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SOME RESULTS ON A GENERALIZED USEFUL INFORMATION MEASURE

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ABSTRACT. A parametric mean length is defined as the quantity

$$_{\alpha\beta}L_u = \frac{\alpha}{\alpha-1}\left[1 - \sum P_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}\right],$$

$$\text{where } \alpha \neq 1, \sum p_i = 1$$

this being the useful mean length of code words weighted by utilities, u_i. Lower and upper bounds for $_{\alpha\beta}L_u$ are derived in terms of useful information for the incomplete power distribution, p^{β}.

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1. INTRODUCTION

Consider the following model for a random experiment S,

$$S_N = [E; P; U]$$

where $E = (E_1, E_2, \ldots, E_n)$ is a finite system of events happening with respective probabilities $P = (p_1, p_2, \ldots, p_N)$, $p_i \geq 0$, and $\sum p_i = 1$ and credited with utilities $U = (u_1, u_2, \ldots, u_N)$,

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$u_i > 0$, $i = 1, 2, \dots, N$. Denote the model by E, where

$$E = \begin{bmatrix} E_1 \; E_2 \; \cdots \; E_N \\ p_1 \; p_2 \; \cdots \; p_N \\ u_1 \; u_2 \; \cdots \; u_N \end{bmatrix} \tag{1.1}$$

We call (1.1) a Utility Information Scheme (UIS). Belis and Guiasu [3] proposed a measure of information called 'useful information' for this scheme, given by

$$H(U;P) = -\sum u_i p_i \log p_i, \tag{1.2}$$

where $H(U;P)$ reduces to Shannon's [8] entropy when the utility aspect of the scheme is ignored i.e., when $u_i = 1$ for each i. Throughout the paper, $\sum$ will stand for $\sum_{i=1}^{N}$ unless otherwise stated and logarithms are taken to base D $(D > 1)$.

Guiasu and Picard [5] considered the problem of encoding the outcomes in (1.1) by means of a prefix code with codewords $w_1, w_2, \dots, w_N$ having lengths $n_1, n_2, \dots, n_N$ and satisfying Kraft's inequality [4]

$$\sum_{i=1}^{N} D^{-n_i} \leq 1, \tag{1.3}$$

where D is the size of the code alphabet. The useful mean length L_u of code was defined as

$$L_u = \frac{\sum u_i n_i p_i}{\sum u_i p_i} \tag{1.4}$$

and the authors obtained bounds for it in terms of $H(U;P)$.

Longo [8], Gurdial and Pessoa [6], Khan and Autar [7], Autar and Khan [2] have studied generalized coding theorems by considering different generalized measures of (1.2) and (1.4) under condition (1.3) of unique decipherability.

In this paper, we study some coding theorems by considering a new function depending on the parameters α and β and a utility function. Our motivation for studying this new function is that it generalizes some entropy functions already existing in the literature (see C. Arndt [1]). The function under study is closely related to Tsallis entropy which is used in physics.

2. CODING THEOREMS

Consider a function

$$_{\alpha\beta}H(U;P) = \frac{\alpha}{\alpha - 1}\left[1 - \left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}}\right], \tag{2.1}$$

where $\alpha > 0$ $(\neq 1)$, $\beta > 0$, $p_i \geq 0$, $i = 1, 2, \dots, N$ and $\sum p_i \leq 1$.

(i) When $\beta = 1$ and $\alpha \to 1$, (2.1) reduces to a measure of useful information for the incomplete distribution due to Belis and Guiasu [3].

(ii) When $u_i = 1$ for each i i.e., when the utility aspect is ignored, $\sum p_i = 1$, $\beta = 1$ and $\alpha \to 1$, the measure (2.1) reduces to Shannon's entropy [10].

(iii) When $u_i = 1$ for each i, the measure (2.1) becomes entropy for the β-power distribution derived from P studied by Roy [9]. We call $_{\alpha\beta}H(U;P)$ in (2.1) the generalized useful measure of information for the incomplete power distribution P^{β}.

Further consider,

$$ {}_{\alpha\beta}L_u = \frac{\alpha}{\alpha - 1}\left[1 - \sum P_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}} D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)}\right], \tag{2.2} $$

where $\alpha > 0\ (\neq 1)$, $\sum p_i \leq 1$.

(i) For $\beta = 1$, $u_i = 1$ for each i and $\alpha \to 1$, ${}_{\alpha\beta}L_u$ in (2.2) reduces to the useful mean length L_u of the code given in (1.4).

(ii) For $\beta = 1$, $u_i = 1$ for each i and $\alpha \to 1$, ${}_{\alpha\beta}L_u$ becomes the optimal code length defined by Shannon [10].

We establish a result, that in a sense, provides a characterization of ${}_{\alpha\beta}H(U;P)$ under the condition of unique decipherability.

Theorem 2.1. *For all integers $D > 1$*

$$ {}_{\alpha\beta}L_u \geq {}_{\alpha\beta}H(U;P) \tag{2.3} $$

under the condition (1.3). Equality holds if and only if

$$ n_i = -\log\left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right). \tag{2.4} $$

Proof. We use Hölder's [11] inequality

$$ \sum x_i y_i \geq \left(\sum x_i^p\right)^{\frac{1}{p}}\left(\sum y_i^q\right)^{\frac{1}{q}} \tag{2.5} $$

for all $x_i \geq 0$, $y_i \geq 0$, $i = 1, 2, \ldots, N$ when $P < 1\ (\neq 1)$ and $p^{-1} + q^{-1} = 1$, with equality if and only if there exists a positive number c such that

$$ x_i^p = c y_i^q. \tag{2.6} $$

Setting

$$ x_i = p_i^{\frac{\alpha\beta}{\alpha-1}}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha-1}} D^{-n_i}, $$

$$ y_i = p_i^{\frac{\alpha\beta}{1-\alpha}}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{1-\alpha}}, $$

$p = 1 - 1/\alpha$ and $q = 1 - \alpha$ in (2.5) and using (1.3) we obtain the result (2.3) after simplification for $\frac{\alpha}{\alpha-1} > 0$ as $\alpha > 1$. □

Theorem 2.2. *For every code with lengths $\{n_i\}$, $i = 1, 2, ..., N$, ${}_{\alpha\beta}L_u$ can be made to satisfy,*

$$ {}_{\alpha\beta}L_u \geq {}_{\alpha\beta}H(U;P) D^{\left(\frac{1-\alpha}{\alpha}\right)} + \frac{\alpha}{1-\alpha}\left[1 - D^{\left(\frac{1-\alpha}{\alpha}\right)}\right]. \tag{2.7} $$

Proof. Let n_i be the positive integer satisfying, the inequality

$$ -\log\left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) \leq n_i < -\log\left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) + 1. \tag{2.8} $$

Consider the intervals

$$ \delta_i = \left[-\log\left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right), -\log\left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) + 1\right] \tag{2.9} $$

of length 1. In every δ_i, there lies exactly one positive number $n_i$such that

$$0 < -\log\left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) \leq n_i < -\log\left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) + 1. \tag{2.10}$$

It can be shown that the sequence $\{n_i\}$, $i = 1, 2, \ldots, N$ thus defined, satisfies (1.3). From (2.10) we have

$$\begin{aligned} n_i &< -\log\left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) + 1 \\ \Rightarrow D^{-n_i} &< \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) D \\ \Rightarrow D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)} &< \left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right)^{\frac{1-\alpha}{\alpha}} D^{\frac{\alpha-1}{\alpha}} \end{aligned} \tag{2.11}$$

Multiplying both sides of (2.11) by $p_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\alpha\beta}}\right)^{\frac{1}{\alpha}}$, summing over $i = 1, 2, \ldots, N$ and simplifying, gives (2.7). □

Theorem 2.3. *For every code with lengths $\{n_i\}$, $i = 1, 2, ..., N$, of Theorem 2.1, ${}_{\alpha\beta}L_u$ can be made to satisfy*

$${}_{\alpha\beta}H(U; P) \leq {}_{\alpha\beta}L_u < {}_{\alpha\beta}H(U; P) + \frac{\alpha}{\alpha - 1}(1 - D) \tag{2.12}$$

Proof. Suppose

$$\overline{n_i} = -\log\left(\frac{u_i P_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) \tag{2.13}$$

Clearly $\overline{n_i}$ and $\overline{n_i} + 1$ satisfy 'equality' in Hölder's inequality (2.5). Moreover, $\overline{n_i}$ satisfies Kraft's inequality (1.3).

Suppose n_i is the unique integer between $\overline{n_i}$ and $\overline{n_i} + 1$, then obviously, n_i satisfies (1.3).

Since $\alpha > 0$ $(\neq 1)$, we have

$$\begin{aligned} &\sum p_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}} D^{\overline{n}_i(\alpha-1)/\alpha} \\ &\qquad \leq \sum p_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}} D^{n_i(\alpha-1)/\alpha} \\ &\qquad < D\left(\sum p_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}} D^{\overline{n}_i(\alpha-1)/\alpha}\right) \end{aligned} \tag{2.14}$$

Since,

$$\sum p_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}} D^{\overline{n}_i(\alpha-1)/\alpha} = \left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}}$$

Hence, (2.14) becomes

$$\left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}} \leq \sum p_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}} D^{-\overline{n}_i(\alpha-1)/\alpha} < D\left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}\right)^{\frac{1}{\alpha}}$$

which gives the result (2.12). □

REFERENCES

[1] C. ARNDT, *Information Measures- Information and its Description in Science and Engineering*, Springer, (2001) Berlin.

[2] R. AUTAR AND A.B. KHAN, On generalized useful information for incomplete distribution, *J. of Comb. Information and Syst. Sci.*, **14**(4) (1989), 187–191.

[3] M. BELIS AND S. GUIASU, A qualitative-quantitative measure of information in Cybernetics Systems, *IEEE Trans. Information Theory*, **IT-14** (1968), 593–594.

[4] A. FEINSTEIN, *Foundation of Information Theory*, McGraw Hill, New York, (1958).

[5] S. GUIASU AND C.F. PICARD, Borne infericutre de la Longuerur utile de certain codes, *C.R. Acad. Sci, Paris*, **273A** (1971), 248–251.

[6] GURDIAL AND F. PESSOA, On useful information of order α, *J. Comb. Information and Syst. Sci.*, **2** (1977), 158–162.

[7] A.B. KHAN AND R. AUTAR, On useful information of order α and β, *Soochow J. Math.*, **5** (1979), 93–99.

[8] G. LONGO, A noiseless coding theorem for sources having utilities, *SIAM J. Appl. Math.*, **30**(4) (1976), 739–748.

[9] L.K. ROY, Comparison of Renyi entropies of power distribution, *ZAMM*, **56** (1976), 217–218.

[10] C.E. SHANNON, A Mathematical Theory of Communication, *Bell System Tech-J.*, **27** (1948), 394–423, 623–656.

[11] O. SHISHA, *Inequalities*, Academic Press, New York, (1967).