equality iff  $p_i = g_i$ , for all  $i$ .*

The following improves this inequality. Indeed, the lower bound is sharpened, an upper bound is provided, and the equality condition is built right in.

**Proposition 1.2.** *Let  $p_i, g_i > 0$ , where  $1 \leq i \leq n$  and  $\sum_{i=1}^n p_i = \sum_{i=1}^n g_i$ . Then the following estimates hold.*

$$\sum_{i=1}^n \frac{g_i(g_i - p_i)^2}{(g_i)^2 + (\max(g_i, p_i))^2} \leq \sum_{i=1}^n p_i \ln \left( \frac{p_i}{g_i} \right) \leq \sum_{i=1}^n \frac{g_i(g_i - p_i)^2}{(g_i)^2 + (\min(g_i, p_i))^2}.$$

*Proof.* We begin with the inequality [6]

$$(1.1) \quad \frac{1}{x^2 + 1} \leq \frac{\ln(x)}{x^2 - 1} \leq \frac{1}{2x}, \text{ for } x > 0.$$

Thus

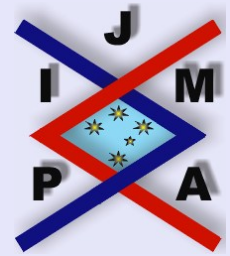
$$\frac{x^2 - 1}{2x} \leq \ln(x) \leq \frac{x^2 - 1}{x^2 + 1} \quad \text{for } 0 < x \leq 1,$$

and

$$\frac{x^2 - 1}{x^2 + 1} \leq \ln(x) \leq \frac{x^2 - 1}{2x} \quad \text{for } 1 < x.$$

Equalities occur only for  $x = 1$ . We rewrite these as

$$(1.2) \quad x - 1 - \frac{(x - 1)^2}{2x} \leq \ln(x) \leq x - 1 - \frac{x(x - 1)^2}{x^2 + 1} \quad \text{for } 0 < x \leq 1,$$



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and

$$(1.3) \quad x - 1 - \frac{x(x-1)^2}{x^2+1} \leq \ln(x) \leq x - 1 - \frac{(x-1)^2}{2x} \quad \text{for } 1 < x.$$

Now, substituting  $g_i/p_i$  for  $x$  in (1.2) and (1.3), and then summing we obtain

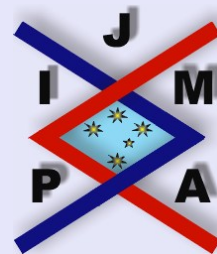
$$\begin{aligned} \sum_{g_i \leq p_i} g_i - \sum_{g_i \leq p_i} p_i - \sum_{g_i \leq p_i} \frac{g_i(g_i - p_i)^2}{(g_i)^2 + (g_i)^2} &\leq \sum_{g_i \leq p_i} p_i \ln\left(\frac{g_i}{p_i}\right) \\ &\leq \sum_{g_i \leq p_i} g_i - \sum_{g_i \leq p_i} p_i - \sum_{g_i \leq p_i} \frac{g_i(g_i - p_i)^2}{(g_i)^2 + (p_i)^2} \end{aligned}$$

and

$$\begin{aligned} \sum_{g_i > p_i} g_i - \sum_{g_i > p_i} p_i - \sum_{g_i > p_i} \frac{g_i(g_i - p_i)^2}{(g_i)^2 + (p_i)^2} &\leq \sum_{g_i > p_i} p_i \ln\left(\frac{g_i}{p_i}\right) \\ &\leq \sum_{g_i > p_i} g_i - \sum_{g_i > p_i} p_i - \sum_{g_i > p_i} \frac{g_i(g_i - p_i)^2}{(g_i)^2 + (g_i)^2} \end{aligned}$$

respectively.

Taking these together and using  $\sum_{i=1}^n p_i = \sum_{i=1}^n g_i$  we have our proposition.  $\square$



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## 2. Remarks

**Remark 2.1.** With  $G = \sqrt{xy}$ ,  $L = (x - y)/(\ln(x) - \ln(y))$ , and  $A = (x + y)/2$ , being the Geometric, Logarithmic, and Arithmetic Means of  $x, y > 0$  respectively, the inequality  $G \leq L \leq A$  is well known [8], [2]. This can be proved by observing (c.f. [5]) that

$$L = \int_0^1 x^t y^{1-t} dt,$$

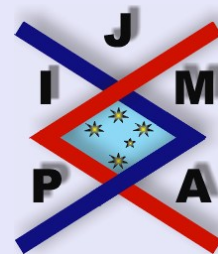
and then applying the following:

**Theorem 2.1 (Hadamard's Inequality).** If  $f$  is a convex function on  $[a, b]$ , then

$$(b - a)f\left(\frac{a + b}{2}\right) \leq \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}(b - a)$$

with the inequalities being strict when  $f$  is not constant.

The inequality in (1.1) now can be obtained by letting  $y = 1/x$  in  $G \leq L \leq A$ . Thus any refinement of  $G \leq L \leq A$  would lead to an improved version of (1.1) and, in principle, to an improvement of Proposition 1.2. For example, it is also known that  $G \leq G^{\frac{2}{3}} A^{\frac{1}{3}} \leq L \leq \frac{2}{3}G + \frac{1}{3}A \leq A$  [3], [8], [2]. The latter can be proved simply by observing that the left side of Hadamard's Inequality is the midpoint approximation  $M$  to  $L$  and the right side is the trapezoid approximation  $T$ . Now  $\frac{2}{3}M + \frac{1}{3}T$  is Simpson's rule and looking at the error term there (e.g. [7]) yields  $L \leq \frac{2}{3}G + \frac{1}{3}A \leq A$ .



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**Remark 2.2.** Using  $G \leq G^{\frac{2}{3}}A^{\frac{1}{3}} \leq L \leq \frac{2}{3}G + \frac{1}{3}A \leq A$ , with  $y = x + 1$  we get

$$\begin{aligned} \sqrt{x(x+1)} &\leq (\sqrt{x(x+1)})^{\frac{2}{3}} \left(\frac{2x+1}{2}\right)^{\frac{1}{3}} \\ &\leq \frac{1}{\ln(1+\frac{1}{x})} \leq \frac{2}{3}\sqrt{x(x+1)} + \frac{1}{3}\frac{2x+1}{2} \leq \frac{2x+1}{2}. \end{aligned}$$

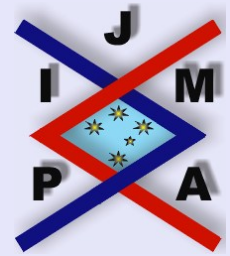
Therefore

$$\left(1 + \frac{1}{x}\right)^{\frac{2}{3}\sqrt{x(x+1)} + \frac{1}{3}\frac{2x+1}{2}} < e < \left(1 + \frac{1}{x}\right)^{(\sqrt{x(x+1)})^{2/3}(\frac{2x+1}{2})^{1/3}}$$

(c.f. [4]). For example  $x = 100$  gives  $2.71828182842204 < e < 2.71828182846830$ . Now  $e = 2.71828182845905\dots$ , so the left and right hand sides are both correct to 10 decimal places. We point out also that  $x$  does not need to be an integer.

**Remark 2.3.** Using  $G \leq G^{\frac{2}{3}}A^{\frac{1}{3}} \leq L \leq \frac{2}{3}G + \frac{1}{3}A \leq A$ , and replacing  $x$  with  $e^x$  and letting  $y = e^{-x}$ , we have

$$1 \leq (\cosh(x))^{1/3} \leq \frac{\sinh(x)}{x} \leq \frac{2}{3} + \frac{1}{3} \cosh(x) \leq \cosh(x).$$



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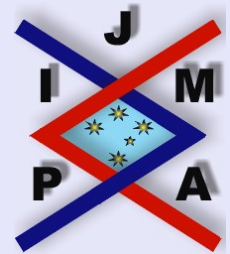
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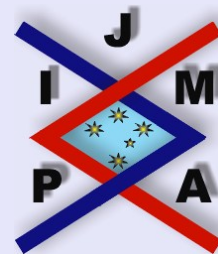
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