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APPROXIMATION OF B-CONTINUOUS AND B-DIFFERENTIABLE FUNCTIONS BY GBS OPERATORS DEFINED BY INFINITE SUM

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ABSTRACT. In this paper we start from a class of linear and positive operators defined by infinite sum. We consider the associated GBS operators and we give an approximation of B-continuous and B-differentiable functions with these operators. Through particular cases, we obtain statements verified by the GBS operators of Mirakjan-Favard-Szász, Baskakov and Meyer-König and Zeller.

Key words and phrases: Linear positive operators, GBS operators, B-continuous and B-differentiable functions, approximation of B-continuous and B-differentiable functions by GBS operators.

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1. INTRODUCTION

In this section, we recall some notions and results which we will use in this article. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In the following, let X and Y be real intervals.

A function $f : X \times Y \to \mathbb{R}$ is called a B-continuous function in $(x_0, y_0) \in X \times Y$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta f[(x, y), (x_0, y_0)] = 0,$$

where

$$\Delta f[(x, y), (x_0, y_0)] = f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0)$$

denotes a so-called mixed difference of f.

A function $f : X \times Y \to \mathbb{R}$ is called a B-continuous function on $X \times Y$ if and only if it is B-continuous in any point of $X \times Y$.

A function $f : X \times Y \to \mathbb{R}$ is called a B-differentiable function in $(x_0, y_0) \in X \times Y$ if and only if it exists and if the limit is finite

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta f[(x, y), (x, y_0)]}{(x - x_0)(y - y_0)}.$$

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This limit is called the B-differential of f in the point (x_0, y_0) and is noted by $D_B f(x_0, y_0)$.

A function $f : X \times Y \to \mathbb{R}$ is called a B-differentiable function on $X \times Y$ if and only if it is B-differentiable in any point of $X \times Y$.

The definition of B-continuity and B-differentiability was introduced by K. Bögel in the papers [8] and [9].

The function $f : X \times Y \to \mathbb{R}$ is B-bounded on $X \times Y$ if and only if there exists $k > 0$ so that $|\Delta f[(x, y), (s, t)]| \leq k$ for any $(x, y), (s, t) \in X \times Y$.

We shall use the function sets $B(X \times Y) = \{f | f : X \times Y \to \mathbb{R},\ f$ bounded on $X \times Y\}$ with the usual sup-norm $\|\cdot\|_\infty$, $B_b(X \times Y) = \{f | f : X \times Y \to \mathbb{R},\ f$ is B-bounded on $X \times Y\}$, $C_b(X \times Y) = \{f | f : X \times Y \to \mathbb{R},\ f$ is B-continuous on $X \times Y\}$ and $D_b(X \times Y) = \{f | f : X \times Y \to \mathbb{R},\ f$ is B-differentiable on $X \times Y\}$.

Let $f \in B_b(X \times Y)$. The function $\omega_{\text{mixed}}(f; \cdot, \cdot) : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by

$$\omega_{\text{mixed}}(f; \delta_1, \delta_2) = \sup\{\Delta f[(x, y), (s, t)]| : |x - s| \leq \delta_1, |y - t| \leq \delta_2\}$$

for any $(\delta_1, \delta_2) \in [0, \infty) \times [0, \infty)$ is called the mixed modulus of smoothness.

Theorem 1.1. *Let X and Y be compact real intervals and $f \in B_b(X \times Y)$.*
Then $\lim_{\delta_1, \delta_2 \to 0} \omega_{mixed}(f; \delta_1, \delta_2) = 0$ if and only if $f \in C_b(X \times Y)$.

For any $x \in X$ consider the function $\varphi_x : X \to \mathbb{R}$, defined by $\varphi_x(t) = |t - x|$, for any $t \in X$. For additional information, see the following papers: [1], [3], [15] and [19].

Let $m \in \mathbb{N}$ and the operator $S_m : C_2([0, \infty)) \to C([0, \infty))$ defined for any function $f \in C_2([0, \infty))$ by

$$(S_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right), \tag{1.1}$$

for any $x \in [0, \infty)$, where $C_2([0, \infty)) = \left\{f \in C([0, \infty)) : \lim_{x \to \infty} \frac{f(x)}{1 + x^2}\right.$ exists and is finite$\left.\right\}$. The operators $(S_m)_{m \geq 1}$ are called the Mirakjan-Favard-Szász operators, introduced in 1941 by G. M. Mirakjan in the paper [13].

These operators were intensively studied by J. Favard in 1944 in the paper [11] and O. Szász in the paper [20].

From [18], the following three lemmas result.

Lemma 1.2. *For any $m \in \mathbb{N}$, we have that*

$$\left(S_m \varphi_x^2\right)(x) = \frac{x}{m}, \tag{1.2}$$

$$\left(S_m \varphi_x^4\right)(x) = \frac{3mx^2 + x}{m^3} \tag{1.3}$$

for any $x \in [0, \infty)$ and

$$\left(S_m \varphi_x^2\right)(x) \leq \frac{a}{m}, \tag{1.4}$$

$$\left(S_m \varphi_x^4\right)(x) \leq \frac{a(3a + 1)}{m^2} \tag{1.5}$$

for any $x \in [0, a]$, where $a > 0$.

Let $m \in \mathbb{N}$ and the operator $V_m : C_2([0,\infty)) \to C([0,\infty))$, defined for any function $f \in C_2([0,\infty))$ by

$$(V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right) \tag{1.6}$$

for any $x \in [0,\infty)$.

The operators $(V_m)_{m\geq 1}$ are called Baskakov operators, introduced in 1957 by V. A. Baskakov in the paper [5].

Lemma 1.3. *For any $m \in \mathbb{N}$, we have that*

$$\left(V_m \varphi_x^2\right)(x) = \frac{x(1+x)}{m}, \tag{1.7}$$

$$\left(V_m \varphi_x^4\right)(x) = \frac{3(m+2)x^4 + 6(m+2)x^3 + (3m+7)x^2 + x}{m^3} \tag{1.8}$$

for any $x \in [0,\infty)$ and

$$\left(V_m \varphi_x^2\right)(x) \leq \frac{a(1+a)}{m}, \tag{1.9}$$

$$\left(V_m \varphi_x^4\right)(x) \leq \frac{a(9a^3 + 18a^2 + 10a + 1)}{m^2} \tag{1.10}$$

for any $x \in [0,a]$, where $a > 0$.

W. Meyer-König and K. Zeller have introduced a sequence of linear positive operators in paper [12]. After a slight adjustment, given by E. W. Cheney and A. Sharma in [10], these operators take the form $Z_m : B([0,1)) \to C([0,1))$, defined for any function $f \in B([0,1))$ by

$$(Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k}\right), \tag{1.11}$$

for any $m \in \mathbb{N}$ and for any $x \in [0,1)$.

These operators are called the Meyer-König and Zeller operators.

In the following we consider $Z_m : C([0,1]) \to C([0,1])$, for any $m \in \mathbb{N}$.

Lemma 1.4. *For any $m \in \mathbb{N}$ and any $x \in [0,1]$, we have that*

$$\left(Z_m \varphi_x^2\right)(x) \leq \frac{x(1-x)^2}{m+1}\left(1 + \frac{2x}{m+1}\right) \tag{1.12}$$

and

$$\left(Z_m \varphi_x^2\right)(x) \leq \frac{2}{m}. \tag{1.13}$$

The inequality of Corollary 5 from [4], in the condition (1.14) becomes inequality (1.15). Inequality (1.16) is demonstrated in [16].

Theorem 1.5. *Let $L : C_b(X \times Y) \to B(X \times Y)$ be a linear positive operator and $UL : C_b(X \times Y) \to B(X \times Y)$ the associated GBS operator. Supposing that the operator L has the property*

$$\left(L(\cdot - x)^{2i}(* - y)^{2j}\right)(x,y) = \left(L(\cdot - x)^{2i}\right)(x,y)\left(L(* - y)^{2j}\right)(x,y) \tag{1.14}$$

for any $(x,y) \in X \times Y$ and any $i,j \in \{1,2\}$, where "$\cdot$" and "$*$" stand for the first and second variable. Then:

(*i*) *For any function* $f \in C_b(X \times Y)$, *any* $(x,y) \in X \times Y$ *and any* $\delta_1, \delta_2 > 0$, *we have that*

$$|f(x,y) - (ULf)(x,y)| \leq |f(x,y)||1 - (Le_{00})(x,y)| + \Big[(Le_{00})(x,y) + \delta_1^{-1}\sqrt{(L(\cdot - x)^2)(x,y)} + \delta_2^{-1}\sqrt{(L(* - y)^2)(x,y)} + \delta_1^{-1}\delta_2^{-1}\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)}\Big]\omega_{mixed}(f;\delta_1,\delta_2). \tag{1.15}$$

(*ii*) *For any* $f \in D_b(X \times Y)$ *with* $D_B f \in B(X \times Y)$, *any* $(x,y) \in X \times Y$ *and any* $\delta_1, \delta_2 > 0$, *we have that*

$$\begin{aligned} &|f(x,y) - (ULf)(x,y)| \\ &\leq |f(x,y)||1 - (Le_{00})(x,y)| + 3\|D_B f\|_\infty \sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)} \\ &\quad + \Big[\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)} \\ &\quad + \delta_1^{-1}\sqrt{(L(\cdot - x)^4)(x,y)(L(* - y)^2)(x,y)} \\ &\quad + \delta_2^{-1}\sqrt{(L(\cdot - x)^2)(x,y)(L(* - y)^4)(x,y)} \\ &\quad + \delta_1^{-1}\delta_2^{-1}(L(\cdot - x)^2)(x,y)(L(* - y)^2)(x,y)\Big]\omega_{mixed}(D_B f;\delta_1,\delta_2). \end{aligned} \tag{1.16}$$

2. PRELIMINARIES

Let $I, J, K \subset \mathbb{R}$ be intervals, $J \subset K$ and $I \cap J \neq \emptyset$. We consider the sequence of nodes $\left((x_{m,k})_{k\in\mathbb{N}_0}\right)_{m\geq 1}$ so that $x_{m,k} \in I \cap J$, $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ and the functions $\varphi_{m,k} : K \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$, for any $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $x \in J$.

Definition 2.1. *If* $m \in \mathbb{N}$, *we define the operator* $L_m^* : E(I) \to F(K)$ *by*

$$(L_m^* f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x) f(x_{m,k}) \tag{2.1}$$

for any function $f \in E(I)$ *and any* $x \in K$, *where* $E(I)$ *and* $F(K)$ *are subsets of the set of real functions defined on* I, *respectively on* K.

Proposition 2.1. *The operators* $(L_m^*)_{m\geq 1}$ *are linear and positive on* $E(I \cap J)$.

Proof. The proof follows immediately. □

Definition 2.2. *If* $m, n \in \mathbb{N}$, *the operator* $L_{m,n}^* : E(I \times I) \to F(K \times K)$ *defined for any function* $f \in E(I \times I)$ *and any* $(x,y) \in K \times K$ *by*

$$\left(L_{m,n}^* f\right)(x,y) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{m,k}(x)\varphi_{n,j}(y) f(x_{m,k}, x_{n,j}) \tag{2.2}$$

is called the bivariate operator of L^* *- type.*

Proposition 2.2. *The operators* $\left(L_{m,n}^*\right)_{m,n\geq 1}$ *are linear and positive on* $E[(I \times I) \cap (J \times J)]$.

Proof. The proof follows immediately. □

Definition 2.3. *If* $m, n \in \mathbb{N}$, *the operator* $UL_{m,n}^* : E(I \times I) \to F(K \times K)$ *defined for any function* $f \in E(I \times I)$ *and any* $(x,y) \in K \times K$ *by*

$$\left(UL_{m,n}^* f\right)(x,y) = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{m,k}(x)\varphi_{n,j}(y)\big[f(x_{m,k}, y) + f(x, x_{n,j}) - f(x_{m,k}, x_{n,j})\big] \tag{2.3}$$

is called a GBS operator of L^* *- type.*

3. MAIN RESULTS

Lemma 3.1. *For any $m, n \in \mathbb{N}$, $i, j \in \mathbb{N}_0$ and $(x, y) \in K \times K$, the identity*

$$\left(L^*_{m,n}(\cdot - x)^{2i}(* - y)^{2j}\right)(x, y) = \left(L^*_m(\cdot - x)^{2i}\right)(x)\left(L^*_n(* - y)^{2j}\right)(y) \tag{3.1}$$

holds.

Proof. We have that

$$\begin{aligned}\left(L^*_{m,n}(\cdot - x)^{2i}(* - y)^{2j}\right)(x, y) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{m,k}(x)\varphi_{n,j}(y)(x_{m,k} - x)^{2i}(x_{n,j} - y)^{2j} \\ &= \sum_{k=0}^{\infty} \varphi_{m,k}(x)(x_{m,k} - x)^{2i} \sum_{j=0}^{\infty} \varphi_{n,j}(y)(x_{n,j} - y)^{2j} \\ &= \left(L^*_m(\cdot - x)^{2i}\right)(x)\left(L^*_n(* - y)^{2j}\right)(y),\end{aligned}$$

so (3.1) holds. □

For the operators constructed in this section, we note that $\delta_m(x) = \sqrt{\left(L^*_m\varphi^2_x\right)(x)}$, $\delta_{m,x} = \sqrt{\left(L^*_m\varphi^4_x\right)(x)}$, where $x \in I \cap J$, $m \in \mathbb{N}$, $m \neq 0$.

Then, by taking Lemma 3.1 into account, Theorem 1.5 becomes:

Theorem 3.2.

(i) *For any function $f \in C_b(I \times I)$, any $(x, y) \in (I \times I) \cap (J \times J)$, any $m, n \in \mathbb{N}$, any $\delta_1, \delta_2 > 0$, we have that*

$$\begin{aligned}&|f(x, y) - (UL^*_{m,n}f)(x, y)| \\ &\quad \leq |f(x, y)||1 - (Le_{00})(x, y)| + \big((Le_{00})(x, y) + \delta_1^{-1}\delta_m(x) + \delta_2^{-1}\delta_n(y) \\ &\qquad + \delta_1^{-1}\delta_2^{-1}\delta_m(x)\delta_n(y)\big)\omega_{\textit{mixed}}(f; \delta_1, \delta_2).\end{aligned} \tag{3.2}$$

(ii) *For any function $f \in D_b(I \times I)$ with $D_Bf \in B(I \times I)$, any $(x, y) \in (I \times I) \cap (J \times J)$, any $m, n \in \mathbb{N}$, any $\delta_1, \delta_2 > 0$, we have that*

$$\begin{aligned}|f(x, y) - (UL^*f)(x, y)| &\leq |f(x, y)||1 - (Le_{00})(x, y)| \\ &+ 3\|D_Bf\|_\infty\delta_m(x)\delta_n(y) + \big[\delta_m(x)\delta_n(y) + \delta_1^{-1}\delta_{m,x}\delta_n(y) \\ &+ \delta_2^{-1}\delta_m(x)\delta_{n,y} + \delta_1^{-1}\delta_2^{-1}\delta^2_m(x)\delta^2_n(y)\big]\omega_{\textit{mixed}}(D_Bf; \delta_1, \delta_2).\end{aligned} \tag{3.3}$$

In the following, we give examples of operators and of the associated GBS operators.

Application 1. *If $I = J = K = [0, \infty)$, $E(I) = C_2([0, \infty))$, $F(K) = C([0, \infty))$, $\varphi_{m,k}(x) = e^{-mx}\frac{(mx)^k}{k!}$, $x_{m,k} = \frac{k}{m}$, $x \in [0, \infty)$, $m, k \in \mathbb{N}_0$, $m \neq 0$, then we obtain the Mirakjan-Favard-Szász operators.*

Theorem 3.3. *Let $a, b \in \mathbb{R}$, $a > 0$ and $b > 0$. Then:*

(i) *For any function $f \in C([0, \infty) \times [0, \infty))$, any $(x, y) \in [0, a] \times [0, b]$ and $m, n \in \mathbb{N}$, we have that*

$$|f(x, y) - (US_{m,n}f)(x, y)| \leq \left(1 + \sqrt{a}\right)\left(1 + \sqrt{b}\right)\omega_{\textit{mixed}}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right). \tag{3.4}$$

(ii) *For any function* $f \in D_b([0,\infty)\times[0,\infty)) \cap C([0,\infty)\times[0,\infty))$ *with* $D_B f \in B([0,a]\times[0,b])$*, any* $(x,y)\in[0,a]\times[0,b]$*, any* $m,n\in\mathbb{N}$*, we have that*

$$|f(x,y)-(US_{m,n}f)(x,y)| \leq \sqrt{ab}\Bigg[3\|D_Bf\|_\infty + \Big(1+\sqrt{3a+1} + \sqrt{3b+1}+\sqrt{ab}\Big)\omega_{mixed}\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\Bigg]\frac{1}{\sqrt{mn}}. \tag{3.5}$$

Proof. It results from Theorem 3.2, by choosing $\delta_1=\frac{1}{\sqrt{m}}$, $\delta_2=\frac{1}{\sqrt{n}}$ and Lemma 1.2. □

Theorem 3.4. *If* $f\in C([0,\infty)\times[0,\infty))$*, then the convergence*

$$\lim_{m,n\to\infty}(US_{m,n}f)(x,y)=f(x,y) \tag{3.6}$$

is uniform on any compact $[0,a]\times[0,b]$*, where* $a,b>0$*.*

Proof. It results from Theorem 1.1 and Theorem 3.3. □

Application 2. *If* $I=J=K=[0,\infty)$*,* $E(I)=C_2([0,\infty))$*,* $F(K)=C([0,\infty))$*,* $\varphi_{m,k}(x)=(1+x)^{-m}\binom{m+k-1}{k}\left(\frac{x}{1+x}\right)^k$*,* $x_{m,k}=\frac{k}{m}$*,* $x\in[0,\infty)$*,* $m,k\in\mathbb{N}_0$*,* $m\neq 0$*, then we obtain the Baskakov operators.*

Theorem 3.5. *Let* $a,b\in\mathbb{R}$*,* $a>0$ *and* $b>0$*. Then:*

(i) *For any function* $f\in C([0,\infty)\times[0,\infty))$*, any* $(x,y)\in[0,a]\times[0,b]$ *and any* $m,n\in\mathbb{N}$*, we have that*

$$|f(x,y)-(UV_{m,n}f)(x,y)| \leq \left(1+\sqrt{a(1+a)}\right)\left(1+\sqrt{b(1+b)}\right)\omega_{mixed}\left(f;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right). \tag{3.7}$$

(ii) *For any function* $f \in D_b([0,\infty)\times[0,\infty)) \cap C([0,\infty)\times[0,\infty))$ *with* $D_B f \in B([0,a]\times[0,b])$*, any* $(x,y)\in[0,a]\times[0,b]$*, any* $m,n\in\mathbb{N}$*, we have that*

$$|f(x,y)-(UV_{m,n}f)(x,y)| \leq \sqrt{ab(1+a)(1+b)}\Bigg\{3\|D_B\|_\infty + \Big[1+\sqrt{9a^3+18a^2+10a+1}+\sqrt{9b^3+18b^2+10b+1} + \sqrt{ab(1+a)(1+b)}\Big]\omega_{mixed}\left(D_Bf;\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right)\Bigg\}\frac{1}{\sqrt{mn}}. \tag{3.8}$$

Proof. It results from Theorem 3.2, by choosing $\delta_1=\frac{1}{\sqrt{m}}$, $\delta_2=\frac{1}{\sqrt{n}}$ and Lemma 1.3. □

Theorem 3.6. *If* $f\in C([0,\infty)\times[0,\infty))$*, then the convergence*

$$\lim_{m,n\to\infty}(UV_{m,n}f)(x,y)=f(x,y) \tag{3.9}$$

is uniform on any compact $[0,a]\times[0,b]$*, where* $a,b>0$*.*

Proof. It results from Theorem 1.1 and Theorem 3.5. □

Application 3. *If* $I=J=K=[0,1]$*,* $E(I)=F(K)=C([0,1])$*,* $\varphi_{m,k}(x)=\binom{m+k}{k}(1-x)^{m+1}x^k$*,* $x_{m,k}=\frac{k}{m}$*,* $x\in[0,1]$*,* $m,k\in\mathbb{N}_0$*,* $m\neq 0$*, then we obtain the Meyer-König and Zeller operators.*

Theorem 3.7. *For any function* $f \in C([0,1] \times [0,1])$*, any* $(x,y) \in [0,1] \times [0,1]$ *and any* $m,n \in \mathbb{N}$*, we have that*

$$|f(x,y) - (UZ_{m,n}f)(x,y)| \leq (3+2\sqrt{2})\omega_{mixed}\left(f; \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right). \tag{3.10}$$

Proof. It results from Theorem 3.2, by choosing $\delta_1 = \frac{1}{\sqrt{m}}$, $\delta_2 = \frac{1}{\sqrt{n}}$ and Lemma 1.4. □

Theorem 3.8. *If* $f \in C([0,1] \times [0,1])$*, then the convergence*

$$\lim_{m,n\to\infty} (UZ_{m,n}f)(x,y) = f(x,y) \tag{3.11}$$

is uniform on $[0,1] \times [0,1]$.

Proof. It results from Theorem 1.1 and Theorem 3.7. □

REFERENCES

[1] I. BADEA, Modulul de continuitate în sens Bögel şi unele aplicaţii în aproximarea printr-un operator Bernstein, *Studia Univ. Babeş-Bolyai, Ser. Math.-Mech.*, **4**(2) (1973), 69–78 (Romanian).

[2] C. BADEA, I. BADEA AND H.H. GONSKA, A test function theorem and approximation by pseudopolynomials, *Bull. Austral. Math. Soc.*, **34** (1986), 53–64.

[3] C. BADEA, I. BADEA, C. COTTIN AND H.H. GONSKA, Notes on the degree of approximation of B-continuous and B-differentiable functions, *J. Approx. Theory Appl.*, **4** (1988), 95–108.

[4] C. BADEA AND C. COTTIN, Korovkin-type theorems for generalized boolean sum operators, *Colloquia Mathematica Societatis János Bolyai*, **58** (1990), Approximation Theory, Kecskemét (Hungary), 51–67.

[5] V.A. BASKAKOV, An example of a sequence of linear positive operators in the space of continuous functions, *Dokl. Acad. Nauk, SSSR*, **113** (1957), 249–251.

[6] D. BĂRBOSU, On the approximation by GBS operators of Mirakjan type, *BAM-1794/2000*, XCIV, 169–176.

[7] M. BECHER AND R.J. NESSEL, A global approximation theorem for Meyer-König and Zeller operators, *Math. Zeitschr.*, **160** (1978), 195–206.

[8] K. BÖGEL, Mehrdimensionale Differentiation von Funktionen mehrer Veränderlicher, *J. Reine Angew. Math.*, **170** (1934), 197–217.

[9] K. BÖGEL, Über mehrdimensionale Differentiation, Integration und beschränkte Variation, *J. Reine Angew. Math.*, **173** (1935), 5–29.

[10] E.W. CHENEY AND A. SHARMA, Bernstein power series, *Canadian J. Math.*, **16** (1964), 241–252.

[11] J. FAVARD, Sur les multiplicateur d'interpolation, *J. Math. Pures Appl.*, **23**(9) (1944), 219–247.

[12] W. MEYER-KÖNIG AND K. ZELLER, Bernsteinsche Potenzreihen, *Studia Math.*, **19** (1960), 89–94.

[13] G.M. MIRAKJAN, Approximation of continuous functions with the aid of polynomials, *Dokl. Acad. Nauk SSSR*, **31** (1941), 201–205.

[14] M.W. MÜLLER, Die Folge der Gammaoperatoren, *Dissertation Stuttgart*, 1967.

[15] M. NICOLESCU, Analiză matematică, **II**, *Editura Didactică şi Pedagogică, Bucureşti*, 1980 (Romanian).

[16] O.T. POP, Approximation of B-differentiable functions by GBS operators, *Anal. Univ. Oradea, Fasc. Matematica*, Tom **XIV** (2007), 15–31.

[17] O.T. POP, Approximation of B-continuous and B-differentiable functions by GBS operators defined by finite sum, *Facta Universitatis* (Niš), *Ser. Math. Inform.*, **22**(1) (2007), 33–41.

[18] O.T. POP, About some linear and positive operators defined by infinite sum, *Dem. Math.*, **39** (2006).

[19] D.D. STANCU, GH. COMAN, O. AGRATINI AND R. TRÎMBIŢAŞ, Analiză numerică şi teoria aproximării, **I**, *Presa Universitară Clujeană*, Cluj-Napoca, 2001 (Romanian).

[20] O. SZÁSZ, Generalization of S. Bernstein's polynomials to the infinite interval, *J. Research National Bureau of Standards*, **45** (1950), 239–245.