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SOME CONSIDERATIONS ON THE MONOTONICITY PROPERTY OF POWER MEANS

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Abstract

If A is an isotonic linear functional and $f : [a, b] \rightarrow (0, \infty)$ is a monotone function then $Q(r, f) = (f^r(a) + f^r(b) - A(f^r))^{1/r}$ is increasing in r .

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1. Introduction

Let $0 < a \leq x_1 \leq x_2 \leq \dots \leq x_n \leq b$ and w_k ($1 \leq k \leq n$) be positive weights associated with these x_k and whose sum is unity. A Mc D. Mercer [3] proved the following variant of Jensen's inequality.

Theorem 1.1. *If f is a convex function on an interval containing the points x_k then*

$$(1.1) \quad f \left(a + b - \sum_{k=1}^n w_k x_k \right) \leq f(a) + f(b) - \sum_{k=1}^n w_k f(x_k).$$

The weighted power means $M_r(x, w)$ of the number x_i with weights w_i are defined as

$$M_r(x, w) = \left(\sum_{k=1}^n w_k x_k^r \right)^{\frac{1}{r}} \text{ for } r \neq 0$$

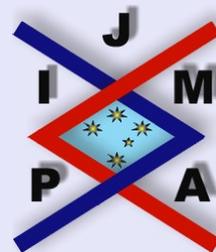
$$M_0(x, w) = \exp \left(\sum_{k=1}^n w_k \ln x_k \right).$$

In [2] Mercer defined the family of functions

$$Q_r(a, b, x) = (a^r + b^r - M_r^r(x, w))^{\frac{1}{r}} \text{ for } r \neq 0$$

$$Q_0(a, b, x) = \frac{ab}{M_0}$$

and proved the following (see also [4]):



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Theorem 1.2. For $r < s$, $Q_r(a, b, x) \leq Q_s(a, b, x)$.

In [3] are given another proofs of the above theorems.

Let us consider a isotonic linear functional A , i.e., a functional $A : C[a, b] \rightarrow \mathbb{R}$ with the properties:

- (i) $A(tf + sg) = tA(f) + sA(g)$ for $t, s \in \mathbb{R}, f, g \in C[a, b]$;
- (ii) $A(f) \geq 0$ of $f(x) \geq 0$ for all $x \in [a, b]$.

In [1] A. Lupaş proved the following result:

“If f is a convex function and A is an isotonic linear functional with $A(e_0) = 1$, then

$$(1.2) \quad f(a_1) \leq A(f) \leq \frac{(b - a_1)f(a) + (a_1 - a)f(b)}{b - a},$$

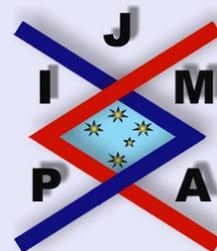
where $e_i : [a, b] \rightarrow \mathbb{R}$, $e_i(x) = x^i$ and $a_1 = A(e_1)$.

Let A be an isotonic linear functional defined on $C[a, b]$ such that $A(e_0) = 1$. For a real number r and positive function f , $f \in C[a, b]$ we define the power mean of order r as

$$(1.3) \quad M(r, f) = \begin{cases} (A(f^r))^{\frac{1}{r}} & \text{for } r \neq 0 \\ \exp(A(\log f)) & \text{for } r = 0 \end{cases}$$

and for every monotone function $f : [a, b] \rightarrow (0, \infty)$

$$(1.4) \quad Q(r; f) = \begin{cases} (f^r(a) + f^r(b) - M^r(r, f))^{\frac{1}{r}}, & r \neq 0 \\ \frac{f(a)f(b)}{\exp(A(\log f))}, & r = 0 \end{cases}.$$



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2. Main results

Our main results are given in the following theorems. Let A be an isotonic linear functional defined on $C[a, b]$ such that $A(e_0) = 1$.

Theorem 2.1. *Let f be a convex function on $[a, b]$. Then*

$$(2.1) \quad f(a + b - a_1) \leq A(g) \leq f(a) + f(b) - f(a) \frac{b - a_1}{b - a} - f(b) \frac{a_1 - a}{b - a} \\ \leq f(a) + f(b) - A(f),$$

where $g = f(a + b - \cdot)$.

Theorem 2.2. *Let $r, s \in \mathbb{R}$ such that $r \leq s$. Then*

$$(2.2) \quad Q(r, f) \leq Q(s, f),$$

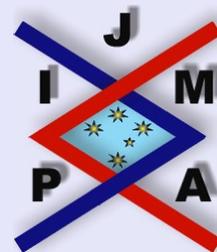
for every monotone positive function.

Proof of Theorem 2.1. The function g is a convex function. From inequality (1.2), written for the function g we get:

$$(2.3) \quad f(a + b - a_1) \leq A(g) \leq \frac{(b - a_1)f(b) + (a_1 - a)f(a)}{b - a}.$$

Using Hadamard's inequality (1.2) relative to the function f we obtain

$$(2.4) \quad A(f) \leq f(a) \frac{b - a_1}{b - a} + f(b) \frac{a_1 - a}{b - a}.$$



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However,

$$(2.5) \quad \frac{(b - a_1)f(b) + (a_1 - a)f(a)}{b - a} = f(a) + f(b) - f(a)\frac{b - a_1}{b - a} - f(b)\frac{a_1 - a}{b - a}.$$

Now (2.1) follows by (2.5), (2.4) and (2.3). □

Proof of Theorem 2.2. Let us denote $\alpha = f^r(a)$, $\beta = f^r(b)$. If $0 < r < s$ then the function $g(x) = x^{s/r}$ is convex. Let us consider the following isotonic linear functional $B : C[\alpha, \beta] \rightarrow \mathbb{R}$ defined by $B(h) = A(h \circ f^r)$, where $\alpha = \min(f^r(a), f^r(b))$, $\beta = \max(f^r(a), f^r(b))$. We have:

$$B(e_1) = A(f^r).$$

From (2.1) we get

$$g(\alpha + \beta - B(e_1)) \leq g(\alpha) + g(\beta) - B(g)$$

or

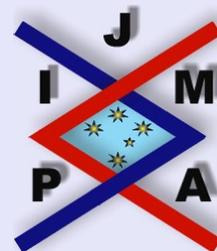
$$(f(a)^r + f^r(b) - A(f^r))^{s/r} \leq f^s(a) + f^s(b) - A(f^s).$$

The last inequality is equivalent to

$$Q(r, f) \leq A(s, f).$$

For $r < s < 0$, g is concave and we obtain

$$(f^r(a) + f^r(b) - A(f^r))^{s/r} \geq f^s(a) + f^s(b) - A(f^s)$$



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which is also equivalent to $Q(r, f) \leq Q(s, f)$. Finally, applying (2.1) to the concave function $\log x$ for the functional

$$B(g) = A(g \circ f^r),$$

we have

$$\log(\alpha + \beta - A(f^r)) \geq \log \alpha + \log \beta - A(\log f^r),$$

or

$$r \log(Q(r, f)) \geq r \log Q(0, f),$$

which shows that for $r > 0$

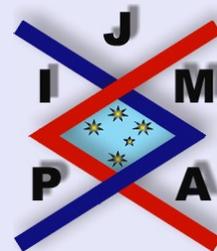
$$Q(-r, f) \leq Q(0, f) \leq Q(r, f).$$

□

Remark 2.1. For the functional A , $A : C[a, b] \rightarrow \mathbb{R}$ defined by

$$A(f) = \sum_{k=1}^n w_k f(x_k),$$

in the particular case when $f(x) = x^r$ we obtain Theorem 1.2.



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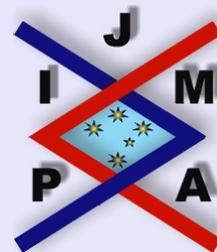
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