SOLUTION OF ONE CONJECTURE ON INEQUALITIES WITH POWER-EXPONENTIAL FUNCTIONS

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Abstract: In this paper, we prove one conjecture presented in the paper [V. Cîrtoaje, *On some inequalities with power-exponential functions*, J. Inequal. Pure Appl. Math. 10 (2009) no. 1, Art. 21. http://jipam.vu.edu.au/article.php?sid=1077].

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1. Introduction

In the paper [1], V. Cîrtoaje posted 5 conjectures on inequalities with power-exponential functions. In this paper, we prove Conjecture 4.6.

Conjecture 4.6. *Let r be a positive real number. The inequality*

$$a^{rb} + b^{ra} \leq 2 \tag{1.1}$$

holds for all nonnegative real numbers a and b with $a + b = 2$, if and only if $r \leq 3$.

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2. Proof of Conjecture 4.6

First, we prove the necessary condition. Put $a = 2 - \frac{1}{x}$, $b = \frac{1}{x}$, $r = 3x$ for $x > 1$. Then we have

$$a^{rb} + b^{ra} > 2. \tag{2.1}$$

In fact,

$$\left(2 - \frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^{3x\left(2-\frac{1}{x}\right)} = 8 - \frac{12}{x} + \frac{6}{x^2} - \frac{1}{x^3} + \left(\frac{1}{x}\right)^{6x-3}$$

and if we show that $\left(\frac{1}{x}\right)^{6x-3} > -6 + \frac{12}{x} - \frac{6}{x^2} + \frac{1}{x^3}$ then the inequality (2.1) will be fulfilled for all $x > 1$. Put $t = \frac{1}{x}$, then $0 < t < 1$. The inequality (2.1) becomes

$$t^{\frac{6}{t}} > t^3(t^3 - 6t^2 + 12t - 6) = t^3\beta(t),$$

where $\beta(t) = t^3 - 6t^2 + 12t - 6$. From $\beta'(t) = 3(t-2)^2$, $\beta(0) = -6$, and from that there is only one real $t_0 = 0.7401$ such that $\beta(t_0) = 0$ and we have that $\beta(t) \le 0$ for $0 \le t \le t_0$. Thus, it suffices to show that $t^{\frac{6}{t}} > t^3\beta(t)$ for $t_0 < t < 1$. Rewriting the previous inequality we get

$$\alpha(t) = \left(\frac{6}{t} - 3\right)\ln t - \ln(t^3 - 6t^2 + 12t - 6) > 0.$$

From $\alpha(1) = 0$, it suffices to show that $\alpha'(t) < 0$ for $t_0 < t < 1$, where

$$\alpha'(t) = -\frac{6}{t^2}\ln t + \left(\frac{6}{t} - 3\right)\frac{1}{t} - \frac{3t^2 - 12t + 12}{t^3 - 6t^2 + 12t - 6}.$$

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$\alpha'(t) < 0$ is equivalent to

$$\gamma(t) = 2\ln t - 2 + t + \frac{t^2(t-2)^2}{t^3 - 6t^2 + 12t - 6} > 0.$$

From $\gamma(1) = 0$, it suffices to show that $\gamma'(t) < 0$ for $t_0 < t < 1$, where

$$\gamma'(t) = \frac{(4t^3 - 12t^2 + 8t)(t^3 - 6t^2 + 12t - 6) - (t^4 - 4t^3 + 4t^2)(3t^2 - 12t + 12)}{(t^3 - 6t^2 + 12t - 6)^2} + \frac{2}{t} + 1$$
$$= \frac{t^6 - 12t^5 + 56t^4 - 120t^3 + 120t^2 - 48t}{(t^3 - 6t^2 + 12t - 6)^2} + \frac{2}{t} + 1.$$

$\gamma'(t) < 0$ is equivalent to

$$p(t) = 2t^7 - 22t^6 + 92t^5 - 156t^4 + 24t^3 + 240t^2 - 252t + 72 < 0.$$

From

$$p(t) = 2(t-1)(t^6 - 10t^5 + 36t^4 - 42t^3 - 30t^2 + 90t - 36),$$

it suffices to show that

$$q(t) = t^6 - 10t^5 + 36t^4 - 42t^3 - 30t^2 + 90t - 36 > 0. \tag{2.2}$$

Since $q(0.74) = 5.893$, $q(1) = 9$ it suffices to show that $q''(t) < 0$ and (2.2) will be proved. Indeed, for $t_0 < t < 1$, we have

$$\begin{aligned} q''(t) &= 2(15t^4 - 100t^3 + 216t^2 - 126t - 30) \\ &< 2(40t^4 - 100t^3 + 216t^2 - 126t - 30) \\ &= 4(t-1)(20t^3 - 30t^2 + 78t + 15) \\ &< 4(t-1)(-30t^2 + 78t) < 0. \end{aligned}$$

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This completes the proof of the necessary condition.

We prove the sufficient condition. Put $a = 1-x$ and $b = 1+x$, where $0 < x < 1$. Since the desired inequality is true for $x = 0$ and for $x = 1$, we only need to show that

$$(2.3)\qquad (1-x)^{r(1+x)} + (1+x)^{r(1-x)} \leq 2 \quad \text{for} \quad 0 < x < 1,\ 0 < r \leq 3.$$

Denote $\varphi(x) = (1-x)^{r(1+x)} + (1+x)^{r(1-x)}$. We show that $\varphi'(x) < 0$ for $0 < x < 1$, $0 < r \leq 3$ which gives that (2.3) is valid ($\varphi(0) = 2$).

$$\varphi'(x) = (1-x)^{r(1+x)} \left(r\ln(1-x) - r\frac{1+x}{1-x} \right) + (1+x)^{r(1-x)} \left(r\frac{1-x}{1+x} - r\ln(1+x) \right).$$

The inequality $\varphi'(x) < 0$ is equivalent to

$$(2.4)\qquad \left(\frac{1+x}{1-x}\right)^r \left(\frac{1-x}{1+x} - \ln(1+x)\right) \leq (1-x^2)^{rx} \left(\frac{1+x}{1-x} - \ln(1-x)\right).$$

If $\delta(x) = \frac{1-x}{1+x} - \ln(1+x) \leq 0$, then (2.4) is evident. Since $\delta'(x) = -\frac{2}{(1+x)^2} - \frac{1}{1+x} < 0$ for $0 \leq x < 1$, $\delta(0) = 1$ and $\delta(1) = -\ln 2$, we have $\delta(x) > 0$ for $0 \leq x < x_0 \cong 0.4547$. Therefore, it suffices to show that $h(x) \geq 0$ for $0 \leq x \leq x_0$, where

$$h(x) = rx\ln(1-x^2) - r\ln\left(\frac{1+x}{1-x}\right) + \ln\left(\frac{1+x}{1-x} - \ln(1-x)\right) - \ln\left(\frac{1-x}{1+x} - \ln(1+x)\right).$$

We show that $h'(x) \geq 0$ for $0 < x < x_0$, $0 < r \leq 3$. Then from $h(0) = 0$ we obtain

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$h(x) \geq 0$ for $0 < x \leq x_0$ and it implies that the inequality (2.4) is valid.

$$h'(x) = r\ln(1-x^2) - 2r\frac{1+x^2}{1-x^2} + \frac{3-x}{(1-x)(1+x-(1-x)\ln(1-x))} + \frac{3+x}{(1+x)(1-x-(1+x)\ln(1+x))}.$$

Put $A = \ln(1+x)$ and $B = \ln(1-x)$. The inequality $h'(x) \geq 0$, $0 < x < x_0$ is equivalent to

$$r(2x^2+2-(1-x^2)(A+B)) \leq \frac{3-2x-x^2}{1-x-(1+x)A} + \frac{3+2x-x^2}{1+x-(1-x)B}. \tag{2.5}$$

Since $2x^2+2-(1-x^2)(A+B) > 0$ for $0 < x < 1$, it suffices to prove that

$$3(2x^2+2-(1-x^2)(A+B)) \leq \frac{3-2x-x^2}{1-x-(1+x)A} + \frac{3+2x-x^2}{1+x-(1-x)B} \tag{2.6}$$

and then the inequality (2.5) will be fulfilled for $0 < r \leq 3$. The inequality (2.6) for $0 < x < x_0$ is equivalent to

$$\begin{aligned}6x^2-6x^4-(9x^4+13x^3+5x^2+7x+6)A-(9x^4-13x^3+5x^2-7x+6)B \\ -(3x^4+6x^3-6x-3)A^2-(3x^4-6x^3+6x-3)B^2 \\ -(12x^4-12)AB-(3x^4-6x^2+3)AB(A+B) \leq 0.\end{aligned} \tag{2.7}$$

It is easy to show that the following Taylor's formulas are valid for $0 < x < 1$:

$$A = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}x^{n+1}, \quad B = -\sum_{n=0}^{\infty} \frac{1}{n+1}x^{n+1},$$

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$$A^2 = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n+1} \left(\sum_{i=1}^{n} \frac{1}{i} \right) x^{n+1}, \quad B^2 = \sum_{n=1}^{\infty} \frac{2}{n+1} \left(\sum_{i=1}^{n} \frac{1}{i} \right) x^{n+1},$$

$$AB = -\sum_{n=0}^{\infty} \frac{1}{n+1} \left(\sum_{i=1}^{2n+1} \frac{(-1)^{i+1}}{i} \right) x^{2n+2}.$$

Since

$$A^2 + B^2 = \sum_{n=1,3,5,\ldots} \frac{4}{n+1} \left(\sum_{i=1}^{n} \frac{1}{i} \right) x^{n+1}$$

and

$$\frac{4}{n+1} \left(\sum_{i=1}^{n} \frac{1}{i} \right) \leq \frac{4}{n+1} \left(1 + \frac{n-1}{2} \right) = 2,$$

we have

$$\begin{aligned} A^2 + B^2 &= \sum_{n=1,3,5,\ldots} \frac{4}{n+1} \left(\sum_{i=1}^{n} \frac{1}{i} \right) x^{n+1} \\ &= 2x^2 + \frac{11}{6}x^4 + \frac{137}{90}x^6 + \sum_{n=7,9,\ldots} \frac{4}{n+1} \left(\sum_{i=1}^{n} \frac{1}{i} \right) x^{n+1} \\ &< 2x^2 + \frac{11}{6}x^4 + \frac{137}{90}x^6 + 2 \sum_{n=7,9,\ldots} x^{n+1} \\ &= 2x^2 + \frac{11}{6}x^4 + \frac{137}{90}x^6 + \frac{2x^8}{1-x^2}. \end{aligned}$$

From this and from the previous Taylor's formulas we have

$$A + B > -x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \frac{1}{4} \left(\frac{x^8}{1-x^2} \right), \tag{2.8}$$

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$$A - B > 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7, \tag{2.9}$$

$$A^2 + B^2 < 2x^2 + \frac{11}{6}x^4 + \frac{137}{90}x^6 + \frac{2x^8}{1 - x^2}, \tag{2.10}$$

$$A^2 - B^2 < -2x^3 - \frac{5}{3}x^5, \tag{2.11}$$

$$AB < -x^2 - \frac{5}{12}x^4 \quad \text{for} \quad 0 < x < 1. \tag{2.12}$$

Now, having in view (2.12) and the obvious inequality $A + B < 0$, to prove (2.7) it suffices to show that

$$\begin{aligned} 6x^2 - 6x^4 - (6 + 5x^2 + 9x^4)(A + B) + (7x + 13x^3)(B - A) + (3 - 3x^4)(A^2 + B^2) \\ + (6x - 6x^3)(A^2 - B^2) - (12 - 12x^4)\left(x^2 + \frac{5}{12}x^4\right) \\ + \left(x^2 + \frac{5}{12}x^4\right)(3 - 6x^2 + 3x^4)(A + B) \leq 0. \end{aligned}$$

By using the inequalities (2.10), (2.11), the previous inequality will be proved if we

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show that

$$\begin{aligned}6x^2 - 6x^4 - (6+5x^2+9x^4)(A+B) + (7x+13x^3)(B-A)\\ + (3-3x^4)\left(2x^2+\frac{11}{6}x^4+\frac{137}{90}x^6+\frac{2x^8}{1-x^2}\right) - (6x-6x^3)\left(2x^3+\frac{5}{3}x^5\right)\\ - (12-12x^4)\left(x^2+\frac{5}{12}x^4\right) + \left(x^2+\frac{5}{12}x^4\right)(3-6x^2+3x^4)(B+A) \le 0,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}(2.13)\quad -\frac{35}{2}x^4+\frac{377}{30}x^6+\frac{19}{2}x^8-\frac{137}{30}x^{10}+6(x^8+x^{10})\\ -(A+B)\left(6+2x^2+\frac{55}{4}x^4-\frac{1}{2}x^6-\frac{5}{4}x^8\right)\\ +(7x+13x^3)(B-A)\le 0.\end{aligned}$$

To prove (2.13) it suffices to show

$$\begin{aligned}(2.14)\quad -8x^2-\frac{259}{6}x^4+\frac{357}{20}x^6+\frac{1841}{120}x^8+\frac{337}{420}x^{10}-\frac{19}{24}x^{12}-\frac{5}{12}x^{14}\\ +\frac{x^8}{1-x^2}\left(\frac{3}{2}+\frac{1}{2}x^2+\frac{55}{16}x^4-\frac{1}{8}x^6-\frac{5}{16}x^8\right)<0.\end{aligned}$$

It follows from (2.8) and (2.9). Since $0<x<\frac{1}{2}$ we have $\frac{1}{1-x^2}<\frac{4}{3}$. If we show

$$\begin{aligned}\varepsilon(x) = -8x^2-\frac{259}{6}x^4+\frac{357}{20}x^6+\frac{1841}{120}x^8+\frac{337}{420}x^{10}-\frac{19}{24}x^{12}\\ -\frac{5}{12}x^{14}+2x^8+\frac{2}{3}x^{10}+\frac{55}{12}x^{12}-\frac{1}{6}x^{14}-\frac{5}{12}x^{16}<0,\end{aligned}$$

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then the inequality (2.14) will be proved. From $x^6 < x^4$, $x^8 < x^4$, $x^{10} < x^4$ and $x^{12} < x^4$, we obtain that

$$\varepsilon(x) < -8x^2 - \frac{19}{7}x^4 - \frac{7}{12}x^{14} - \frac{5}{12}x^{16} < 0.$$

This completes the proof.

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