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AN EXPLICIT MERTENS' TYPE INEQUALITY FOR ARITHMETIC PROGRESSIONS

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Abstract

We give an explicit Mertens type formula for primes in arithmetic progressions using mean values of Dirichlet L-functions at $s = 1$.

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Contents

1. Introduction and Main Result

The very useful Mertens' formula states that

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)=\frac{e^{-\gamma}}{\log x}\left\{1+O\left(\frac{1}{\log x}\right)\right\}$$

for any real number $x \geq 2$, where $\gamma \approx 0.577215664\ldots$ denotes the Euler constant. Some explicit inequalities have been given in [4] where it is showed for example that

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} < e^{\gamma}\delta\left(x\right)\log x, \tag{1.1}$$

where

$$\delta\left(x\right) := 1+\frac{1}{\left(\log x\right)^{2}}. \tag{1.2}$$

Let $1 \leq l \leq k$ be positive integers satisfying $(k,l) = 1$. The aim of this paper is to provide an explicit upper bound for the product

$$\prod_{\substack{p\le x \\ p\equiv l \,(\mathrm{mod}\, k)}}\left(1-\frac{1}{p}\right)^{-1}. \tag{1.3}$$

In [2, 5], the authors gave asymptotic formulas for (1.3) in the form

$$\prod_{\substack{p\le x \\ p\equiv l \,(\mathrm{mod}\, k)}}\left(1-\frac{1}{p}\right)\sim c\left(k,l\right)\left(\log x\right)^{-1/\varphi(k)},$$

where φ is the Euler totient function and $c(k,l)$ is a constant depending on l and k. Nevertheless, because of the non-effectivity of the Siegel-Walfisz theorem, one cannot compute the implied constant in the error term. Moreover, the constant $c(k,l)$ is given only for some particular cases in [2], whereas K.S. Williams established a quite complicated expression of $c(k,l)$ involving a product of Dirichlet $L-$functions $L(s;\chi)$ and a function $K(s;\chi)$ at $s=1$, where $K(s;\chi)$ is the generating Dirichlet series of the completely multiplicative function k_χ defined by

$$k_\chi(p) := p\left\{1-\left(1-\frac{\chi(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\chi(p)}\right\}$$

for any prime number p and any Dirichlet character χ modulo k. The author then gave explicit expressions of $c(k,l)$ in the case $k=24$.

It could be useful to have an explicit upper bound for (1.3) valid for a large range of k and x. Indeed, we shall see in a forthcoming paper that such a bound could be used to estimate class numbers of certain cyclic number fields. We prove the following result:

Theorem 1.1. *Let $1 \le l \le k$ be positive integers satisfying $(k,l)=1$ and $k \ge 37$, and x be a positive real number such that $x > k$. We have:*

$$\prod_{\substack{p\le x\\ p\equiv l \,(\mathrm{mod}\, k)}}\left(1-\frac{1}{p}\right)^{-1}$$

$$< e^{2(\gamma - B)} \sqrt{\zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right)} \cdot \left(\frac{e^{\gamma}\varphi(k)}{k} \log x\right)^{\frac{1}{\varphi(k)}} \cdot \Phi(x,k),$$

where

$$\Phi(x,k) := \exp\left\{\frac{2}{\log x}\left(\frac{2\sqrt{k}\log k}{\varphi(k)} \sum_{\chi \neq \chi_0} \left|\frac{L'}{L}(1;\chi)\right| + 2\sqrt{k}\log k + E - \gamma\right)\right\},$$

$B \approx 0.261497212847643\ldots$ *and* $E \approx 1.332582275733221\ldots$

The restriction $k \geq 37$ is given here just to use a simpler expression of the Polyá-Vinogradov inequality, but one can prove a similar result with $k \geq 9$ only, the constants in $\Phi(x,k)$ being slightly larger.

2. Notation

p denotes always a prime, $1 \leq l \leq k$ are positive integers satisfying $(k,l) = 1$ and $k \geq 37$, $x > k$ is a real number,

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \approx 0.5772156649015328\ldots$$

is the Euler constant and

$$\gamma_1 := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\log k}{k} - \frac{(\log n)^2}{2} \right) \approx -0.07281584548367\ldots$$

is the first Stieltjes constant. Similarly,

$$E := \lim_{n\to\infty} \left(\log n - \sum_{p\leq n} \frac{\log p}{p} \right) \approx 1.332582275733221\ldots$$

and

$$B := \lim_{n\to\infty} \left(\sum_{p\leq n} \frac{1}{p} - \log\log n \right) \approx 0.261497212847643\ldots$$

χ denotes a Dirichlet character modulo k and χ_0 is the principal character modulo k. For any Dirichlet character χ modulo k and any $s \in \mathbb{C}$ such that $\operatorname{Re} s > 1$, $L(s;\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ is the Dirichlet $L-$ function associated to χ. $\sum_{\chi\neq\chi_0}$ means that the sum is taken over all non-principal characters modulo k. Λ is the Von Mangoldt function and $f * g$ denotes the usual Dirichlet convolution product.

3. Sums with Primes

From [4] we get the following estimates:

Lemma 3.1.

$$\sum_p \log p \sum_{\alpha=2}^{\infty} \frac{1}{p^\alpha} = E - \gamma \quad \textit{and} \quad \sum_p \sum_{\alpha=2}^{\infty} \frac{1}{\alpha p^\alpha} = \gamma - B.$$

4. The Polyá-Vinogradov Inequality and Character Sums with Primes

Lemma 4.1. *Let χ be any non-principal Dirichlet character modulo $k \geq 37$.*

(i) *For any real number $x \geq 1$,*

$$\left|\sum_{n \leq x} \chi(n)\right| < \frac{9}{10}\sqrt{k}\log k.$$

(ii) *Let $F \in C^1\left([1;+\infty[\,,\,[0;+\infty[\right)$ such that $F(t) \underset{t\to\infty}{\searrow} 0$. For any real number $x \geq 1$,*

$$\left|\sum_{n > x} \chi(n)\, F(n)\right| \leq \frac{9}{5} F(x)\sqrt{k}\log k.$$

(iii) *For any real number $x > k$,*

$$\left|\sum_{p > x} \frac{\chi(p)}{p}\right| < \frac{2}{\log x}\left\{2\sqrt{k}\log k\left(\left|\frac{L'}{L}(1;\chi)\right| + 1\right) + E - \gamma\right\}.$$

Proof.

(i) The result follows from Qiu's improvement of the Polyá-Vinogradov inequality (see [3, p. 392]).

(ii) Abel summation and (i).

(iii) Let $\chi \neq \chi_0$ be a Dirichlet character modulo $k \geq 37$ and $x > k$ be any real number.

(a) Since $\chi(\mu * \mathbf{1}) = \varepsilon$ where $\varepsilon(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{otherwise} \end{cases}$ and $\mathbf{1}(n) = 1$, we get:

$$\sum_{d \leq x} \frac{\mu(d)\chi(d)}{d} \sum_{m \leq x/d} \frac{\chi(m)}{m} = 1$$

and hence, since $\chi \neq \chi_0$,

$$\sum_{d \leq x} \frac{\mu(d)\chi(d)}{d} = \frac{1}{L(1;\chi)} \left(\sum_{d \leq x} \frac{\mu(d)\chi(d)}{d} \sum_{m > x/d} \frac{\chi(m)}{m} + 1 \right)$$

and thus, using (ii),

$$\left| \sum_{d \leq x} \frac{\mu(d)\chi(d)}{d} \right| \leq \frac{\frac{9}{5}\sqrt{k}\log k + 1}{|L(1;\chi)|} < \frac{2\sqrt{k}\log k}{|L(1;\chi)|}. \tag{4.1}$$

(b) Since $\log = \Lambda * \mathbf{1}$, we get:

$$\begin{aligned} \sum_{n \leq x} \frac{\chi(n)\Lambda(n)}{n} &= \sum_{d \leq x} \frac{\mu(d)\chi(d)}{d} \sum_{m \leq x/d} \frac{\chi(m)\log m}{m} \\ &= \left(\sum_{d \leq x/e} + \sum_{x/e < d \leq x} \right) \sum_{m \leq x/d} \frac{\chi(m)\log m}{m} \end{aligned}$$

$$= \sum_{d \le x/e} \frac{\mu(d)\chi(d)}{d} \sum_{m \le x/d} \frac{\chi(m)\log m}{m} + \frac{\chi(2)\log 2}{2} \sum_{x/e < d \le x} \frac{\mu(d)\chi(d)}{d}$$

$$= -L'(1;\chi) \sum_{d \le x/e} \frac{\mu(d)\chi(d)}{d} - \sum_{d \le x/e} \frac{\mu(d)\chi(d)}{d} \sum_{m > x/d} \frac{\chi(m)\log m}{m} + \frac{\chi(2)\log 2}{2} \sum_{x/e < d \le x} \frac{\mu(d)\chi(d)}{d}$$

and, by using (ii), (4.1) and the trivial bound for the third sum, we get:

$$\begin{aligned} (4.2) \quad & \left| \sum_{n \le x} \frac{\chi(n)\Lambda(n)}{n} \right| \\ & < \sqrt{k}\log k \left\{ 2\left|\frac{L'}{L}(1;\chi)\right| + \frac{9}{5x}\sum_{d \le x/e} \log\left(\frac{x}{d}\right) + \frac{\log 2}{2}\left(1+\frac{e}{x}\right) \right\} \\ & \le \sqrt{k}\log k \left\{ 2\left|\frac{L'}{L}(1;\chi)\right| + \frac{18}{5e} + \frac{\log 2}{2}\left(1+\frac{e}{37}\right) \right\} \\ & < 2\sqrt{k}\log k \left\{ \left|\frac{L'}{L}(1;\chi)\right| + 1 \right\} \end{aligned}$$

since $x > q \ge 37$.

(c) By Abel summation, we get:

$$\left|\sum_{p>x}\frac{\chi(p)}{p}\right| \le \frac{2}{\log x}\max_{t\ge x}\left|\sum_{p\le t}\frac{\chi(p)\log p}{p}\right|.$$

Moreover,

$$\sum_{p\le t}\frac{\chi(p)\log p}{p} = \sum_{n\le t}\frac{\chi(n)\Lambda(n)}{n} - \sum_{\substack{p \\ p^\alpha\le t}}\sum_{\alpha=2}^{\infty}\frac{\chi(p^\alpha)\log p}{p^\alpha}$$

and then:

$$\begin{aligned}\left|\sum_{p\le t}\frac{\chi(p)\log p}{p}\right| &\le \left|\sum_{n\le t}\frac{\chi(n)\Lambda(n)}{n}\right| + \sum_{p}\log p\sum_{\alpha=2}^{\infty}\frac{1}{p^\alpha} \\ &= \left|\sum_{n\le t}\frac{\chi(n)\Lambda(n)}{n}\right| + E - \gamma \\ &< 2\sqrt{k}\log k\left\{\left|\frac{L'}{L}(1;\chi)\right| + 1\right\} + E - \gamma\end{aligned}$$

by (4.2). This concludes the proof of Lemma 4.1. □

5. Mean Value Estimates of Dirichlet $L-$functions

Lemma 5.1.

(i) *For any positive integers* j,k,

$$\sum_{\substack{n=1\\(n,k)=1}}^{jk}\frac{1}{n}=\frac{\varphi(k)}{k}\left\{\log(jk)+\gamma+\sum_{p\mid k}\frac{\log p}{p-1}\right\}+\frac{c_0(j,k)\,2^{\omega(k)}}{jk}$$

where $\omega(k):=\sum_{p\mid k}1$ *and* $|c_0(j,k)|\le 1$.

(ii) *For any positive integer* $k\ge 9$,

$$\left(\frac{k}{\varphi(k)}\right)^2\zeta(2)\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)+2\gamma_1+\gamma+\frac{\pi^2}{3}-\left(\log k+\sum_{p\mid k}\frac{\log p}{p-1}\right)^2\le 0.$$

(iii) *For any positive integer* $k\ge 9$,

$$\prod_{\chi\ne\chi_0}|L(1;\chi)|^{1/\varphi(k)}\le\sqrt{\zeta(2)}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)^{\frac12}.$$

Proof.

(i)

$$\sum_{\substack{n=1\\(n,k)=1}}^{jk}\frac{1}{n}=\sum_{d\mid k}\frac{\mu(d)}{d}\sum_{n\le jk/d}\frac{1}{n}=\sum_{d\mid k}\frac{\mu(d)}{d}\left(\log\left(\frac{jk}{d}\right)+\gamma+\frac{\varepsilon(d)\,d}{jk}\right)$$

where $|\varepsilon(d)| \leq 1$ and hence:

$$\sum_{\substack{n=1 \\ (n,k)=1}}^{jk} \frac{1}{n} = \{\log(jk) + \gamma\} \sum_{d|k} \frac{\mu(d)}{d} - \sum_{d|k} \frac{\mu(d)\log d}{d} + \frac{1}{jk} \sum_{d|k} \varepsilon(d)\mu(d)$$

and we conclude by noting that

$$\sum_{d|k} \frac{\mu(d)}{d} = \frac{\varphi(k)}{k},$$

$$\sum_{d|k} \frac{\mu(d)\log d}{d} = -\frac{\varphi(k)}{k} \sum_{p|k} \frac{\log p}{p-1}$$

and

$$\left| \sum_{d|k} \varepsilon(d)\mu(d) \right| \leq \sum_{d|k} \mu^2(d) = 2^{\omega(k)}.$$

(ii) Define

$$A(k) := \left(\frac{k}{\varphi(k)}\right)^2 \zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) + 2\gamma_1 + \gamma + \frac{\pi^2}{3} - \left(\log k + \sum_{p|k} \frac{\log p}{p-1}\right)^2.$$

Using [1] we check the inequality for $9 \leq k \leq 513$ and then suppose $k \geq 514$. Since

$$\frac{k}{\varphi(k)} = \prod_{p|k} \frac{p}{p-1} \leq \prod_{p|k} p^{\frac{1}{p-1}}$$

we have taking logarithms

$$\sum_{p|k} \frac{\log p}{p-1} \geq \log\left(\frac{k}{\varphi(k)}\right) \geq \log\left(\frac{k}{k-1}\right)$$

and from the inequality ([4])

$$\frac{k}{\varphi(k)} < e^{\gamma} \log\log k + \frac{2.50637}{\log\log k}$$

valid for any integer $k \geq 3$, we obtain

$$A(k) \leq \zeta(2)\left(e^{\gamma}\log\log k + \frac{2.50637}{\log\log k}\right)^2 + 2\gamma_1 + \gamma + \frac{\pi^2}{3} - \left(\log\left(\frac{k^2}{k-1}\right)\right)^2 < 0$$

if $k \geq 514$.

(iii) First,

$$\sum_{\chi \neq \chi_0} |L(1;\chi)|^2 = \lim_{N \to \infty} S(N)$$

where

$$S(N) := \sum_{m,n=1}^{Nk} \frac{\chi(n)\overline{\chi}(m)}{nm} - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2 .$$

Following a standard argument, we have using (i):

$$\begin{aligned}
&S(N) \\
&= \varphi(k) \sum_{\substack{m \neq n=1 \\ m \equiv n \pmod k \\ (n,k)=(m,k)=1}}^{Nk} \frac{1}{mn} - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2 \\
&= \varphi(k) \sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n^2} + \varphi(k) \sum_{\substack{m \neq n=1 \\ m \equiv n \pmod k \\ (n,k)=(m,k)=1}}^{Nk} \frac{1}{mn} - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2 \\
&\leq \varphi(k)\zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \\
&\qquad + 2\varphi(k) \sum_{j=1}^{N} \sum_{\substack{n=1 \\ (n,k)=1}}^{(N-j)k} \frac{1}{n(n+jk)} - \left(\sum_{\substack{n=1 \\ (n,k)=1}}^{Nk} \frac{1}{n} \right)^2
\end{aligned}$$

$$= \varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)$$
$$+\frac{2\varphi(k)}{k}\sum_{j=1}^{N}\frac{1}{j}\left(\sum_{\substack{n=1\\(n,k)=1}}^{(N-j)k}\frac{1}{n}-\sum_{\substack{n=1+jk\\(n,k)=1}}^{Nk}\frac{1}{n}\right)-\left(\sum_{\substack{n=1\\(n,k)=1}}^{Nk}\frac{1}{n}\right)^2$$
$$\leq \varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)+\frac{2\varphi(k)}{k}\sum_{j=1}^{N}\frac{1}{j}\sum_{\substack{n=1\\(n,k)=1}}^{jk}\frac{1}{n}-\left(\sum_{\substack{n=1\\(n,k)=1}}^{Nk}\frac{1}{n}\right)^2$$
$$= \varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)$$
$$+\frac{2\varphi(k)}{k}\sum_{j=1}^{N}\frac{1}{j}\left(\frac{\varphi(k)}{k}\left\{\log(jk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right\}+\frac{c_0(j,k)\,2^{\omega(k)}}{jk}\right)$$
$$-\left(\frac{\varphi(k)}{k}\left\{\log(Nk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right\}+\frac{c_0(N,k)\,2^{\omega(k)}}{Nk}\right)^2.$$

We now neglect the dependance of c_0 in k. Since

$$\sum_{m=1}^{M}\frac{1}{m}=\log M+\gamma+\frac{c_1(M)}{M}$$

and

$$\sum_{m=1}^{M}\frac{\log m}{m}=\frac{(\log M)^2}{2}+\gamma_1+\frac{c_2(M)\log M}{M},$$

where $0<c_1(M)\leq\frac{1}{2}$ and $|c_2(M)|\leq 1$, we get:

$$\begin{aligned}
S(N) &\leq \varphi(k)\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)\\
&\quad+\left(\frac{\varphi(k)}{k}\right)^2\left\{(\log N)^2+2\gamma_1+\frac{2c_2(N)\log N}{N}\right.\\
&\quad+2\left(\log k+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)\left(\log N+\gamma+\frac{c_1(N)}{N}\right)\\
&\quad\left.-\left(\log(Nk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)^2\right\}\\
&\quad+\frac{2^{\omega(k)+1}\varphi(k)}{k^2}\left\{\sum_{j=1}^{N}\frac{c_0(j)}{j^2}\right.\\
&\quad\left.-\frac{c_0(N)}{N}\left(\log(Nk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)\right\}-\frac{2^{2\omega(k)}c_0^2(N)}{N^2k^2}
\end{aligned}$$

$$= \varphi(k)\,\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)$$

$$+\left(\frac{\varphi(k)}{k}\right)^2\left\{2\gamma_1+\gamma-\left(\log k+\sum_{p|k}\frac{\log p}{p-1}\right)^2\right.$$

$$+\frac{2c_1(N)}{N}\left(\log k+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)+\left.\frac{2c_2(N)\log N}{N}\right\}$$

$$+\frac{2^{\omega(k)+1}\varphi(k)}{k^2}\left\{\sum_{j=1}^{N}\frac{c_0(j)}{j^2}\right.$$

$$\left.-\frac{c_0(N)}{N}\left(\log(Nk)+\gamma+\sum_{p|k}\frac{\log p}{p-1}\right)\right\}-\frac{2^{2\omega(k)}c_0^2(N)}{N^2k^2}$$

and then

$$\lim_{N\to\infty} S(N)\leq \varphi(k)\,\zeta(2)\prod_{p|k}\left(1-\frac{1}{p^2}\right)$$

$$+\left(\frac{\varphi(k)}{k}\right)^2\left\{2\gamma_1+\gamma-\left(\log k+\sum_{p|k}\frac{\log p}{p-1}\right)^2\right\}$$

$$+\frac{2^{\omega(k)}\varphi(k)\,\pi^2}{3k^2}$$

and the inequality $2^{\omega(k)} \leq \varphi(k)$ (valid for any integer $k \geq 3$ and $\neq 6$) implies

$$\begin{aligned}\lim_{N\to\infty} S(N) &\leq \varphi(k)\,\zeta(2)\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)\\ &\qquad + \left(\frac{\varphi(k)}{k}\right)^2 \left\{2\gamma_1+\gamma+\frac{\pi^2}{3}-\left(\log k+\sum_{p\mid k}\frac{\log p}{p-1}\right)^2\right\}\\ &= (\varphi(k)-1)\,\zeta(2)\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)\\ &\qquad + \left(\frac{\varphi(k)}{k}\right)^2 \left\{\left(\frac{k}{\varphi(k)}\right)^2 \zeta(2)\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)\right.\\ &\qquad \left. +2\gamma_1+\gamma+\frac{\pi^2}{3}-\left(\log k+\sum_{p\mid k}\frac{\log p}{p-1}\right)^2\right\}\\ &\leq (\varphi(k)-1)\,\zeta(2)\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)\end{aligned}$$

if $k \geq 9$ by (ii). Hence

$$\frac{1}{\varphi(k)-1}\sum_{\chi\neq\chi_0}|L(1;\chi)|^2 \leq \zeta(2)\prod_{p\mid k}\left(1-\frac{1}{p^2}\right).$$

Now the IAG inequality implies:

$$\prod_{\chi \neq \chi_0} |L(1;\chi)|^{\frac{1}{\varphi(k)}} = \exp\left\{ \frac{1}{2\varphi(k)} \sum_{\chi \neq \chi_0} \log |L(1;\chi)|^2 \right\}$$

$$\leq \exp\left\{ \frac{\varphi(k)-1}{2\varphi(k)} \log\left(\frac{1}{\varphi(k)-1} \sum_{\chi \neq \chi_0} |L(1;\chi)|^2 \right) \right\}$$

$$\leq \left(\zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \right)^{\frac{\varphi(k)-1}{2\varphi(k)}}$$

$$\leq \left(\zeta(2) \prod_{p|k} \left(1 - \frac{1}{p^2}\right) \right)^{\frac{1}{2}}.$$

□

6. Proof of the Theorem

Lemma 6.1. *If χ_0 is the principal character modulo k and if $x > k$, then:*

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-\chi_0(p)} < \frac{e^{\gamma} \varphi(k)\, \delta(x)}{k} \cdot \log x,$$

where δ is the function defined in (1.2).

Proof. Since $x > k$,

$$\prod_{\substack{p \le x \\ p | k}} \left(1 - \frac{1}{p}\right) = \prod_{p | k} \left(1 - \frac{1}{p}\right) = \frac{\varphi(k)}{k}$$

and then

$$\begin{aligned} \prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-\chi_0(p)} &= \prod_{\substack{p \le x \\ p \nmid k}} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1} \prod_{\substack{p \le x \\ p | k}} \left(1 - \frac{1}{p}\right) \\ &= \frac{\varphi(k)}{k} \prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1} \end{aligned}$$

and we use (1.1) . □

Proof of the theorem. Let $1 \le l \le k$ be positive integers satisfying $(k,l) = 1$ and $k \ge 37$, and x be a positive real number such that $x > k$. We have:

$$\prod_{\substack{p \le x \\ p \equiv l \,(\mathrm{mod}\, k)}} \left(1 - \frac{1}{p}\right)^{-\varphi(k)} = \prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-\chi_0(p)} \cdot \prod_{\chi \ne \chi_0} \left(\prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-\chi(p)} \right)^{\overline{\chi}(l)}$$
$$:= \Pi_1 \times \Pi_2$$

with $\Pi_1 < \dfrac{e^{\gamma} \varphi(k)\, \delta(x)}{k} \cdot \log x$ by Lemma 6.1. Moreover,

$$\begin{aligned}
\Pi_2 &= \exp \left\{ \sum_{\chi \ne \chi_0} \overline{\chi}(l) \left(- \sum_{p \le x} \chi(p) \log \left(1 - \frac{1}{p}\right) \right) \right\} \\
&= \exp \left(\sum_{\chi \ne \chi_0} \overline{\chi}(l) \sum_{p \le x} \sum_{\alpha = 1}^{\infty} \frac{\chi(p)}{\alpha p^{\alpha}} \right) \\
&= \exp \left\{ \sum_{\chi \ne \chi_0} \overline{\chi}(l) \left(\sum_{p \le x} \frac{\chi(p)}{p} + \sum_{p \le x} \sum_{\alpha = 2}^{\infty} \frac{\chi(p)}{\alpha p^{\alpha}} \right) \right\}
\end{aligned}$$

and if $\chi \ne \chi_0$, we have

$$\begin{aligned}
L(1;\chi) &= \prod_{p} \left(1 - \frac{\chi(p)}{p}\right)^{-1} \\
&= \exp \left(\sum_{p \le x} \frac{\chi(p)}{p} + \sum_{p > x} \frac{\chi(p)}{p} + \sum_{p} \sum_{\alpha = 2}^{\infty} \frac{\chi(p^{\alpha})}{\alpha p^{\alpha}} \right)
\end{aligned}$$

and thus

$$\Pi_2 = \prod_{\chi \neq \chi_0} L(1;\chi)^{\overline{\chi}(l)} \times \exp\left\{ \sum_{\chi \neq \chi_0} \overline{\chi}(l) \left(-\sum_{p>x} \frac{\chi(p)}{p} + \sum_{p \leq x} \sum_{\alpha=2}^{\infty} \frac{\chi(p)}{\alpha p^{\alpha}} - \sum_{p} \sum_{\alpha=2}^{\infty} \frac{\chi(p^{\alpha})}{\alpha p^{\alpha}} \right) \right\}$$

and hence

$$|\Pi_2| \leq \prod_{\chi \neq \chi_0} |L(1;\chi)| \cdot \exp\left\{ \sum_{\chi \neq \chi_0} \left| \sum_{p>x} \frac{\chi(p)}{p} \right| + 2(\varphi(k)-1) \sum_{p} \sum_{\alpha=2}^{\infty} \frac{1}{\alpha p^{\alpha}} \right\}$$
$$= e^{2(\varphi(k)-1)(\gamma-B)} \prod_{\chi \neq \chi_0} |L(1;\chi)| \cdot \exp\left\{ \sum_{\chi \neq \chi_0} \left(\left| \sum_{p>x} \frac{\chi(p)}{p} \right| \right) \right\}$$

and we use Lemma 4.1 (iii) and Lemma 5.1 (iii). We conclude the proof by noting that, if $x > 37$, $\frac{\delta(x)}{e^{2(\gamma-B)}} < 1$. □

References

[1] PARI/GP, Available by anonymous ftp from the URL: ftp://megrez.math.u-bordeaux.fr/pub/pari.

[2] E. GROSSWALD, Some number theoretical products, *Rev. Colomb. Mat.*, **21** (1987), 231–242.

[3] D.S. MITRINOVIĆ AND J. SÁNDOR (in cooperation with B. CRSTICI), *Handbook of Number Theory*, Kluwer Acad. Publishers, ISBN: 0-7923-3823-5.

[4] J.B. ROSSER AND L. SCHŒNFELD, Approximate formulas for some functions of prime numbers, *Illinois J. Math.*, **6** (1962), 64–94.

[5] K.S. WILLIAMS, Mertens' theorem for arithmetic progressions, *J. Number Theory*, **6** (1974), 353–359.