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A NOTE ON A WEIGHTED OSTROWSKI TYPE INEQUALITY FOR CUMULATIVE DISTRIBUTION FUNCTIONS

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ABSTRACT. In this note, a weighted Ostrowski type inequality for the cumulative distribution function and expectation of a random variable is established.

Key words and phrases: Weight function, Ostrowski type inequality, Cumulative distribution functions.

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1. INTRODUCTION

In [1], N. S. Barnett and S. S. Dragomir established the following Ostrowski type inequality for cumulative distribution functions.

Theorem 1.1. *Let X be a random variable taking values in the finite interval $[a,b]$, with cumulative distribution function $F(x) = \Pr(X \le x)$, then,*

$$\begin{aligned}\left|\Pr(X \le x) - \frac{b - E(X)}{b-a}\right| &\le \frac{1}{b-a}\left[[2x-(a+b)]\ \Pr(X \le x) + \int_a^b \operatorname{sgn}(t-x)F(t)dt\right] \\ &\le \frac{1}{b-a}\left[(b-x))\ \Pr(X \ge x) + (x-a)\ \Pr(X \le x)\right] \\ &\le \frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{(b-a)} \qquad (1.1)\end{aligned}$$

for all $x \in [a,b]$. All the inequalities in (1.1) are sharp and the constant $\frac{1}{2}$ the best possible.

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In this paper, we establish a weighted version of this result using similar methods to those used in [1]. The results of [1] are then retrieved by taking the weight function to be 1.

2. MAIN RESULTS

We assume that the weight function $w:(a,b)\longrightarrow[0,\infty)$, is integrable, nonnegative and

$$\int_a^b w(t)dt<\infty.$$

The domain of w may be finite or infinite and w may vanish at the boundary points. We denote

$$m(a,b)=\int_a^b w(t)dt.$$

We also know that the expectation of any function $\varphi(X)$ of the random variable X is given by:

$$E\left[\varphi\left(X\right)\right]=\int_a^b \varphi(t)dF\left(t\right). \tag{2.1}$$

Taking $\varphi\left(X\right)=\displaystyle\int w(X)dX,$ then from (2.1) and integrating by parts, we get,

$$\begin{aligned} E_W &= E\left[\int w(X)dX\right] \\ &= \int_a^b \left(\int w(t)dt\right)dF\left(t\right) \\ &= W(b)-\int_a^b w(t)F(t)dt, \end{aligned} \tag{2.2}$$

where $W(b)=\left[\int w(t)dt\right]_{t=b}.$

Theorem 2.1. *Let X be a random variable taking values in the finite interval $[a,b]$, with cumulative distribution function $F(x)=Pr(X\le x)$, then,*

$$\begin{aligned} &\left|\Pr(X\le x)-\frac{W(b)-E_W}{m(a,b)}\right| \\ &\qquad\le\frac{1}{m(a,b)}\left[\left[m(a,x)-m(x,b)\right]\ \Pr(X\le x)+\int_a^b \operatorname{sgn}(t-x)w(t)F(t)dt\right] \\ &\qquad\le\frac{1}{m(a,b)}\left[m(a,x)\ \Pr(X\le x)+m(x,b)\ \Pr(X\ge x)\right] \\ &\qquad\le\frac{1}{2}+\frac{\left|\frac{m(x,b)-m(a,x)}{2}\right|}{m(a,b)} \end{aligned} \tag{2.3}$$

for all $x\in[a,b]$. All the inequalities in (2.3) are sharp and the constant $\frac{1}{2}$ is the best possible.

Proof. Consider the Kernel $p:[a,b]^2\longrightarrow\mathbb{R}$ defined by

$$p(x,t)=\begin{cases}\int_a^t w(u)du & \text{if } t\in[a,x]\\[2ex] \int_b^t w(u)du & \text{if } t\in(x,b],\end{cases} \tag{2.4}$$

then the Riemann-Stieltjes integral $\int_a^b p(x,t)dF\,(t)$ exists for any $x \in [a,b]$ and the following identity holds:

$$\int_a^b p(x,t)dF\,(t) = m(a,b)F(x) - \int_a^b w(t)F(t)dt\,. \tag{2.5}$$

Using (2.2) and (2.5), we get (see [2, p. 452]),

$$m(a,b)F(x) + E_W - W(b) = \int_a^b p(x,t)dF\,(t)\,. \tag{2.6}$$

As shown in [1], if $p : [a,b] \;\to \mathbb{R}$ is continuous on $[a,b]$ and $\nu : [a,b] \;\to \mathbb{R}$ is monotonic non-decreasing, then the Riemann -Stieltjes integral $\int_a^b p\,(x)\;\, d\nu\,(x)$ exists and

$$\left|\int_a^b p\,(x)\;\, d\nu\,(x)\right| \le \int_a^b |p\,(x)|\, d\nu\,(x)\,. \tag{2.7}$$

Using (2.7) we have

$$\begin{aligned}
\left|\int_a^b p(x,t)dF\,(t)\right| &= \left|\int_a^x \left(\int_a^t w(u)du\right)\; dF\,(t) + \int_x^b \left(\int_b^t w(u)du\right) dF\,(t)\right| \\
&\le \int_a^x \left|\int_a^t w(u)du\right|\; dF\,(t) + \int_x^b \left|\int_b^t w(u)du\right|\; dF\,(t) \\
&= \left(\int_a^t w(u)du\right)\; F(t)\Big|_a^x - \int_a^x F(t)\frac{d}{dt}\left(\int_a^t w(u)du\right) dt \\
&\quad + \left(\int_t^b w(u)du\right) F(t)\Big|_x^b - \int_x^b F(t)\frac{d}{dt}\left(\int_t^b w(u)du\right) dt \\
&= \left[m(a,x) - m(x,b)\right] F(x) + \int_a^b \operatorname{sgn}(t-x)w(t)F(t)dt. \qquad (2.8)
\end{aligned}$$

Using the identity (2.6) and the inequality (2.8), we deduce the first part of (2.3).

We know that

$$\int_a^b \operatorname{sgn}(t-x)w(t)F(t)dt = -\int_a^x w(t)F(t)dt + \int_x^b w(t)F(t)dt.$$

As $F(\cdot)$ is monotonic non-decreasing on $[a,b]$,

$$\int_a^x w(t)F(t)dt \ge m(a,x)F(a) = 0,$$

$$\int_x^b w(t)F(t)dt \le m(x,b)F(b) = m(x,b)$$

and

$$\int_a^b \operatorname{sgn}(t-x)w(t)F(t)dt \le m(x,b) \text{ for all } x \in [a,b].$$

Consequently,

$$\begin{aligned}
[m(a,x) - m(x,b)] \Pr\left(X \le x\right) &+ \int_a^b \operatorname{sgn}(t-x)w(t)F(t)dt \\
&\le [m(a,x) - m(x,b)] \Pr(X \le x) + m(x,b) \\
&= m(a,x)\Pr(X \le x) + m(x,b)\Pr(X \ge x)
\end{aligned}$$

and the second part of (2.3) is proved.

Finally,

$$\begin{aligned}
m(a,x)\Pr(X\le x)+m(x,b)\Pr(X\ge x)
&\le \max\{m(a,x),m(x,b)\}\left[\Pr(X\le x)+\Pr(X\ge x)\right]\\
&=\frac{m(a,b)+|m(x,b)-m(a,x)|}{2}
\end{aligned}$$

and the last part of (2.3) follows. □

Remark 2.2. Since

$$\Pr(X\ge x)=1-\Pr(X\le x),$$

we can obtain an equivalent to (2.3) for

$$\left|\Pr(X\ge x)-\frac{E_W+m(a,b)-W(b)}{m(a,b)}\right|.$$

Following the same style of argument as in Remark 2.3 and Corollary 2.4 of [1], we have the following two corollaries.

Corollary 2.3.

$$\begin{aligned}
&\left|\Pr\left(X\le\frac{a+b}{2}\right)-\frac{W(b)-E_W}{m(a,b)}\right|\\
&\quad\le\frac{1}{m(a,b)}\left[\left[m\left(a,\frac{a+b}{2}\right)-m\left(\frac{a+b}{2},b\right)\right]\ \Pr(X\le x)\right.\\
&\qquad\left.+\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)w(t)F(t)dt\right]\\
&\quad\le\frac{1}{2}+\frac{\left|\frac{m\left(\frac{a+b}{2},b\right)-m\left(a,\frac{a+b}{2}\right)}{2}\right|}{m(a,b)}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\Pr\left(X\ge\frac{a+b}{2}\right)-\frac{E_W+m(a,b)-W(b)}{m(a,b)}\right|\\
&\quad\le\frac{1}{m(a,b)}\left[\left[m\left(a,\frac{a+b}{2}\right)-m\left(\frac{a+b}{2},b\right)\right]\ \Pr\left(X\le\frac{a+b}{2}\right)\right.\\
&\qquad\left.+\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)w(t)F(t)dt\right]\\
&\quad\le\frac{1}{2}+\frac{\left|\frac{m\left(\frac{a+b}{2},b\right)-m\left(a,\frac{a+b}{2}\right)}{2}\right|}{m(a,b)}.
\end{aligned}$$

Corollary 2.4.

$$\begin{aligned}
&\frac{1}{m(a,b)}\left[\frac{2W(b)-m(a,b)-\left|m\left(\frac{a+b}{2},b\right)-m\left(a,\frac{a+b}{2}\right)\right|}{2}-E_W\right]\\
&\le\Pr\left(X\le\frac{a+b}{2}\right)\\
&\le 1+\frac{1}{m(a,b)}\left[\frac{2W(b)-m(a,b)+\left|m\left(\frac{a+b}{2},b\right)-m\left(a,\frac{a+b}{2}\right)\right|}{2}-E_W\right].
\end{aligned}$$

Additional inequalities for $\Pr[X \le x]$ and $\Pr\left(X \le \frac{a+b}{2}\right)$ are obtainable in the style of Corollary 2.6 and Remarks 2.5 and 2.7 of [1] using (2.3).

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