

# Journal of Inequalities in Pure and Applied Mathematics

## ON THE HEISENBERG-WEYL INEQUALITY

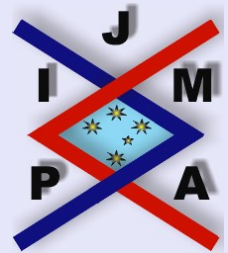
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ISSN (electronic): 1443-5756  
169-04



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volume 6, issue 1, article 11,  
2005.

*Received 20 September, 2004;*  
*accepted 25 November, 2004.*

*Communicated by: G. Anastassiou*

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Abstract

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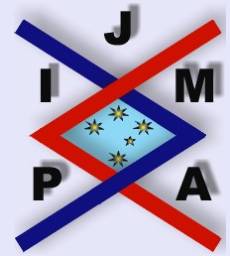
## Abstract

In 1927, W. Heisenberg demonstrated the impossibility of specifying simultaneously the position and the momentum of an electron within an atom. The well-known *second moment Heisenberg-Weyl inequality* states: Assume that  $f : \mathbb{R} \rightarrow \mathbb{C}$  is a complex valued function of a random real variable  $x$  such that  $f \in L^2(\mathbb{R})$ . Then the product of the second moment of the random real  $x$  for  $|f|^2$  and the second moment of the random real  $\xi$  for  $|\hat{f}|^2$  is at least  $E_{|f|^2}/4\pi$ , where  $\hat{f}$  is the Fourier transform of  $f$ , such that  $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x) dx$  and  $f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi) d\xi$ ,  $i = \sqrt{-1}$  and  $E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2 dx$ . In 2004, the author generalized the afore-mentioned result to *the higher order absolute moments for  $L^2$  functions  $f$*  with orders of moments in the set of natural numbers. In this paper, a new generalization proof is established with orders of absolute moments in the set of non-negative real numbers. Afterwards, an application is provided by means of the well-known Euler gamma function and the Gaussian function and an open problem is proposed on some pertinent extremum principle. This inequality can be applied in harmonic analysis and quantum mechanics.

**2000 Mathematics Subject Classification:** 26Dxx, 30Xxx, 33Xxx, 42Xxx, 43Xxx, 60Xxx, 62Xxx, 81Xxx.

**Key words:** Heisenberg-Weyl Inequality, Uncertainty Principle, Absolute Moment, Gaussian, Extremum Principle.

We are grateful to Professors George Anastassiou and Bill Beckner for their great suggestions.



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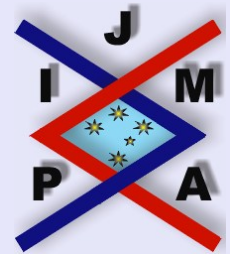
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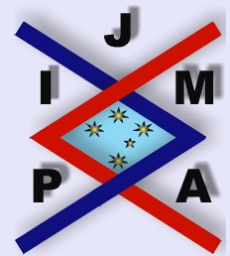
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# 1. Introduction

The serious question of certainty in science was high-lighted by Heisenberg (1901-1976), in 1927, via his “uncertainty principle” [7]. He demonstrated, for instance, the impossibility of specifying simultaneously the position and the speed (or the momentum) of an electron within an atom. In 1933, according to Wiener (1894-1964) [10] *a pair of transforms cannot both be very small.*

This uncertainty principle was stated in 1925 by Wiener, according to Wiener’s autobiography [11, p. 105–107], in a lecture in Göttingen. In 1997, according to Folland and Sitaram [5] the uncertainty principle in harmonic analysis says: *A nonzero function and its Fourier transform cannot both be sharply localized.* The following result of the *Heisenberg-Weyl Inequality* is credited to Pauli (1900 – 1958) according to Weyl [9, p. 77, p. 393–394]. In 1928, according to Pauli [9], *the less the uncertainty in  $|f|^2$ , the greater the uncertainty in  $|\hat{f}|^2$ , and conversely.* This result does not actually appear in Heisenberg’s seminal paper [7] (in 1927). In 1997 Battle [1] proved a number of excellent uncertainty results for wavelet states. Coifman et al. [3] established important results in signal processing and compression with wavelet packets. For fundamental accounts of the construction of orthonormal wavelets we refer the reader to Daubechies [4]. In 1998, Burke Hubbard [2] wrote a remarkable book on wavelets. According to her, most people first learn the Heisenberg uncertainty principle in connection with quantum mechanics, but it is also a central statement of information processing. According to Folland and Sitaram [5] (in 1997), Heisenberg gave an incisive analysis of the physics of the uncertainty principle but contains little mathematical precision. The following second order moment Heisenberg-Weyl inequality provides a precise quantitative formulation of the above-mentioned



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uncertainty principle according to W. Pauli.

## 1.1. Second Moment Heisenberg-Weyl Inequality ([2] – [5])

For any  $f \in L^2(\mathbb{R})$ ,  $f : \mathbb{R} \rightarrow \mathbb{C}$ , such that  $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 dx = E_{|f|^2}$ , any fixed but arbitrary constants  $x_m, \xi_m \in \mathbb{R}$ , and for the second order moments (variances)

$$(\mu_2)_{|f|^2} = \sigma_{|f|^2}^2 = \int_{\mathbb{R}} (x - x_m)^2 |f(x)|^2 dx$$

and

$$(\mu_2)_{|\hat{f}|^2} = \sigma_{|\hat{f}|^2}^2 = \int_{\mathbb{R}} (\xi - \xi_m)^2 |\hat{f}(\xi)|^2 d\xi,$$

the second order moment Heisenberg-Weyl inequality

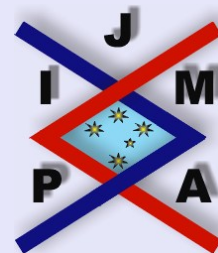
$$(H_1) \quad \sigma_{|f|^2}^2 \cdot \sigma_{|\hat{f}|^2}^2 \geq \frac{\|f\|_2^4}{16\pi^2},$$

holds. Equality holds in  $(H_1)$  if and only if the generalized Gaussians

$$f(x) = c_0 \exp(2\pi i x \xi_m) \exp(-c(x - x_m)^2)$$

hold for some constants  $c_0 \in \mathbb{C}$  and  $c > 0$ .

The *Heisenberg-Weyl inequality* in mathematical statistics and Fourier analysis asserts that: The product of the variances of the probability measures  $|f(x)|^2 dx$  and  $|\hat{f}(\xi)|^2 d\xi$  is larger than an absolute constant. Parts of harmonic analysis on euclidean spaces can naturally be expressed in terms of  $a$



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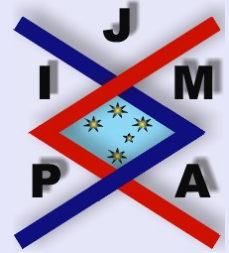


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*Gaussian measure*; that is, a measure of the form  $c_0 e^{-c|x|^2} dx$ , where  $dx$  is the Lebesgue measure and  $c, c_0 (> 0)$  constants. Among these are: Logarithmic Sobolev inequalities, and Hermite expansions. In 1999, according to Gasquet and Witomski [6] the Heisenberg-Weyl inequality in *spectral analysis* says that the product of the effective duration  $\Delta x$  and the effective bandwidth  $\Delta \xi$  of a signal cannot be less than the value  $1/4\pi$  =Heisenberg lower bound, where  $\Delta x^2 = \sigma_{|f|^2}^2 / E_{|f|^2}$  and  $\Delta \xi^2 (= \sigma_{|\hat{f}|^2}^2 / E_{|\hat{f}|^2}) = \sigma_{|\hat{f}|^2}^2 / E_{|\hat{f}|^2}$  with  $f : \mathbb{R} \rightarrow \mathbb{C}, \hat{f} : \mathbb{R} \rightarrow \mathbb{C}$  defined as in  $(H_1)$ , and

$$(PPR) \quad E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = E_{|\hat{f}|^2},$$

according to the Plancherel-Parseval-Rayleigh identity [6].

## 1.2. Fourth Moment Heisenberg-Weyl Inequality ( [8, p. 26] )

For any  $f \in L^2(\mathbb{R}), f : \mathbb{R} \rightarrow \mathbb{C}$ , such that  $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 dx = E_{|f|^2}$ , any fixed but arbitrary constants  $x_m, \xi_m \in \mathbb{R}$ , and for the fourth order moments

$$(\mu_4)_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^4 |f(x)|^2 dx$$

and

$$(\mu_4)_{|\hat{f}|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^4 |\hat{f}(\xi)|^2 d\xi,$$

the fourth order moment Heisenberg - Weyl inequality

$$(H_2) \quad (\mu_4)_{|f|^2} \cdot (\mu_4)_{|\hat{f}|^2} \geq \frac{1}{64\pi^4} E_{2,f}^2,$$

holds, where

$$E_{2,f} = 2 \int_{\mathbb{R}} \left[ (1 - 4\pi^2 \xi_m^2 x_\delta^2) |f(x)|^2 - x_\delta^2 |f'(x)|^2 - 4\pi \xi_m x_\delta^2 \operatorname{Im}(f(x) \overline{f'(x)}) \right] dx,$$

with  $x_\delta = x - x_m$ ,  $\xi_\delta = \xi - \xi_m$ ,  $\operatorname{Im}(\cdot)$  is the imaginary part of  $(\cdot)$ , and  $|E_{2,f}| < \infty$ .

The “inequality” ( $H_2$ ) holds, unless  $f(x) = 0$ .

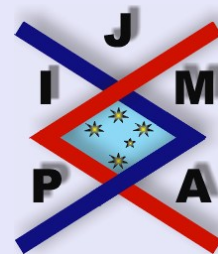
We note that if the ordinary differential equation of second order

$$(ODE) \quad f''_\alpha(x) = -2c_2 x_\delta^2 f_\alpha(x)$$

holds, with  $\alpha = -2\pi \xi_m i$ ,  $f_\alpha(x) = e^{\alpha x} f(x)$ , and a constant  $c_2 = \frac{1}{2} k_2^2 > 0$ ,  $k_2 \in \mathbb{R}$  and  $k_2 \neq 0$ , then “equality” in ( $H_2$ ) seems to occur. However, the solution of this differential equation (ODE), given by the function

$$f(x) = \sqrt{|x_\delta|} e^{2\pi i x \xi_m} \left[ c_{20} J_{-1/4} \left( \frac{1}{2} |k_2| x_\delta^2 \right) + c_{21} J_{1/4} \left( \frac{1}{2} |k_2| x_\delta^2 \right) \right],$$

in terms of the Bessel functions  $J_{\pm 1/4}$  of the first kind of orders  $\pm 1/4$ , leads to a contradiction, because this  $f \notin L^2(\mathbb{R})$ . Furthermore, a limiting argument is required for this problem. For the proof of this inequality see [8]. In 2004, we [8] generalized the Heisenberg-Weyl inequality with orders of moments in the set of natural numbers. In this paper we establish a new generalization proof with orders of absolute moments in the set of non-negative real numbers. It is open to investigate cases, where the integrand on the right-hand side of integrals of  $E_{2,f}$  will be nonnegative. For instance, for  $x_m = \xi_m = 0$ , this integrand is:  $|f(x)|^2 - x^2 |f'(x)|^2$  ( $\geq 0$ ).




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## 2. Heisenberg-Weyl Inequality

If  $\int_{\mathbb{R}} |f(x)|^2 dx = E_{|f|^2}$ , then we state and prove the following new theorem.

**Theorem 2.1.** *If  $f \in L^2(\mathbb{R})$  and  $\rho \geq 2$ , then the Heisenberg-Weyl inequality*

$$(2.1) \quad (\mu_{\rho}^*)_{|f|^2}^{1/\rho} (\mu_{\rho}^*)_{|\hat{f}|^2}^{1/\rho} \geq E_{|f|^2}^{2/\rho} / 4\pi,$$

*holds for any fixed but arbitrary real constants  $x_m, \xi_m$  and the higher order absolute moments*

$$(\mu_{\rho}^*)_{|f|^2} = \int_{\mathbb{R}} |x_{\delta}|^{\rho} |f(x)|^2 dx$$

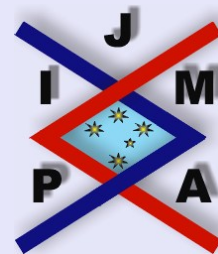
*with  $x_{\delta} = x - x_m$  and*

$$(\mu_{\rho}^*)_{|\hat{f}|^2} = \int_{\mathbb{R}} |\xi_{\delta}|^{\rho} |\hat{f}(\xi)|^2 d\xi$$

*with  $\xi_{\delta} = \xi - \xi_m$ . The “inequality” (2.1) holds, unless  $f(x) = 0$ . Equality in (2.1) holds for  $\rho = 2$  and all the Gaussian mappings of the form  $f(x) = c_0 \exp(-cx^2)$ , where  $c_0, c$  are constants and  $c_0 \in \mathbb{C}, c > 0$ , or for  $\rho \geq 2$  and all mappings  $f \in L^2(\mathbb{R})$ , such that  $|x_{\delta}| = |\xi_{\delta}| = \sqrt{1/4\pi}$ .*

*Proof.* Applying the inequality ( $H_1$ ), the Hölder inequality and the Plancherel-Parseval-Rayleigh identity one gets

$$\begin{aligned} & (\mu_{\rho}^*)_{|f|^2}^{\frac{2}{\rho}} \left( E_{|f|^2} \right)^{1 - \frac{2}{\rho}} \\ &= \left( \int_{\mathbb{R}} |x_{\delta}|^{\rho} |f(x)|^2 dx \right)^{\frac{2}{\rho}} \left( \int_{\mathbb{R}} |f(x)|^2 dx \right)^{1 - \frac{2}{\rho}} \end{aligned}$$



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$$\begin{aligned}
&= \left[ \int_{\mathbb{R}} \left( |x_{\delta}|^2 |f(x)|^{4/\rho} \right)^{\rho/2} dx \right]^{\frac{2}{\rho}} \left[ \int_{\mathbb{R}} \left( |f(x)|^{2(1-\frac{2}{\rho})} \right)^{1/(1-\frac{2}{\rho})} dx \right]^{1-\frac{2}{\rho}} \\
&\geq \int_{\mathbb{R}} \left[ \left( x_{\delta}^2 |f(x)|^{4/\rho} \right) \left( |f(x)|^{2(1-\frac{2}{\rho})} \right) \right] dx \\
&= \int_{\mathbb{R}} x_{\delta}^2 |f(x)|^2 dx = \sigma_{|f|^2}^2,
\end{aligned}$$

or

$$(2.2) \quad (\mu_{\rho}^*)_{|f|^2}^{1/\rho} \geq \sigma_{|f|^2} / \left( E_{|f|^2} \right)^{(1-\frac{2}{\rho})/2}.$$

Equality in (2.2) holds if and only if

$$|x_{\delta}|^{\rho} E_{|f|^2} = (\mu_{\rho}^*)_{|f|^2}.$$

Similarly, we prove from (2.2) and (PPR) that

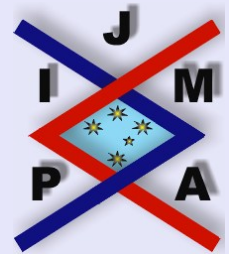
$$(\mu_{\rho}^*)_{|\hat{f}|^2}^{2/\rho} \left( E_{|\hat{f}|^2} \right)^{1-\frac{2}{\rho}} \geq \sigma_{|\hat{f}|^2}^2,$$

or

$$(2.3) \quad (\mu_{\rho}^*)_{|\hat{f}|^2}^{1/\rho} \geq \sigma_{|\hat{f}|^2} / \left( E_{|\hat{f}|^2} \right)^{(1-\frac{2}{\rho})/2}.$$

Equality in (2.3) holds if and only if

$$|\xi_{\delta}|^{\rho} E_{|\hat{f}|^2} = (\mu_{\rho}^*)_{|\hat{f}|^2}.$$



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Multiplying (2.2) and (2.3) one finds

$$(2.4) \quad M_\rho^* = (\mu_\rho^*)_{|f|^2}^{1/\rho} (\mu_\rho^*)_{|\hat{f}|^2}^{1/\rho} \geq \sigma_{|f|^2} \cdot \sigma_{|\hat{f}|^2} / \left( E_{|f|^2} \right)^{1-\frac{2}{\rho}}.$$

It is now clear, from (2.4) and the classical Heisenberg-Weyl inequality ( $H_1$ ), the complete proof of the above theorem.  $\square$

## 2.1. Euler gamma function and Gaussian function

Assume *the Gaussian function* of the form

$$(2.5) \quad f(x) = c_0 \exp(-cx^2),$$

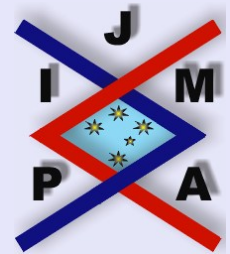
where  $c_0, c$  are constants and  $c_0 \in \mathbb{C}, c > 0$ . Besides consider that  $x_m, \xi_m$ , are *means* of  $x$  for  $|f|^2$  and of  $\xi$  for  $|\hat{f}|^2$ , respectively. If  $\Gamma$  is *the Euler gamma function* and  $\rho = 2, 3, 4, \dots$ , then  $x_m = \int_{\mathbb{R}} x |f(x)|^2 dx = 0$ . We claim that *the Fourier transform*  $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$  is of the form

$$(2.6) \quad \hat{f}(\xi) = c_0 \left( \frac{\pi}{c} \right)^{\frac{1}{2}} \exp\left(-\frac{\pi^2}{c} \xi^2\right),$$

by applying a direct computation using a differential equation ([6, p. 159–161]).

In fact, differentiating the Gaussian function  $f : \mathbb{R} \rightarrow \mathbb{C}$  of the form  $f(x) = c_0 e^{-cx^2}$  with respect to  $x$ , one gets the ordinary differential equation  $f'(x) = -2cx f(x)$ . Thus the Fourier transform of  $f'$  is

$$F f'(\xi) = F [f'(x)](\xi) = [f'(x)]^\wedge(\xi) = [-2cx f(x)]^\wedge(\xi),$$



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or

$$2i\pi\xi\hat{f}(\xi) = \frac{-2c}{-2i\pi} [(-2i\pi x) f(x)]^\wedge(\xi),$$

by standard formulas on differentiation. Thus

$$2i\pi\xi\hat{f}(\xi) = \frac{c}{i\pi} (\hat{f}(\xi))',$$

$$\text{or } -2\pi^2\xi\hat{f}(\xi) = c\hat{f}'(\xi),$$

$$\text{or } (\hat{f}(\xi))' = \hat{f}'(\xi) = -\frac{2\pi}{c} (\pi\xi) \hat{f}(\xi).$$

Solving this *first order differential equation* by the method of the separation of variables we get the general solution

$$(2.7) \quad \hat{f}(\xi) = K(\xi) e^{-\frac{\pi^2}{c}\xi^2},$$

such that  $\hat{f}(0) = K(0)$ . Differentiating the above formula with respect to  $\xi$  one finds

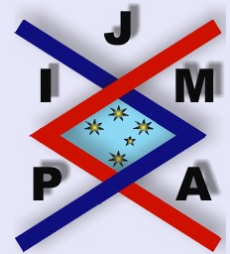
$$\hat{f}'(\xi) = e^{-\frac{\pi^2}{c}\xi^2} \left[ K'(\xi) + K(\xi) \left( -\frac{2\pi^2}{c}\xi \right) \right].$$

Therefore we find  $0 = K'(\xi) e^{-\frac{\pi^2}{c}\xi^2}$ , or  $K'(\xi) = 0$ , or

$$(2.8) \quad K(\xi) = K,$$

which is a constant. But from (2.7) and (2.8) one gets

$$(2.9) \quad \hat{f}(0) = K(0) = K.$$



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Besides from the definition of the Fourier transform we get

$$\begin{aligned}\hat{f}(0) &= \int_{\mathbb{R}} e^{-2i\pi \cdot 0 \cdot x} f(x) dx \\ &= \int_{\mathbb{R}} f(x) dx \\ &= c_0 \int_{\mathbb{R}} e^{-cx^2} dx \\ &= \frac{c_0}{\sqrt{c}} \int_{\mathbb{R}} e^{-[\sqrt{c}x]^2} d(\sqrt{c}x),\end{aligned}$$

or

$$(2.10) \quad \hat{f}(0) = c_0 \sqrt{\frac{\pi}{c}}, \quad c_0 \in \mathbb{C}, \quad c > 0.$$

From (2.9) and (2.10) one finds  $K = c_0 \sqrt{\frac{\pi}{c}}$ ,  $c_0 \in \mathbb{C}$ ,  $c > 0$ .

Therefore we complete the proof of the formula (2.6). Moreover,

$$\xi_m = \int_{\mathbb{R}} \xi \left| \hat{f}(\xi) \right|^2 d\xi = |c_0|^2 \frac{\pi}{c} \int_{\mathbb{R}} \xi \cdot e^{-2\frac{\pi^2}{c}\xi^2} d\xi = 0.$$

Therefore

$$(M_{\rho}^*)^{\rho} = (\mu_{\rho}^*)_{|f|^2} \cdot (\mu_{\rho}^*)_{|f|^2} = (H_{\rho/2}^*)^{\rho} 2\Gamma^2 \left( \frac{\rho+1}{2} \right) \left( \frac{|c_0|^4}{c} \right),$$

or

$$M_{\rho}^* = H_{\rho/2}^* \left[ \frac{4}{\pi} \Gamma^2 \left( \frac{\rho+1}{2} \right) \right]^{\frac{1}{\rho}} E_{|f|^2}^{2/\rho},$$



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because  $E_{|f|^2} = |c_0|^2 (\pi/2c)^{1/2}$ ,  $H_{\rho/2}^* = 1 / ((2\pi) 4^{1/\rho})$ , and

$$\int_{\mathbb{R}} |x|^\rho \exp(-2cx^2) dx = \frac{\Gamma\left(\frac{\rho+1}{2}\right)}{(2c)^{\frac{\rho+1}{2}}}, \quad c > 0, \rho \in N_0.$$

But we have for  $\rho = 2p$ ,  $p \in \mathbb{N}$  that

$$\Gamma\left(\frac{\rho+1}{2}\right) = (\rho-1)!! \left(\frac{\pi}{2^\rho}\right)^{\frac{1}{2}} \geq \left(\frac{\pi}{2^\rho}\right)^{\frac{1}{2}},$$

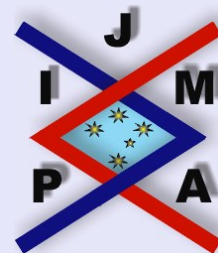
where  $(\rho-1)!! = 1 \cdot 3 \cdot 5 \cdots (\rho-1)$  (for  $\rho = 2p$ ,  $p \in \mathbb{N}$ ). It is clear that this holds as well for  $\rho = 2q+1$ ,  $q \in \mathbb{N}$ . Thus one gets

$$M_\rho^* \geq \left(\frac{1}{(2\pi) 4^{1/\rho}}\right) \left[\frac{4}{\pi} \left(\frac{\pi}{2^\rho}\right)\right]^{\frac{1}{\rho}} E_{|f|^2}^{2/\rho} = E_{|f|^2}^{2/\rho} / 4\pi,$$

verifying (2.1) for all  $\rho = 2, 3, 4, \dots$ . We note that if  $\rho = 2$ ,  $p = 1$  then the equality in (2.1) holds for these Gaussian mappings.

**Queries.** Concerning our Section 8.1 on pp. 26-27 of [8], further investigation is needed for the case of the fundamental “equality” in  $(H_2)$ . As a matter of fact, our function  $f$  is not in  $L^2(\mathbb{R})$ , leading the left-hand side to be infinite in that “equality”. A limiting argument is required for this problem. On the other hand, why doesn’t the corresponding “inequality”  $(H_2)$  attain an extremal in  $L^2(\mathbb{R})$ ?

Here are some of our old results [8] related to the above *Queries*. In particular, if we take into account these results contained in Section 9 on pp. 46-70 [8],



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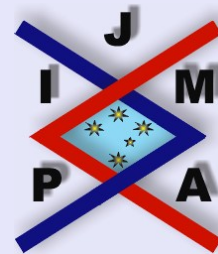
where the Gaussian function and the Euler gamma function  $\Gamma$  are employed, then via Corollary 9.1 on pp. 50-51 [8] we conclude that “equality” in  $(H_p)$ ,  $p \in \mathbb{N} = \{1, 2, 3, \dots\}$ , holds only for  $p = 1$ . Furthermore, employing the above Gaussian function, we established the following *extremum principle* (via (9.33) on p. 51 [8]):

$$(R) \quad R(p) \geq \frac{1}{2\pi}, \quad p \in \mathbb{N}$$

for the corresponding “inequality”  $(H_p)$ ,  $p \in \mathbb{N}$ , where the constant  $1/2\pi$  “on the right-hand side” is the best lower bound for  $p \in \mathbb{N}$ . Therefore “equality” in  $(H_p)$ ,  $p \in \mathbb{N} - \{1\}$ , in Section 8.1 on pp. 19-46 [8] cannot occur under the afore-mentioned well-known functions. On the other hand, there is a lower bound “on the right-hand side” of the corresponding “inequality” in  $(H_2)$  on p. 26 and pp. 54-55 [8] if we employ the above Gaussian function, which equals to  $\frac{1}{64\pi^4} E_{2,f}^2 = \frac{1}{512\pi^3} \cdot \frac{|c_0|^4}{c}$ , with  $c_0, c$  constants and  $c_0 \in \mathbb{C}$ ,  $c > 0$ , because  $E_{|f|^2} = |c_0|^2 \sqrt{\frac{\pi}{2c}}$  and  $E_{2,f} = \frac{1}{2} E_{|f|^2}$ .

Analogous pertinent results are investigated via our Corollaries 9.2-9.6 on pp. 53-68 [8].

**Open Problem And Extremum Principle.** Employing our Theorem 8.1 on p. 20 [8], the Gaussian function, the Euler gamma function  $\Gamma$ , and other related “special functions”, we established and explicitly proved *the extremum principle*



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ple ( $\mathbf{R}$ ):  $R(p) \geq 1/2\pi$ ,  $p \in \mathbb{N}$ , where

$$R(p) = \frac{\Gamma\left(p + \frac{1}{2}\right)}{\left| \sum_{q=0}^{\lfloor \frac{p}{2} \rfloor} (-1)^{p-q} \frac{p}{p-q} \cdot \frac{p!}{(2q)!} \binom{p-q}{q} \Gamma_q \right|},$$

with

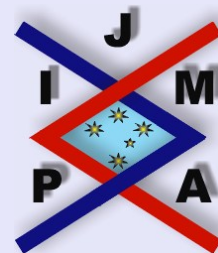
$$\begin{aligned} \Gamma_q &= \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} 2^{2k} \binom{q}{2k}^2 \Gamma^2\left(k + \frac{1}{2}\right) \Gamma\left(2q - 2k + \frac{1}{2}\right) \\ &\quad + 2 \sum_{0 \leq k \leq j \leq \lfloor \frac{q}{2} \rfloor} (-1)^{k+j} 2^{k+j} \binom{q}{2k} \binom{q}{2j} \\ &\quad \times \Gamma\left(k + \frac{1}{2}\right) \Gamma\left(j + \frac{1}{2}\right) \Gamma\left(2q - k - j + \frac{1}{2}\right), \end{aligned}$$

$0 \leq \lfloor \frac{q}{2} \rfloor$  is the greatest integer  $\leq \frac{q}{2}$  for  $q \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$ ,  $\binom{p}{q} = \frac{p!}{q!(p-q)!}$  for  $p \in \mathbb{N}$ ,  $q \in \mathbb{N}_0$  and  $0 \leq q \leq p$ ,  $p! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (p-1) \cdot p$  and  $0! = 1$ , as well as

$$\Gamma\left(p + \frac{1}{2}\right) = \frac{1}{2^{2p}} \cdot \frac{(2p)!}{p!} \sqrt{\pi}, \quad p \in \mathbb{N}$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$



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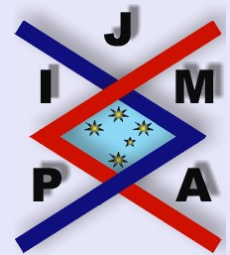
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In addition, we [8] analytically verified this extremum principle for  $p = 1, 2, \dots, 9$  by carrying out all the involved operations. In particular, if we denote  $L = 1/2\pi (\cong 0.159)$ , then the first nine exact values of  $R(p)$  are, as follows:  $\mathbb{R}(1) = L$ ,  $\mathbb{R}(2) = 3L$ ,  $\mathbb{R}(3) = 5L$ ,  $\mathbb{R}(4) = \frac{35}{13}L$ ,  $\mathbb{R}(5) = \frac{63}{17}L$ ,  $\mathbb{R}(6) = \frac{231}{19}L$ ,  $\mathbb{R}(7) = \frac{429}{23}L$ ,  $\mathbb{R}(8) = \frac{495}{47}L$ ,  $\mathbb{R}(9) = \frac{12155}{827}L$ .

Furthermore, by employing computer techniques, this principle was verified for  $p = 1, 2, 3, \dots, 32, 33$ , as well. It now remains *open* to give an explicit second proof of verification for the extremum principle (R) through a much shorter and more elementary method, without applying our Heisenberg-Pauli-Weyl inequality [8].




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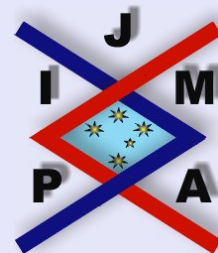
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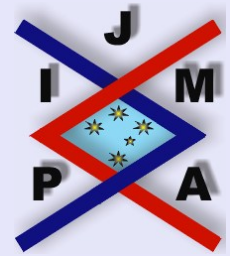
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