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ESTIMATION OF THE DERIVATIVE OF THE CONVEX FUNCTION BY MEANS OF ITS UNIFORM APPROXIMATION

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ABSTRACT. Suppose given the uniform approximation to the unknown convex function on a bounded interval. Starting from it the objective is to estimate the derivative of the unknown convex function. We propose the method of estimation that can be applied to evaluate optimal hedging strategies for the American contingent claims provided that the value function of the claim is convex in the state variable.

Key words and phrases: Convex function, Energy estimate, Left-derivative, Lower convex envelope, Hedging strategies the American contingent claims.

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1. INTRODUCTION

Consider the continuous convex function $f(x)$ on a bounded interval $[a,b]$ and suppose that its explicit analytical form is unknown to us, whereas there is the possibility to construct its continuous uniform approximation f_δ, where δ is a small parameter. Our objective consists in constructing the approximation to the unknown left-derivative $f'(x-)$ based on the known function $f_\delta(x)$. For this purpose we consider the lower convex envelope $\smile{f}_\delta(x)$ of $f_\delta(x)$, that is the maximal convex function, less than or equal to $f_\delta(x)$. Geometrically it represents the thread stretched from below over the graph of the function $f_\delta(x)$. Now our main idea consists of exploiting the left-derivative $\smile{f}'_\delta(x-)$ as a reasonable approximation to the unknown $f'(x-)$. The justification of this method of estimation is the main topic of this article.

These kinds of problems arise naturally in mathematical finance. Indeed, consider the value function $v(t,x)$ of the American contingent claim and suppose we have already constructed its uniform approximation $u(t,x)$, where $0 \le t \le T$, $0 < x \le L$ (for example, by discrete Markov Chain approximation developed by Kushner [1]). The problem is to find the estimation method

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of the partial derivative $\frac{\partial v}{\partial x}(t,x)$ that can be used to construct the optimal hedging strategy (we should note here that the explicit form of the value function $v(t,x)$ is typically unknown for one in most problems of the option pricing). It is well-known (see, for example El Karoui, Jeanblanc-Picque, Shreve [2]) that for a variety of practical problems in the pricing of American contingent claims that utilize one-dimensional diffusion models, the value function $v(t,x)$ is convex in the state variable and hence we can apply the above mentioned method of estimation. Thus, consider first for fixed $t \in [0,T]$ the lower convex envelope $\breve{u}(t,x)$ of the function $u(t,x)$ and then use its left-derivative $\frac{\partial \breve{u}}{\partial x}(t,x-)$ instead of the unknown $\frac{\partial v}{\partial x}(t,x)$. Then in this way we construct the approximation to the optimal hedging strategy.

2. THE ENERGY ESTIMATE FOR CONVEX FUNCTIONS

Consider the arbitrary finite convex function $f(x)$ on a bounded interval $[a,b]$. It is well known that it is continuous inside the interval, and at the endpoints a and b it has finite limits $f(a+)$ and $f(b-)$. Moreover it has finite left and right-derivatives $f'(x-)$ and $f'(x+)$ in the interval (a,b) (see, for example Schwartz [3, p. 205]).

We will use the following inequality several times (Schwartz [3, p. 205]) concerning convex function $f(x)$ and its left-derivative $f'(x-)$

$$f'(x_1-) \le \frac{f(x_2)-f(x_1)}{x_2-x_1} \le f'(x_2-) \tag{2.1}$$

for arbitrary x_1, x_2 with $a < x_1 < x_2 < b$.

Letting x_2 tend to b we get

$$f'(x_1-) \le \frac{f(b-)-f(x_1)}{b-x_1},$$

and similarly letting x_1 tend to a, we have

$$\frac{f(x_2)-f(a+)}{x_2-a} \le f'(x_2-).$$

From here we get

$$\frac{f(x)-f(a+)}{x-a} \le f'(x-) \le \frac{f(b-)-f(x)}{b-x} \quad \text{for} \quad a < x < b.$$

Multiplying the latter inequality by $(x-a)(b-x)$ we obtain the following crucial estimate

$$\begin{aligned}(b-x)\big(f(x)-f(a+)\big) &\le (x-a)(b-x)f'(x-)\\ &\le (x-a)\big(f(b-)-f(x)\big) \quad \text{for} \quad a<x<b.\end{aligned} \tag{2.2}$$

From this estimate we see, that the function

$$w_1(x) = (x-a)(b-x)f'(x-)$$

is bounded on the interval (a,b), moreover

$$w_1(a+) = 0, \quad w_1(b-) = 0,$$

hence it is natural to extend this function at the endpoints a and b by the relations

$$w_1(a) = 0, \quad w_1(b) = 0.$$

Thus we obtain the function $w_1(x)$ defined on the closed interval $[a,b]$, which is left-continuous with right-hand limits and is bounded on the interval $[a,b]$. Similarly for another finite convex function $\varphi(x)$ defined on $[a,b]$ we may denote

$$w_2(x) = (x-a)(b-x)\varphi'(x-) \quad \text{for} \quad a \le x \le b$$

Finally introduce the following function

$$\begin{aligned}(2.3)\qquad w(x) &= w_1(x) - w_2(x)\\ &= (x-a)(b-x)\big(f'(x-) - \varphi'(x-)\big) \text{ for } a \le x \le b\end{aligned}$$

and note that it is bounded, left-continuous (with right-hand limits) and also

$$w(a) = 0, \quad w(b) = 0.$$

Now our objective in this section is to bound the Riemann integral $\int_a^b w^2(x)dx$, it holds the key to our method of estimation of the derivative of the arbitrary convex function. This bound is given in the theorem below.

Theorem 2.1. *For arbitrary two finite convex functions $f(x)$ and $\varphi(x)$ defined on a closed interval $[a,b]$ the following energy estimate is valid*

$$\begin{aligned}(2.4)\quad \int_a^b (x-a)^2(b-x)^2\big(f'(x-) - \varphi'(x-)\big)^2\,dx \\ \le \frac{8}{9}\cdot\sqrt{3}\cdot \sup_{x\in(a,b)}\big|f(x)-\varphi(x)\big| \sup_{x\in(a,b)}\big|f(x)+\varphi(x)\big|(b-a)^3 \\ + \frac{4}{3}\left(\sup_{x\in(a,b)}\big|f(x)-\varphi(x)\big|\right)^2 (b-a)^3.\end{aligned}$$

Proof. The proof is lengthy and therefore divided in two stages. At the first stage we verify the validity of the statement for smooth (twice continuously differentiable) convex functions. At the second stage we approximate arbitrary finite convex functions inside the interval $[a,b]$ by smooth ones in an appropriate manner and afterwards pass on limit in the previously obtained estimate.

Thus, at first we assume that the convex functions $f(x)$ and $\varphi(x)$ are twice continuously differentiable on $[a,b]$ in which case we obviously have

$$f''(x) \ge 0, \quad \varphi''(x) \ge 0, \quad a < x < b.$$

Introduce the functions

$$u(x) = f(x) - \varphi(x), \quad v(x) = (x-a)^2(b-x)^2\big(f(x)-\varphi(x)\big).$$

Consider the following integral and use in it the integration by parts formula

$$\begin{aligned}\int_a^b u'(x)v'(x)dx &= u'(x)v(x)\Big|_a^b - \int_a^b v(x)u''(x)\,dx\\ &= -\int_a^b (x-a)^2(b-x)^2\big(f(x)-\varphi(x)\big)\big(f''(x)-\varphi''(x)\big)\,dx,\end{aligned}$$

as $v(a) = v(b) = 0$.

From here we get the estimate

$$\left|\int_a^b u'(x)v'(x)\,dx\right| \le \sup_{x\in[a,b]} |f(x)-\varphi(x)| \int_a^b (x-a)^2(b-x)^2\,|f''(x)-\varphi''(x)|\,dx$$

However, as pointed above $f''(x) \ge 0$, $\varphi''(x) \ge 0$, hence

$$\big|f''(x) - \varphi''(x)\big| \le f''(x) + \varphi''(x)$$

and from the previous estimate we obtain the bound

$$\left|\int_a^b u'(x)v'(x)\,dx\right| \le \sup_{x\in[a,b]}\left|f(x)-\varphi(x)\right|\int_a^b (x-a)^2(b-x)^2\big(f''(x)+\varphi''(x)\big)\,dx.$$

Next we have to transform the integral

$$\begin{aligned}\int_a^b &(x-a)^2(b-x)^2\big(f''(x)+\varphi''(x)\big)\,dx\\ &= (x-a)^2(b-x)^2\big(f'(x)+\varphi'(x)\big)\Big|_a^b\\ &\quad - \int_a^b \big((x-a)^2(b-x)^2\big)'\big(f'(x)+\varphi'(x)\big)\,dx\\ &= -\int_a^b 2(x-a)(b-x)(-2x+a+b)\big(f'(x)+\varphi'(x)\big)\,dx\\ &= -2(x-a)(b-x)(-2x+a+b)\big(f(x)+\varphi(x)\big)\Big|_a^b\\ &\quad + \int_a^b \big(2(x-a)(b-x)(-2x+a+b)\big)'\big(f(x)+\varphi(x)\big)\,dx.\end{aligned}$$

Therefore

$$\begin{aligned}\int_a^b &(x-a)^2(b-x)^2\big(f''(x)+\varphi''(x)\big)\,dx\\ &\le \sup_{x\in[a,b]}\left|f(x)+\varphi(x)\right|\int_a^b \left|\big(2(x-a)(b-x)(-2x+a+b)\big)'\right|dx\end{aligned}$$

Evaluating the last integral we get

$$\int_a^b \left|\big(2(x-a)(b-x)(-2x+a+b)\big)'\right|dx = \frac{4}{9}\cdot\sqrt{3}\cdot(b-a)^3.$$

Whence we come to the estimate

$$\left|\int_a^b u'(x)v'(x)\,dx\right| \le \frac{4}{9}\cdot\sqrt{3}\cdot\sup_{x\in[a,b]}\left|f(x)-\varphi(x)\right|\cdot\sup_{x\in[a,b]}\left|f(x)+\varphi(x)\right|\cdot(b-a)^3. \tag{2.5}$$

On the other hand

$$\begin{aligned}\int_a^b u'(x)v'(x)\,dx &= \int_a^b \big(f'(x)-\varphi'(x)\big)\left[(x-a)^2(b-x)^2\big(f(x)-\varphi(x)\big)\right]'dx\\ &= 2\int_a^b (x-a)(b-x)(-2x+a+b)\big(f(x)-\varphi(x)\big)\big(f'(x)-\varphi'(x)\big)\,dx\\ &\quad + \int_a^b (x-a)^2(b-x)^2\big(f'(x)-\varphi'(x)\big)^2\,dx.\end{aligned}$$

Therefore we get the equality

$$\begin{aligned}\int_a^b &(x-a)^2(b-x)^2\big(f'(x)-\varphi'(x)\big)^2\,dx\\ &= \int_a^b u'(x)v'(x)\,dx - \int_a^b (x-a)(b-x)\left(f'(x)-\varphi'(x)\right)\\ &\qquad\times 2(-2x+a+b)\left(f(x)-\varphi(x)\right)\,dx.\end{aligned} \tag{2.6}$$

Bound now the last term

$$\begin{aligned}
&\left|\int_a^b (x-a)(b-x)\big(f'(x)-\varphi'(x)\big)2(-2x+a+b)\big(f(x)-\varphi(x)\big)\,dx\right| \\
&\qquad \le \frac{1}{2}\int_a^b (x-a)^2(b-x)^2\big(f'(x)-\varphi'(x)\big)^2\,dx \\
&\qquad\qquad + 2\int_a^b (-2x+a+b)^2\big(f(x)-\varphi(x)\big)^2\,dx \\
&\qquad \le \frac{1}{2}\int_a^b (x-a)^2(b-x)^2\big(f'(x)-\varphi'(x)\big)^2\,dx \\
&\qquad\qquad + 2\left(\sup_{x\in[a,b]}\big|f(x)-\varphi(x)\big|\right)^2\frac{1}{3}(b-a)^3,
\end{aligned}$$

as

$$\int_a^b (-2x+a+b)^2\,dx = \frac{1}{3}(b-a)^3.$$

Therefore we obtain

$$\begin{aligned}
&\left|\int_a^b (x-a)(b-x)\big(f'(x)-\varphi'(x)\big)\cdot 2(-2x+a+b)\big(f(x)-\varphi(x)\big)\,dx\right| \\
&\qquad \le \frac{1}{2}\int_a^b (x-a)^2(b-x)^2\big(f'(x)-\varphi'(x)\big)^2\,dx \\
&\qquad\qquad + \frac{2}{3}\left(\sup_{x\in[a,b]}|f(x)-\varphi(x)|\right)^2\cdot(b-a)^3.
\end{aligned}\tag{2.7}$$

Finally, if we use the bounds (2.5) and (2.7) in the equality (2.6), we come to the following estimate

$$\begin{aligned}
&\int_a^b (x-a)^2(b-x)^2\left(f'(x)-\varphi'(x)\right)^2\,dx \\
&\qquad \le \frac{8}{9}\cdot\sqrt{3}\cdot\sup_{x\in[a,b]}|f(x)-\varphi(x)|\cdot\sup_{x\in[a,b]}|f(x)+\varphi(x)|\cdot(b-a)^3 \\
&\qquad\qquad + \frac{4}{3}\left(\sup_{x\in[a,b]}|f(x)-\varphi(x)|\right)^2\cdot(b-a)^3.
\end{aligned}\tag{2.8}$$

Let us pass to the second stage of the proof. Consider two arbitrary finite convex functions $f(x)$ and $\varphi(x)$ on the closed interval $[a,b]$. We have to construct the sequences of smooth convex functions $f_n(x)$ and $\varphi_n(x)$ approximating, respectively, the functions $f(x)$ and $\varphi(x)$ inside the interval $[a,b]$ in an appropriate manner.

For this purpose we will use the following smoothing function

$$\rho(x) = \begin{cases} c\cdot e^{\frac{1}{x(x-2)}} & \text{for } 0<x<2 \\ 0, & \text{otherwise} \end{cases}, \tag{2.9}$$

where the factor c is chosen to satisfy the equality

$$\int_0^2 \rho(x)\,dx = 1.$$

Define

$$f_n(x) = \int_a^b n \cdot \rho\big(n \cdot (x-y)\big) \cdot f(y)\, dy, \tag{2.10}$$
$$\varphi_n(x) = \int_a^b n \cdot \rho\big(n \cdot (x-y)\big) \cdot \varphi(y)\, dy,$$

where $n = 1, 2, \dots$ and $x \in (-\infty, +\infty)$.

For arbitrary fixed $\delta > 0$ consider the restriction of functions $f_n(x)$ and $\varphi_n(x)$ on the interval $[a+\delta, b-\delta]$ and let $n \geq \frac{4}{\delta}$. Then $n \cdot (x-a) \geq 4$ and $n \cdot (x-b) \leq 0$ for $x \in [a+\delta, b-\delta]$.

Perform in (2.10) the change of variable $z = n \cdot (x-y)$, then we'll have

$$f_n(x) = \int_{n\cdot(x-b)}^{n\cdot(x-a)} \rho(z) \cdot f\left(x - \frac{z}{n}\right)\, dz,$$
$$\varphi_n(x) = \int_{n\cdot(x-b)}^{n\cdot(x-a)} \rho(z) \cdot \varphi\left(x - \frac{z}{n}\right)\, dz.$$

But the function $\rho(z)$ is equal to zero outside the interval $(0,2)$ and hence we obtain

$$f_n(x) = \int_0^2 \rho(z) \cdot f\left(x - \frac{z}{n}\right)\, dz, \tag{2.11}$$
$$\varphi_n(x) = \int_0^2 \rho(z) \cdot \varphi\left(x - \frac{z}{n}\right)\, dz,$$

if $n \geq \frac{4}{\delta}$. From the definition (2.10) it is obvious, that the functions $f_n(x)$ and $\varphi_n(x)$ are infinitely differentiable, while the convexity of these functions simply follows from the representation (2.11).

Next we show the uniform convergence of the sequence $f_n(x)$ to $f(x)$ on the interval $[a+\delta, b-\delta]$ (similarly for $\varphi_n(x)$ to $\varphi(x)$). For this purpose we use the uniform continuity of the function $f(x)$ on the interval $[a+\frac{\delta}{2}, b-\delta]$. For fix $\varepsilon > 0$ there exists $\widehat{\delta}$ such that we have

$$\big|f(x_2) - f(x_1)\big| \leq \varepsilon \;\text{ if }\; |x_2 - x_1| < \widehat{\delta} \;\text{ and }\; x_1, x_2 \in \left[a + \frac{\delta}{2}, b - \delta\right].$$

Take $n \geq \max\{\frac{4}{\widehat{\delta}}, \frac{4}{\delta}\}$. Then for $0 \leq z \leq 2$ and $x \in [a+\delta, b-\delta]$ we get

$$\frac{z}{n} \leq \min\left\{\frac{\widehat{\delta}}{2}, \frac{\delta}{2}\right\}, \quad x - \frac{z}{n} \geq a + \delta - \frac{\delta}{2} = a + \frac{\delta}{2}.$$

Hence

$$\left| f\left(x - \frac{z}{n}\right) - f(x)\right| \leq \varepsilon \;\text{ for }\; n \geq \max\left\{\frac{4}{\widehat{\delta}}, \frac{4}{\delta}\right\}$$

and consequently

$$|f_n(x) - f(x)| = \left|\int_0^2 \rho(z) \cdot \left(f\left(x - \frac{z}{n}\right) - f(x)\right)\, dz\right| \leq \varepsilon$$

for $x \in [a+\delta, b-\delta]$ and $n \geq \max\{\frac{4}{\widehat{\delta}}, \frac{4}{\delta}\}$.

Thus we've shown the uniform convergence of the sequence $f_n(x)$ to $f(x)$ on the interval $[a+\delta, b-\delta]$.

Now we need to differentiate the relations (2.11). We'll use again the basic inequality (2.1) on convex functions. Take therein

$$x_1 = \left(x - \frac{z}{n}\right) - h, \quad x_2 = x - \frac{z}{n},$$

where $0 < h < \frac{\delta}{4}$.

We will have

$$f'\Big(\Big(x - \frac{z}{n} - h\Big) - \Big) \le \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h} \le f'\Big(\Big(x - \frac{z}{n}\Big) - \Big) \tag{2.12}$$

if $x \in [a+\delta, b-\delta]$, $0 \le z \le 2$, $0 < h < \frac{\delta}{4}$ and $n \ge \frac{4}{\delta}$.

Taking into account that the left-derivative of the convex function is nondecreasing and that

$$x - \frac{z}{n} - h \ge a + \frac{\delta}{4}, \quad x - \frac{z}{n} \le b - \delta$$

we get

$$f'\Big(\Big(a + \frac{\delta}{4}\Big) - \Big) \le \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h} \le f'((b-\delta)-). \tag{2.13}$$

It follows from here that the family of functions

$$\Phi_h^{n,x}(z) = \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h}$$

is uniformly bounded by the constant

$$c = \max\left(\left|f'\Big(\Big(a + \frac{\delta}{4}\Big) - \Big)\right|, \left|f'((b-\delta)-)\right|\right)$$

if $x \in [a+\delta, b-\delta]$, $0 \le z \le 2$, $0 < h < \frac{\delta}{4}$ and $n \ge \frac{4}{\delta}$.

We write from the representation (2.11)

$$\frac{f_n(x) - f_n(x-h)}{h} = \int_0^2 \rho(z) \cdot \frac{f(x - \frac{z}{n}) - f(x - \frac{z}{n} - h)}{h}\, dz.$$

Now letting h to zero and using the bounded convergence theorem we come to the following formula

$$f_n'(x) = \int_0^2 \rho(z) \cdot f'\Big(\Big(x - \frac{z}{n}\Big) - \Big)\, dz \tag{2.14}$$

for $x \in [a+\delta, b-\delta]$ and $n \ge \frac{4}{\delta}$.

From this formula it is easy to see, that for fixed $x \in [a+\delta, b-\delta]$ the sequence $f_n'(x)$ converges to the left-derivative $f'(x-)$. Indeed consider the difference

$$f_n'(x) - f'(x-) = \int_0^2 \rho(z) \cdot \left(f'\Big(\Big(x - \frac{z}{n}\Big) - \Big) - f'(x-)\right) dz,$$

where we assume that $n \ge \frac{4}{\delta}$ and choose arbitrary $\varepsilon > 0$. As the left-derivative $f'(x-)$ is left-continuous we can find $N(\varepsilon)$ such that (for $0 \le z \le 2$)

$$\left|f'\Big(\Big(x - \frac{z}{n}\Big) - \Big) - f'(x-)\right| \le \varepsilon \text{ if only } n > N(\varepsilon).$$

Hence we get

$$\left|f_n'(x) - f'(x-)\right| \le \int_0^2 \rho(z) \cdot \varepsilon\, dz = \varepsilon \text{ if } n > \max\left(\frac{4}{\delta}, N(\varepsilon)\right).$$

Thus for any fixed $x \in [a+\delta, b-\delta]$ we have

$$\lim_{n\to\infty} f_n'(x) = f'(x-),$$

and similarly for functions $\varphi_n'(x)$, $\varphi'(x-)$

$$\lim_{n\to\infty} \varphi_n'(x) = \varphi'(x-).$$

Next write the estimate (2.8) for functions $f_n(x)$, $\varphi_n(x)$ restricted to the interval $[a+\delta, b-\delta]$

$$\begin{aligned} (2.15)\quad \int_{a+\delta}^{b-\delta} &\big(x-(a+\delta)\big)^2 \cdot \big((b-\delta)-x\big)^2 \cdot \big(f_n'(x)-\varphi_n'(x)\big)^2\,dx \\ &\le \frac{8}{9}\cdot\sqrt{3}\cdot \sup_{x\in[a+\delta,b-\delta]} \big|f_n(x)-\varphi_n(x)\big| \\ &\qquad \times \sup_{x\in[a+\delta,b-\delta]} \big|f_n(x)+\varphi_n(x)\big|\cdot(b-a-2\cdot\delta)^3 \\ &\qquad\qquad + \frac{4}{3}\cdot\left(\sup_{x\in[a+\delta,b-\delta]} \big|f_n(x)-\varphi_n(x)\big|\right)^2\cdot(b-a-2\cdot\delta)^3. \end{aligned}$$

For $x\in[a+\delta,b-\delta]$, $0\le z\le 2$ and $n\ge\frac{4}{\delta}$ we have

$$f'\left(\left(a+\frac{\delta}{2}\right)-\right) \le f'\left(\left(x-\frac{z}{n}\right)-\right) \le f'((b-\delta)-),$$

multiplying this inequality by $\rho(z)$ and integrating by z over $(0,2)$ from the equality (2.14) we obtain

$$f'\left(\left(a+\frac{\delta}{2}\right)-\right) \le f_n'(x) \le f'((b-\delta)-).$$

Similarly for the functions $\varphi_n'(x)$

$$\varphi'\left(\left(a+\frac{\delta}{2}\right)-\right) \le \varphi_n'(x) \le \varphi'((b-\delta)-).$$

Hence the sequences of the functions $f_n'(x)$ and $\varphi_n'(x)$ are uniformly bounded on the interval $[a+\delta,b-\delta]$ for $n\ge\frac{4}{\delta}$. Thus we can apply the bounded convergence theorem in the left-hand side of the inequality (2.15) passing to limit when $n\to\infty$ and we get

$$\begin{aligned} (2.16)\quad \int_{a+\delta}^{b-\delta} &\big(x-(a+\delta)\big)^2 \cdot \big((b-\delta)-x\big)^2 \cdot \big(f'(x-)-\varphi'(x-)\big)^2\,dx \\ &\le \frac{8}{9}\cdot\sqrt{3}\cdot \sup_{x\in[a+\delta,b-\delta]} \big|f(x)-\varphi(x)\big| \\ &\qquad \times \sup_{x\in[a+\delta,b-\delta]} \big|f(x)+\varphi(x)\big|\cdot(b-a-2\cdot\delta)^3 \\ &\qquad\qquad + \frac{4}{3}\cdot\left(\sup_{x\in[a+\delta,b-\delta]} \big|f(x)-\varphi(x)\big|\right)^2\cdot(b-a-2\cdot\delta)^3. \end{aligned}$$

Finally it remains to pass onto limit when $\delta\to 0$ in the inequality (2.16). Introduce the following function

$$u_\delta(x) = \begin{cases} \chi_{(a+\delta,b-\delta]}(x)\cdot\left(\frac{x-a-\delta}{x-a}\right)^2\cdot\left(\frac{b-\delta-x}{b-x}\right)^2 & \text{for } a<x<b \\ 0 & \text{otherwise} \end{cases},$$

where $\chi_{(a+\delta,b-\delta]}(x)$ is the characteristic function of the interval $(a+\delta, b-\delta]$. Evidently

$$0 \le u_\delta(x) \le 1, \quad \lim_{\delta\downarrow 0} u_\delta(x) = \begin{cases} 1, & a < x < b \\ 0, & \text{otherwise} \end{cases}.$$

We remind now the definition (2.3) of the function $w(x)$ and the fact that it is a bounded function on the closed interval $[a, b]$ and we rewrite the inequality (2.16) in terms of the functions $w(x)$ and $u_\delta(x)$.

$$\begin{aligned} \int_a^b u_\delta(x) \cdot w^2(x)\,dx \le \frac{8}{9} \cdot \sqrt{3} \cdot \sup_{x\in[a+\delta,b-\delta]} \left|f(x) - \varphi(x)\right| \\ \times \sup_{x\in[a+\delta,b-\delta]} \left|f(x) + \varphi(x)\right| \cdot (b-a-2\cdot\delta)^3 \\ + \frac{4}{3} \cdot \left(\sup_{x\in[a+\delta,b-\delta]} \left|f(x)-\varphi(x)\right| \right)^2 \cdot (b-a-2\cdot\delta)^3. \end{aligned} \tag{2.17}$$

We use again the bounded convergence theorem in this inequality when $\delta \downarrow 0$ and at last get the desired energy estimate (2.4). □

3. THE MAIN RESULT

The following proposition is the basic result of this article though its proof is a simple consequence of the previous theorem

Theorem 3.1. *Let $f(x)$ be the unknown continuous convex function defined on the bounded interval $[a, b]$ and suppose we have at hand its some continuous uniform approximation $f_\delta(x)$. Consider the lower convex envelope $\breve{f}_\delta(x)$ of the function $f_\delta(x)$. Then for the unknown left-derivative $f'(x-)$ the following estimate through $\breve{f}'_\delta(x-)$ does hold*

$$\begin{aligned} \int_a^b (x-a)^2 \cdot (b-x)^2 \cdot \left(f'(x-) - \breve{f}'_\delta(x-)\right)^2 dx \\ \le \frac{8}{9} \cdot \sqrt{3} \cdot \sup_{x\in[a,b]} \left|f_\delta(x) - f(x)\right| \left(\sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f_\delta(x)| \right) \cdot (b-a)^3 \\ + \frac{4}{3} \left(\sup_{x\in[a,b]} \left|f_\delta(x) - f(x)\right| \right)^2 \cdot (b-a)^3. \end{aligned} \tag{3.1}$$

Proof. Introduce the notation

$$\sup_{x\in[a,b]} \left|f_\delta(x) - f(x)\right| = c_\delta.$$

It is clear that

$$f(x) - c_\delta \le f_\delta(x), \quad f_\delta(x) - c_\delta \le f(x), \ \text{ if } \ x \in [a, b].$$

Therefore we get that the convex function $f(x) - c_\delta$ is less or equal than $f_\delta(x)$ and hence

$$f(x) - c_\delta \le \breve{f}_\delta(x), \ \ x \in [a, b].$$

On the other hand we have

$$\breve{f}_\delta(x) - c_\delta \le f_\delta(x) - c_\delta \le f(x), \ \ x \in [a, b],$$

therefore

$$\left|\breve{f}_\delta(x) - f(x)\right| \le c_\delta, \quad x \in [a,b],$$

that is

$$\sup_{x\in[a,b]} \left|\breve{f}_\delta(x) - f(x)\right| \le \sup_{x\in[a,b]} \left|f_\delta(x) - f(x)\right|. \tag{3.2}$$

Denote $\sup\limits_{x\in[a,b]} |f_\delta(x)| = \widetilde{c}_\delta$, then obviously

$$-\widetilde{c}_\delta \le f_\delta(x) \le \widetilde{c}_\delta, \quad x \in [a,b]$$

and hence

$$-\widetilde{c}_\delta \le \breve{f}_\delta(x) \le f_\delta(x) \le \widetilde{c}_\delta, \quad x \in [a,b],$$

that is

$$\sup_{x\in[a,b]} \left|\breve{f}_\delta(x)\right| \le \sup_{x\in[a,b]} \left|f_\delta(x)\right|. \tag{3.3}$$

Take now $\breve{f}_\delta(x)$ instead of convex function $\varphi(x)$ in the formulation of Theorem 2.1 and use the inequalities (3.2) – (3.3) in the right-hand side of the estimate (2.4), then we directly come to the estimate (3.1). □

Remark 3.2. As the left and the right-hand derivatives of convex function coincide everywhere except on the countable set, Theorems 2.1 and 3.1 are obviously true for the right-derivatives instead of the left-ones.

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