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# ON THE HEISENBERG-WEYL INEQUALITY

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## Abstract

In 1927, W. Heisenberg demonstrated the impossibility of specifying simultaneously the position and the momentum of an electron within an atom.The well-known *second moment Heisenberg-Weyl inequality* states: Assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a random real variable $x$ such that $f \in L^2(\mathbb{R})$. Then the product of the second moment of the random real $x$ for $|f|^2$ and the second moment of the random real $\xi$ for $\left|\hat{f}\right|^2$ is at least $E_{|f|^2} \Big/ 4\pi$, where $\hat{f}$ is the Fourier transform of $f$, such that $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\, dx$ and $f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi)\, d\xi$, $i = \sqrt{-1}$ and $E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2\, dx$. In 2004, the author generalized the afore-mentioned result to *the higher order absolute moments for $L^2$ functions $f$* with orders of moments in the set of natural numbers . In this paper, a new generalization proof is established with orders of absolute moments in the set of non-negative real numbers. Afterwards, an application is provided by means of the well-known Euler gamma function and the Gaussian function and an open problem is proposed on some pertinent extremum principle. This inequality can be applied in harmonic analysis and quantum mechanics.

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*Key words:* Heisenberg-Weyl Inequality, Uncertainty Principle, Absolute Moment, Gaussian, Extremum Principle.

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# Contents

# 1. Introduction

The serious question of certainty in science was high-lighted by Heisenberg (1901-1976), in 1927, via his "uncertainty principle" [7]. He demonstrated, for instance, the impossibility of specifying simultaneously the position and the speed (or the momentum) of an electron within an atom. In 1933, according to Wiener (1894-1964) [10] *a pair of transforms cannot both be very small*.

This uncertainty principle was stated in 1925 by Wiener, according to Wiener's autobiography [11, p. 105–107], in a lecture in Göttingen. In 1997, according to Folland and Sitaram [5] the uncertainty principle in harmonic analysis says: *A nonzero function and its Fourier transform cannot both be sharply localized*. The following result of the *Heisenberg-Weyl Inequality* is credited to Pauli (1900 – 1958) according to Weyl [9, p. 77, p. 393–394]. In 1928, according to Pauli [9], *the less the uncertainty in* $|f|^2$, *the greater the uncertainty in* $\left|\hat{f}\right|^2$, *and conversely*. This result does not actually appear in Heisenberg's seminal paper [7] (in 1927). In 1997 Battle [1] proved a number of excellent uncertainty results for wavelet states. Coifman et al. [3] established important results in signal processing and compression with wavelet packets. For fundamental accounts of the construction of orthonormal wavelets we refer the reader to Daubechies [4]. In 1998, Burke Hubbard [2] wrote a remarkable book on wavelets. According to her, most people first learn the Heisenberg uncertainty principle in connection with quantum mechanics, but it is also a central statement of information processing. According to Folland and Sitaram [5] (in 1997), Heisenberg gave an incisive analysis of the physics of the uncertainty principle but contains little mathematical precision. The following second order moment Heisenberg-Weyl inequality provides a precise quantitative formulation of the above-mentioned uncertainty principle according to W. Pauli.

## 1.1. Second Moment Heisenberg-Weyl Inequality ([2] – [5])

*For any* $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$, *such that* $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx = E_{|f|^2}$, *any fixed but arbitrary constants* $x_m$, $\xi_m \in \mathbb{R}$, *and for the second order moments (variances)*

$$(\mu_2)_{|f|^2} = \sigma_{|f|^2}^2 = \int_{\mathbb{R}} (x - x_m)^2 |f(x)|^2 \, dx$$

*and*

$$(\mu_2)_{|\hat{f}|^2} = \sigma_{|\hat{f}|^2}^2 = \int_{\mathbb{R}} (\xi - \xi_m)^2 \left|\hat{f}(\xi)\right|^2 d\xi,$$

*the second order moment Heisenberg-Weyl inequality*

$$(H_1) \qquad \sigma_{|f|^2}^2 \cdot \sigma_{|\hat{f}|^2}^2 \geq \frac{\|f\|_2^4}{16\pi^2},$$

*holds. Equality holds in* ($H_1$) *if and only if the generalized Gaussians*

$$f(x) = c_0 \exp(2\pi i x \xi_m) \exp\left(-c(x - x_m)^2\right)$$

*hold for some constants* $c_0 \in \mathbb{C}$ *and* $c > 0$.

The *Heisenberg-Weyl inequality* in mathematical statistics and Fourier analysis asserts that: The product of the variances of the probability measures $|f(x)|^2 \, dx$ and $\left|\hat{f}(\xi)\right|^2 d\xi$ is larger than an absolute constant. Parts of harmonic analysis on euclidean spaces can naturally be expressed in terms of $a$

*Gaussian measure*; that is, a measure of the form $c_0 e^{-c|x|^2} dx$, where $dx$ is the Lebesgue measure and $c$, $c_0$ $(> 0)$ constants. Among these are: Logarithmic Sobolev inequalities, and Hermite expansions. In 1999, according to Gasquet and Witomski [6] the Heisenberg-Weyl inequality in *spectral analysis* says that the product of the effective duration $\Delta x$ and the effective bandwidth $\Delta\xi$ of a signal cannot be less than the value $1/4\pi$ =Heisenberg lower bound, where $\Delta x^2 = \sigma^2_{|f|^2} \Big/ E_{|f|^2}$ and $\Delta\xi^2 \left(= \sigma^2_{\left|\hat{f}\right|^2} \Big/ E_{\left|\hat{f}\right|^2}\right) = \sigma^2_{\left|\hat{f}\right|^2} \Big/ E_{|f|^2}$ with $f : \mathbb{R} \to \mathbb{C}$, $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined as in ($H_1$), and

$$\text{(PPR)} \qquad E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2 d\xi = E_{\left|\hat{f}\right|^2},$$

according to the Plancherel-Parseval-Rayleigh identity [6].

## 1.2. Fourth Moment Heisenberg-Weyl Inequality ( [8, p. 26])

*For any* $f \in L^2(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$, *such that* $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 dx = E_{|f|^2}$, *any fixed but arbitrary constants* $x_m, \xi_m \in \mathbb{R}$, *and for the fourth order moments*

$$(\mu_4)_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^4 |f(x)|^2 dx$$

*and*

$$(\mu_4)_{\left|\hat{f}\right|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^4 \left|\hat{f}(\xi)\right|^2 d\xi,$$

*the fourth order moment Heisenberg - Weyl inequality*

$$(H_2) \qquad (\mu_4)_{|f|^2} \cdot (\mu_4)_{\left|\hat{f}\right|^2} \geq \frac{1}{64\pi^4} \mathrm{E}^2_{2,f},$$

*holds, where*

$$E_{2,f} = 2\int_{\mathbb{R}} \left[(1-4\pi^2\xi_m^2 x_\delta^2)\left|f(x)\right|^2 - x_\delta^2\left|f'(x)\right|^2 - 4\pi\xi_m x_\delta^2 \operatorname{Im}(f(x)\overline{f'(x)})\right] dx,$$

*with* $x_\delta = x - x_m$, $\xi_\delta = \xi - \xi_m$, $\operatorname{Im}(\cdot)$ *is the imaginary part of* $(\cdot)$, *and* $|E_{2,f}| < \infty$.

*The "inequality" ($H_2$) holds, unless* $f(x) = 0$.

We note that if the ordinary differential equation of second order

$$f_\alpha''(x) = -2c_2 x_\delta^2 f_\alpha(x) \tag{ODE}$$

holds, with $\alpha = -2\pi\xi_m i$, $f_\alpha(x) = e^{\alpha x} f(x)$, and a constant $c_2 = \frac{1}{2}k_2^2 > 0$, $k_2 \in \mathbb{R}$ and $k_2 \neq 0$, then "equality" in ($H_2$) seems to occur. However, the solution of this differential equation (ODE), given by the function

$$f(x) = \sqrt{|x_\delta|}e^{2\pi i x\xi_m}\left[c_{20}J_{-1/4}\left(\frac{1}{2}\left|k_2\right| x_\delta^2\right) + c_{21}J_{1/4}\left(\frac{1}{2}\left|k_2\right| x_\delta^2\right)\right],$$

in terms of the Bessel functions $J_{\pm 1/4}$ of the first kind of orders $\pm 1/4$, leads to a contradiction, because this $f \notin L^2(\mathbb{R})$. Furthermore, a limiting argument is required for this problem. For the proof of this inequality see [8]. In 2004, we [8] generalized the Heisenberg-Weyl inequality with orders of moments in the set of natural numbers. In this paper we establish a new generalization proof with orders of absolute moments in the set of non-negative real numbers. It is *open* to investigate cases, where the integrand on the right-hand side of integrals of $E_{2,f}$ will be nonnegative. For instance, for $x_m = \xi_m = 0$, this integrand is:$= |f(x)|^2 - x^2|f'(x)|^2$ $(\geq 0)$.

## 2. Heisenberg-Weyl Inequality

If $\int_{\mathbb{R}} |f(x)|^2 \, dx = E_{|f|^2}$, then we state and prove the following new theorem.

**Theorem 2.1.** *If $f \in L^2(\mathbb{R})$ and $\rho \geq 2$, then the Heisenberg-Weyl inequality*

$$(\mu_\rho^*)^{1/\rho}_{|f|^2} (\mu_\rho^*)^{1/\rho}_{|\hat{f}|^2} \geq E^{2/\rho}_{|f|^2} \Big/ 4\pi, \tag{2.1}$$

*holds for any fixed but arbitrary real constants $x_m$, $\xi_m$ and the higher order absolute moments*

$$(\mu_\rho^*)_{|f|^2} = \int_{\mathbb{R}} |x_\delta|^\rho |f(x)|^2 \, dx$$

*with $x_\delta = x - x_m$ and*

$$(\mu_\rho^*)_{|\hat{f}|^2} = \int_{\mathbb{R}} |\xi_\delta|^\rho \left|\hat{f}(\xi)\right|^2 d\xi$$

*with $\xi_\delta = \xi - \xi_m$. The "inequality" (2.1) holds, unless $f(x) = 0$. Equality in (2.1) holds for $\rho = 2$ and all the Gaussian mappings of the form $f(x) = c_0 \exp(-cx^2)$, where $c_0$, $c$ are constants and $c_0 \in \mathbb{C}$, $c > 0$, or for $\rho \geq 2$ and all mappings $f \in L^2(\mathbb{R})$, such that $|x_\delta| = |\xi_\delta| = \sqrt{1/4\pi}$.*

*Proof.* Applying the inequality ($H_1$), *the Hölder inequality* and *the Plancherel-Parseval-Rayleigh identity* one gets

$$(\mu_\rho^*)^{\frac{2}{\rho}}_{|f|^2} \left(E_{|f|^2}\right)^{1-\frac{2}{\rho}} = \left(\int_{\mathbb{R}} |x_\delta|^\rho |f(x)|^2 \, dx\right)^{\frac{2}{\rho}} \left(\int_{\mathbb{R}} |f(x)|^2 \, dx\right)^{1-\frac{2}{\rho}}$$

$$= \left[\int_{\mathbb{R}} \left(|x_\delta|^2 \, |f(x)|^{4/\rho}\right)^{\rho/2} dx\right]^{\frac{2}{\rho}} \left[\int_{\mathbb{R}} \left(|f(x)|^{2\left(1-\frac{2}{\rho}\right)}\right)^{1/\left(1-\frac{2}{\rho}\right)} dx\right]^{1-\frac{2}{\rho}}$$

$$\geq \int_{\mathbb{R}} \left[\left(x_\delta^2 \, |f(x)|^{4/\rho}\right)\left(|f(x)|^{2\left(1-\frac{2}{\rho}\right)}\right)\right] dx$$

$$= \int_{\mathbb{R}} x_\delta^2 \, |f(x)|^2 \, dx = \sigma^2_{|f|^2},$$

or

$$(2.2) \qquad \left(\mu_\rho^*\right)^{1/\rho}_{|f|^2} \geq \sigma_{|f|^2} \bigg/ \left(E_{|f|^2}\right)^{\left(1-\frac{2}{\rho}\right)/2}.$$

*Equality* in (2.2) holds if and only if

$$|x_\delta|^\rho \, E_{|f|^2} = \left(\mu_\rho^*\right)_{|f|^2}.$$

Similarly, we prove from (2.2) and (PPR) that

$$\left(\mu_\rho^*\right)^{2/\rho}_{|\hat{f}|^2} \left(E_{|\hat{f}|^2}\right)^{1-\frac{2}{\rho}} \geq \sigma^2_{|\hat{f}|^2},$$

or

$$(2.3) \qquad \left(\mu_\rho^*\right)^{1/\rho}_{|\hat{f}|^2} \geq \sigma_{|\hat{f}|^2} \bigg/ \left(E_{|f|^2}\right)^{\left(1-\frac{2}{\rho}\right)/2}.$$

*Equality* in (2.3) holds if and only if

$$|\xi_\delta|^\rho \, E_{|f|^2} = \left(\mu_\rho^*\right)_{|\hat{f}|^2}.$$

Multiplying (2.2) and (2.3) one finds

$$M_\rho^* = \left(\mu_\rho^*\right)_{|f|^2}^{1/\rho} \left(\mu_\rho^*\right)_{|\hat{f}|^2}^{1/\rho} \geq \sigma_{|f|^2} \cdot \sigma_{|\hat{f}|^2} \Big/ \left(E_{|f|^2}\right)^{1-\frac{2}{\rho}}. \tag{2.4}$$

It is now clear, from (2.4) and the classical Heisenberg-Weyl inequality ($H_1$), the complete proof of the above theorem. □

## 2.1. Euler gamma function and Gaussian function

Assume *the Gaussian function* of the form

$$f(x) = c_0 \exp\left(-cx^2\right), \tag{2.5}$$

where $c_0$, $c$ are constants and $c_0 \in \mathbb{C}$, $c > 0$. Besides consider that $x_m$, $\xi_m$, are *means* of $x$ for $|f|^2$ and of $\xi$ for $\left|\hat{f}\right|^2$, respectively. If $\Gamma$ is *the Euler gamma function* and $\rho = 2, 3, 4, \ldots$, then $x_m = \int_{\mathbb{R}} x\, |f(x)|^2\, dx = 0$ .We claim that *the Fourier transform* $\hat{f} : \mathbb{R} \to \mathbb{C}$ is of the form

$$\hat{f}(\xi) = c_0 \left(\frac{\pi}{c}\right)^{\frac{1}{2}} \exp\left(-\frac{\pi^2}{c}\xi^2\right), \tag{2.6}$$

by applying a direct computation using a differential equation ([6, p. 159–161]).

In fact, differentiating the Gaussian function $f : \mathbb{R} \to \mathbb{C}$ of the form $f(x) = c_0 e^{-cx^2}$ with respect to $x$, one gets the ordinary differential equation $f'(x) = -2cxf(x)$. Thus the Fourier transform of $f'$ is

$$Ff'(\xi) = F\left[f'(x)\right](\xi) = \left[f'(x)\right]^\wedge(\xi) = \left[-2cxf(x)\right]^\wedge(\xi),$$

or

$$2i\pi\xi\hat{f}(\xi) = \frac{-2c}{-2i\pi}\left[(-2i\pi x) f(x)\right]^{\wedge}(\xi),$$

by standard formulas on differentiation. Thus

$$2i\pi\xi\hat{f}(\xi) = \frac{c}{i\pi}\left(\hat{f}(\xi)\right)',$$
$$\text{or } -2\pi^2\xi\hat{f}(\xi) = c\hat{f}'(\xi),$$
$$\text{or } \left(\hat{f}(\xi)\right)' = \hat{f}'(\xi) = -\frac{2\pi}{c}(\pi\xi)\hat{f}(\xi).$$

Solving this *first order differential equation* by the method of the separation of variables we get the general solution

$$\hat{f}(\xi) = K(\xi)e^{-\frac{\pi^2}{c}\xi^2}, \tag{2.7}$$

such that $\hat{f}(0) = K(0)$. Differentiating the above formula with respect to $\xi$ one finds

$$\hat{f}'(\xi) = e^{-\frac{\pi^2}{c}\xi^2}\left[K'(\xi) + K(\xi)\left(-\frac{2\pi^2}{c}\xi\right)\right].$$

Therefore we find $0 = K'(\xi)e^{-\frac{\pi^2}{c}\xi^2}$, or $K'(\xi) = 0$, or

$$K(\xi) = K, \tag{2.8}$$

which is a constant. But from (2.7) and (2.8) one gets

$$\hat{f}(0) = K(0) = K. \tag{2.9}$$

Besides from the definition of the Fourier transform we get

$$\begin{aligned}\hat{f}(0) &= \int_{\mathbb{R}} e^{-2i\pi\cdot 0\cdot x} f(x)\,dx \\ &= \int_{\mathbb{R}} f(x)\,dx \\ &= c_0 \int_{\mathbb{R}} e^{-cx^2}\,dx \\ &= \frac{c_0}{\sqrt{c}} \int_{\mathbb{R}} e^{-\left[\sqrt{c}x\right]^2} d\left(\sqrt{c}x\right),\end{aligned}$$

or

$$\hat{f}(0) = c_0\sqrt{\frac{\pi}{c}}, \quad c_0 \in \mathbb{C}, \quad c > 0. \tag{2.10}$$

From (2.9) and (2.10) one finds $K = c_0\sqrt{\frac{\pi}{c}}$, $c_0 \in \mathbb{C}$, $c > 0$.
Therefore we complete the proof of the formula (2.6). Moreover,

$$\xi_m = \int_{\mathbb{R}} \xi \left|\hat{f}(\xi)\right|^2 d\xi = |c_0|^2 \frac{\pi}{c} \int_{\mathbb{R}} \xi \cdot e^{-2\frac{\pi^2}{c}\xi^2} d\xi = 0.$$

Therefore

$$(M_\rho^*)^\rho = (\mu_\rho^*)_{|f|^2} \cdot (\mu_\rho^*)_{|\hat{f}|^2} = (H_{\rho/2}^*)^\rho\, 2\Gamma^2\left(\frac{\rho+1}{2}\right)\left(\frac{|c_0|^4}{c}\right),$$

or

$$M_\rho^* = H_{\rho/2}^* \left[\frac{4}{\pi}\Gamma^2\left(\frac{\rho+1}{2}\right)\right]^{\frac{1}{\rho}} E_{|f|^2}^{2/\rho},$$

because $E_{|f|^2} = |c_0|^2 (\pi/2c)^{1/2}$, $H^*_{\rho/2} = 1 \big/ \left( (2\pi) \, 4^{1/\rho} \right)$, and

$$\int_{\mathbb{R}} |x|^\rho \exp\left(-2cx^2\right) dx = \frac{\Gamma\left(\frac{\rho+1}{2}\right)}{(2c)^{\frac{\rho+1}{2}}}, \quad c > 0, \ \rho \in N_0.$$

But we have for $\rho = 2p$, $p \in \mathbb{N}$ that

$$\Gamma\left(\frac{\rho+1}{2}\right) = (\rho-1)!! \left(\frac{\pi}{2^\rho}\right)^{\frac{1}{2}} \geq \left(\frac{\pi}{2^\rho}\right)^{\frac{1}{2}},$$

where $(\rho-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (\rho-1)$ (for $\rho = 2p$, $p \in \mathbb{N}$). It is clear that this holds as well for $\rho = 2q+1$, $q \in \mathbb{N}$. Thus one gets

$$M^*_\rho \geq \left( \frac{1}{(2\pi) \, 4^{1/\rho}} \right) \left[ \frac{4}{\pi} \left( \frac{\pi}{2^\rho} \right) \right]^{\frac{1}{\rho}} E^{2/\rho}_{|f|^2} = E^{2/\rho}_{|f|^2} \Big/ 4\pi,$$

verifying (2.1) for all $\rho = 2, 3, 4, \ldots$. We note that if $\rho = 2$, $p = 1$ then the equality in (2.1) holds for these Gaussian mappings.

**Queries.** *Concerning our Section 8.1 on pp. 26-27 of [8], further investigation is needed for the case of the fundamental "equality" in ($H_2$). As a matter of fact, our function $f$ is not in $L^2(\mathbb{R})$, leading the left-hand side to be infinite in that "equality". A limiting argument is required for this problem. On the other hand, why doesn't the corresponding "inequality" ($H_2$) attain an extremal in $L^2(\mathbb{R})$?*

Here are some of our old results [8] related to the above *Queries*. In particular, if we take into account these results contained in Section 9 on pp. 46-70 [8], where the Gaussian function and the Euler gamma function $\Gamma$ are employed, then via Corollary 9.1 on pp. 50-51 [8] we conclude that "equality" in $(H_p)$, $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$, holds only for $p = 1$. Furthermore, employing the above Gaussian function, we established the following *extremum principle* (via (9.33) on p. 51 [8]):

$$R(p) \geq \frac{1}{2\pi}, \quad p \in \mathbb{N} \tag{R}$$

for the corresponding "inequality" $(H_p)$, $p \in \mathbb{N}$, where the constant $1/2\pi$ "on the right-hand side" is the best lower bound for $p \in \mathbb{N}$. Therefore "equality" in $(H_p)$, $p \in \mathbb{N} - \{1\}$, in Section 8.1 on pp. 19-46 [8] cannot occur under the afore-mentioned well-known functions. On the other hand, there is a lower bound "on the right-hand side" of the corresponding "inequality" in $(H_2)$ on p. 26 and pp. 54-55 [8] if we employ the above Gaussian function, which equals to $\frac{1}{64\pi^4} E_{2,f}^2 = \frac{1}{512\pi^3} \cdot \frac{|c_0|^4}{c}$, with $c_0$, $c$ constants and $c_0 \in \mathbb{C}$, $c > 0$, because $E_{|f|^2} = |c_0|^2 \sqrt{\frac{\pi}{2c}}$ and $E_{2,f} = \frac{1}{2} E_{|f|^2}$.

Analogous pertinent results are investigated via our Corollaries 9.2-9.6 on pp. 53-68 [8].

**Open Problem And Extremum Principle.** Employing our Theorem 8.1 on p. 20 [8], the Gaussian function, the Euler gamma fuction $\Gamma$, and other related *"special functions"*, we established and explicitly proved *the extremum princi-*

*ple* (R): $R(p) \geq 1/2\pi$, $p \in \mathbb{N}$, where

$$R(p) = \frac{\Gamma\left(p+\frac{1}{2}\right)}{\left|\sum_{q=o}^{\left[\frac{p}{2}\right]} (-1)^{p-q} \frac{p}{p-q} \cdot \frac{p!}{(2q)!} \binom{p-q}{q} \Gamma_q\right|},$$

with

$$\Gamma_q = \sum_{k=0}^{\left[\frac{q}{2}\right]} 2^{2k} \binom{q}{2k}^2 \Gamma^2\left(k+\frac{1}{2}\right) \Gamma\left(2q-2k+\frac{1}{2}\right)$$
$$+ 2 \sum_{0 \leq k \leq j \leq \left[\frac{q}{2}\right]} (-1)^{k+j} 2^{k+j} \binom{q}{2k} \binom{q}{2j}$$
$$\times \Gamma\left(k+\frac{1}{2}\right) \Gamma\left(j+\frac{1}{2}\right) \Gamma\left(2q-k-j+\frac{1}{2}\right),$$

$0 \leq \left[\frac{q}{2}\right]$ is the greatest integer $\leq \frac{q}{2}$ for $q \in \mathbb{N} \cup \{0\} = \mathbb{N}_0$, $\binom{p}{q} = \frac{p!}{q!(p-q)!}$ for $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $0 \leq q \leq p$, $p! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (p-1) \cdot p$ and $0! = 1$, as well as

$$\Gamma\left(p+\frac{1}{2}\right) = \frac{1}{2^{2p}} \cdot \frac{(2p)!}{p!}\sqrt{\pi}, \quad p \in \mathbb{N}$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

In addition, we [8] analytically verified this extremum principle for $p = 1, 2, \ldots, 9$ by carrying out all the involved operations. In particular, if we denote $L = 1/2\pi (\cong 0.159)$, then the first nine exact values of $R(p)$ are, as follows: $\mathbb{R}(1) = L$, $\mathbb{R}(2) = 3L$, $\mathbb{R}(3) = 5L$, $\mathbb{R}(4) = \frac{35}{13}L$, $\mathbb{R}(5) = \frac{63}{17}L$, $\mathbb{R}(6) = \frac{231}{19}L$, $\mathbb{R}(7) = \frac{429}{23}L$, $\mathbb{R}(8) = \frac{495}{47}L$, $\mathbb{R}(9) = \frac{12155}{827}L$.

Furthermore, by employing computer techniques, this principle was verified for $p = 1, 2, 3, \ldots, 32, 33$, as well. It now remains *open* to give an explicit second proof of verification for the extremum principle (R) through a much shorter and more elementary method, without applying our Heisenberg-Pauli-Weyl inequality [8].

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