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STOLARSKY MEANS OF SEVERAL VARIABLES

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ABSTRACT. A generalization of the Stolarsky means to the case of several variables is presented. The new means are derived from the logarithmic mean of several variables studied in [9]. Basic properties and inequalities involving means under discussion are included. Limit theorems for these means with the underlying measure being the Dirichlet measure are established.

Key words and phrases: Stolarsky means, Dresher means, Dirichlet averages, Totally positive functions, Inequalities.

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1. INTRODUCTION AND NOTATION

In 1975 K.B. Stolarsky [16] introduced a two-parameter family of bivariate means named in mathematical literature as the Stolarsky means. Some authors call these means the extended means (see, e.g., [6, 7]) or the difference means (see [10]). For $r, s \in \mathbb{R}$ and two positive numbers x and y ($x \neq y$) they are defined as follows [16]

$$E_{r,s}(x,y) = \begin{cases} \left[\dfrac{s}{r}\,\dfrac{x^r - y^r}{x^s - y^s}\right]^{\frac{1}{r-s}}, & rs(r-s) \neq 0; \\ \exp\left(-\dfrac{1}{r} + \dfrac{x^r \ln x - y^r \ln y}{x^r - y^r}\right), & r = s \neq 0; \\ \left[\dfrac{x^r - y^r}{r(\ln x - \ln y)}\right]^{\frac{1}{r}}, & r \neq 0,\ s = 0; \\ \sqrt{xy}, & r = s = 0. \end{cases} \tag{1.1}$$

The mean $E_{r,s}(x,y)$ is symmetric in its parameters r and s and its variables x and y as well. Other properties of $E_{r,s}(x,y)$ include homogeneity of degree one in the variables x and y and

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monotonicity in r and s. It is known that $E_{r,s}$ increases with an increase in either r or s (see [6]). It is worth mentioning that the Stolarsky mean admits the following integral representation ([16])

$$\ln E_{r,s}(x,y) = \frac{1}{s-r}\int_r^s \ln I_t\,dt \tag{1.2}$$

($r \neq s$), where $I_t \equiv I_t(x,y) = E_{t,t}(x,y)$ is the identric mean of order t. J. Pečarić and V. Šimić [15] have pointed out that

$$E_{r,s}(x,y) = \left[\int_0^1 \left(tx^s + (1-t)y^s\right)^{\frac{r-s}{s}}\,dt\right]^{\frac{1}{r-s}} \tag{1.3}$$

($s(r-s) \neq 0$). This representation shows that the Stolarsky means belong to a two-parameter family of means studied earlier by M.D. Tobey [18]. A comparison theorem for the Stolarsky means have been obtained by E.B. Leach and M.C. Sholander in [7] and independently by Zs. Páles in [13]. Other results for the means (1.1) include inequalities, limit theorems and more (see, e.g., [17, 4, 6, 10, 12]).

In the past several years researchers made an attempt to generalize Stolarsky means to several variables (see [6, 18, 15, 8]). Further generalizations include so-called functional Stolarsky means. For more details about the latter class of means the interested reader is referred to [14] and [11].

To facilitate presentation let us introduce more notation. In what follows, the symbol E_{n-1} will stand for the Euclidean simplex, which is defined by

$$E_{n-1} = \left\{(u_1,\ldots,u_{n-1}) : u_i \geq 0,\ 1 \leq i \leq n-1,\ u_1+\cdots+u_{n-1} \leq 1\right\}.$$

Further, let $X = (x_1,\ldots,x_n)$ be an n-tuple of positive numbers and let $X_{\min} = \min(X)$, $X_{\max} = \max(X)$. The following

$$L(X) = (n-1)!\int_{E_{n-1}} \prod_{i=1}^{n} x_i^{u_i}\,du = (n-1)!\int_{E_{n-1}} \exp(u\cdot Z)\,du \tag{1.4}$$

is the special case of the logarithmic mean of X which has been introduced in [9]. Here $u = (u_1,\ldots,u_{n-1},1-u_1-\cdots-u_{n-1})$ where $(u_1,\ldots,u_{n-1}) \in E_{n-1}$, $du = du_1\ldots du_{n-1}$, $Z = \ln(X) = (\ln x_1,\ldots,\ln x_n)$, and $x\cdot y = x_1y_1+\cdots+x_ny_n$ is the dot product of two vectors x and y. Recently J. Merikowski [8] has proposed the following generalization of the Stolarsky mean $E_{r,s}$ to several variables

$$E_{r,s}(X) = \left[\frac{L(X^r)}{L(X^s)}\right]^{\frac{1}{r-s}} \tag{1.5}$$

($r \neq s$), where $X^r = (x_1^r,\ldots,x_n^r)$. In the paper cited above, the author did not prove that $E_{r,s}(X)$ is the mean of X, i.e., that

$$X_{\min} \leq E_{r,s}(X) \leq X_{\max} \tag{1.6}$$

holds true. If $n=2$ and $rs(r-s) \neq 0$ or if $r \neq 0$ and $s=0$, then (1.5) simplifies to (1.1) in the stated cases.

This paper deals with a two-parameter family of multivariate means whose prototype is given in (1.5). In order to define these means let us introduce more notation. By μ we will denote a probability measure on E_{n-1}. The logarithmic mean $\mathcal{L}(\mu;X)$ with the underlying measure μ is defined in [9] as follows

$$\mathcal{L}(\mu; X) = \int_{E_{n-1}} \prod_{i=1}^{n} x_i^{u_i} \mu(u)\, du = \int_{E_{n-1}} \exp(u \cdot Z)\mu(u)\, du. \tag{1.7}$$

We define

$$\mathcal{E}_{r,s}(\mu; X) = \begin{cases} \left[\dfrac{\mathcal{L}(\mu; X^r)}{\mathcal{L}(\mu; X^s)}\right]^{\frac{1}{r-s}}, & r \neq s \\ \exp\left[\dfrac{d}{dr} \ln \mathcal{L}(\mu; X^r)\right], & r = s. \end{cases} \tag{1.8}$$

Let us note that for $\mu(u) = (n-1)!$, the Lebesgue measure on E_{n-1}, the first part of (1.8) simplifies to (1.5).

In Section 2 we shall prove that $\mathcal{E}_{r,s}(\mu; X)$ is the mean value of X, i.e., it satisfies inequalities (1.6). Some elementary properties of this mean are also derived. Section 3 deals with the limit theorems for the new mean, with the probability measure being the Dirichlet measure. The latter is denoted by μ_b, where $b = (b_1, \ldots, b_n) \in \mathbb{R}^n_+$, and is defined as [2]

$$\mu_b(u) = \frac{1}{B(b)} \prod_{i=1}^{n} u_i^{b_i - 1}, \tag{1.9}$$

where $B(\cdot)$ is the multivariate beta function, $(u_1, \ldots, u_{n-1}) \in E_{n-1}$, and $u_n = 1 - u_1 - \cdots - u_{n-1}$. In the Appendix we shall prove that under certain conditions imposed on the parameters r and s, the function $E^{r-s}_{r,s}(x, y)$ is strictly totally positive as a function of x and y.

2. ELEMENTARY PROPERTIES OF $\mathcal{E}_{r,s}(\mu; X)$

In order to prove that $\mathcal{E}_{r,s}(\mu; X)$ is a mean value we need the following version of the Mean-Value Theorem for integrals.

Proposition 2.1. *Let* $\alpha := X_{\min} < X_{\max} =: \beta$ *and let* $f, g \in C\big([\alpha, \beta]\big)$ *with* $g(t) \neq 0$ *for all* $t \in [\alpha, \beta]$. *Then there exists* $\xi \in (\alpha, \beta)$ *such that*

$$\frac{\int_{E_{n-1}} f(u \cdot X)\mu(u)\, du}{\int_{E_{n-1}} g(u \cdot X)\mu(u)\, du} = \frac{f(\xi)}{g(\xi)}. \tag{2.1}$$

Proof. Let the numbers γ and δ and the function ϕ be defined in the following way

$$\gamma = \int_{E_{n-1}} g(u \cdot X)\mu(u)\, du, \qquad \delta = \int_{E_{n-1}} f(u \cdot X)\mu(u)\, du,$$

$$\phi(t) = \gamma f(t) - \delta g(t).$$

Letting $t = u \cdot X$ and, next, integrating both sides against the measure μ, we obtain

$$\int_{E_{n-1}} \phi(u \cdot X)\mu(u)\, du = 0.$$

On the other hand, application of the Mean-Value Theorem to the last integral gives

$$\phi(c \cdot X) \int_{E_{n-1}} \mu(u)\, du = 0,$$

where $c = (c_1, \ldots, c_{n-1}, c_n)$ with $(c_1, \ldots, c_{n-1}) \in E_{n-1}$ and $c_n = 1 - c_1 - \cdots - c_{n-1}$. Letting $\xi = c \cdot X$ and taking into account that

$$\int_{E_{n-1}} \mu(u)\, du = 1$$

we obtain $\phi(\xi) = 0$. This in conjunction with the definition of ϕ gives the desired result (2.1). The proof is complete. $\square$

The author is indebted to Professor Zsolt Páles for a useful suggestion regarding the proof of Proposition 2.1.

For later use let us introduce the symbol $\mathcal{E}^{(p)}_{r,s}(\mu; X)$ ($p \neq 0$), where

$$\mathcal{E}^{(p)}_{r,s}(\mu; X) = \big[\mathcal{E}_{r,s}(\mu; X^p)\big]^{\frac{1}{p}}. \tag{2.2}$$

We are in a position to prove the following.

Theorem 2.2. *Let $X \in \mathbb{R}^n_+$ and let $r, s \in \mathbb{R}$. Then*

(i) $X_{\min} \le \mathcal{E}_{r,s}(\mu; X) \le X_{\max}$,
(ii) $\mathcal{E}_{r,s}(\mu; \lambda X) = \lambda \mathcal{E}_{r,s}(\mu; X)$, $\lambda > 0$, $(\lambda X := (\lambda x_1, \ldots, \lambda x_n))$,
(iii) $\mathcal{E}_{r,s}(\mu; X)$ *increases with an increase in either r and s,*
(iv) $\ln \mathcal{E}_{r,s}(\mu; X) = \dfrac{1}{r-s} \int_s^r \ln \mathcal{E}_{t,t}(\mu; X)\, dt$, $r \neq s$,
(v) $\mathcal{E}^{(p)}_{r,s}(\mu; X) = \mathcal{E}_{pr,ps}(\mu; X)$,
(vi) $\mathcal{E}_{r,s}(\mu; X)\mathcal{E}_{-r,-s}(\mu; X^{-1}) = 1$, $(X^{-1} := (1/x_1, \ldots, 1/x_n))$,
(vii) $\mathcal{E}^{s-r}_{r,s}(\mu; X) = \mathcal{E}^{p-r}_{r,p}(\mu; X)\mathcal{E}^{s-p}_{p,s}(\mu; X)$.

Proof of (*i*). Assume first that $r \neq s$. Making use of (1.8) and (1.7) we obtain

$$\mathcal{E}_{r,s}(\mu; X) = \left[\frac{\int_{E_{n-1}} \exp\big[r(u \cdot Z)\big]\mu(u)\, du}{\int_{E_{n-1}} \exp\big[s(u \cdot Z)\big]\mu(u)\, du}\right]^{\frac{1}{r-s}}.$$

Application of (2.1) with $f(t) = \exp(rt)$ and $g(t) = \exp(st)$ gives

$$\mathcal{E}_{r,s}(\mu; X) = \left[\frac{\exp\big[r(c \cdot Z)\big]}{\exp\big[s(c \cdot Z)\big]}\right]^{\frac{1}{r-s}} = \exp(c \cdot Z),$$

where $c = (c_1, \ldots, c_{n-1}, c_n)$ with $(c_1, \ldots, c_{n-1}) \in E_{n-1}$ and $c_n = 1 - c_1 - \cdots - c_{n-1}$. Since $c \cdot Z = c_1 \ln x_1 + \cdots + c_n \ln x_n$, $\ln X_{\min} \le c \cdot Z \le \ln X_{\max}$. This in turn implies that $X_{\min} \le \exp(c \cdot Z) \le X_{\max}$. This completes the proof of (i) when $r \neq s$. Assume now that $r = s$. It follows from (1.8) and (1.7) that

$$\ln \mathcal{E}_{r,r}(\mu; X) = \left[\frac{\int_{E_{n-1}} (u \cdot Z) \exp\big[r(u \cdot Z)\big]\mu(u)\, du}{\int_{E_{n-1}} \exp\big[r(u \cdot Z)\big]\mu(u)\, du}\right].$$

Application of (2.1) to the right side with $f(t) = t\exp(rt)$ and $g(t) = \exp(rt)$ gives

$$\ln \mathcal{E}_{r,r}(\mu; X) = \left[\frac{(c \cdot Z)\exp\big[r(c \cdot Z)\big]}{\exp\big[r(c \cdot Z)\big]}\right] = c \cdot Z.$$

Since $\ln X_{\min} \le c \cdot Z \le \ln X_{\max}$, the assertion follows. This completes the proof of (i).

Proof of (ii). The following result

$$\mathcal{L}\big(\mu; (\lambda x)^r\big) = \lambda^r \mathcal{L}(\mu; X^r) \tag{2.3}$$

($\lambda > 0$) is established in [9, (2.6)]. Assume that $r \neq s$. Using (1.8) and (2.3) we obtain

$$\mathcal{E}_{r,s}(\mu; \lambda x) = \left[\frac{\lambda^r \mathcal{L}(\mu; X^r)}{\lambda^s \mathcal{L}(\mu; X^s)}\right]^{\frac{1}{r-s}} = \lambda \mathcal{E}_{r,s}(\mu; X).$$

Consider now the case when $r = s \neq 0$. Making use of (1.8) and (2.3) we obtain

$$\begin{aligned}\mathcal{E}_{r,r}(\mu; \lambda X) &= \exp\left[\frac{d}{dr} \ln \mathcal{L}\big(\mu; (\lambda X)^r\big)\right] \\ &= \exp\left[\frac{d}{dr} \ln \big(\lambda^r \mathcal{L}(\mu; X^r)\big)\right] \\ &= \exp\left[\frac{d}{dr}\big(r \ln \lambda + \ln \mathcal{L}(\mu; X^r)\big)\right] = \lambda \mathcal{E}_{r,r}(\mu; X).\end{aligned}$$

When $r = s = 0$, an easy computation shows that

$$\mathcal{E}_{0,0}(\mu; X) = \prod_{i=1}^{n} x_i^{w_i} \equiv G(w; X), \tag{2.4}$$

where

$$w_i = \int_{E_{n-1}} u_i \mu(u)\, du \tag{2.5}$$

($1 \leq i \leq n$) are called the natural weights or partial moments of the measure μ and $w = (w_1, \ldots, w_n)$. Since $w_1 + \cdots + w_n = 1$, $\mathcal{E}_{0,0}(\mu; \lambda X) = \lambda \mathcal{E}_{0,0}(\mu; X)$. The proof of (ii) is complete.

Proof of (iii). In order to establish the asserted property, let us note that the function $r \rightarrow \exp(rt)$ is logarithmically convex (log-convex) in r. This in conjunction with Theorem B.6 in [2], implies that a function $r \rightarrow \mathcal{L}(\mu; X^r)$ is also log-convex in r. It follows from (1.8) that

$$\ln \mathcal{E}_{r,s}(\mu; X) = \frac{\ln \mathcal{L}(\mu; X^r) - \ln \mathcal{L}(\mu; X^s)}{r - s}.$$

The right side is the divided difference of order one at r and s. Convexity of $\ln \mathcal{L}(\mu; X^r)$ in r implies that the divided difference increases with an increase in either r and s. This in turn implies that $\ln \mathcal{E}_{r,s}(\mu; X)$ has the same property. Hence the monotonicity property of the mean $\mathcal{E}_{r,s}$ in its parameters follows. Now let $r = s$. Then (1.8) yields

$$\ln \mathcal{E}_{r,r}(\mu; X) = \frac{d}{dr}\big[\ln \mathcal{L}(\mu; X^r)\big].$$

Since $\ln \mathcal{L}(\mu; X^r)$ is convex in r, its derivative with respect to r increases with an increase in r. This completes the proof of (iii).

Proof of (iv). Let $r \neq s$. It follows from (1.8) that

$$\begin{aligned}\frac{1}{r-s}\int_s^r \ln \mathcal{E}_{t,t}(\mu; X)\, dt &= \frac{1}{r-s}\int_s^r \frac{d}{dt}\big[\ln \mathcal{L}(\mu; X^t)\big]\, dt \\ &= \frac{1}{r-s}\big[\ln \mathcal{L}(\mu; X^r) - \ln \mathcal{L}(\mu; X^s)\big] \\ &= \ln \mathcal{E}_{r,s}(\mu; X).\end{aligned}$$

Proof of (v). Let $r \neq s$. Using (2.2) and (1.8) we obtain

$$\mathcal{E}^{(p)}_{r,s}(\mu; X) = \left[\mathcal{E}_{r,s}(\mu; X^p)\right]^{\frac{1}{p}} = \left[\frac{\mathcal{L}(X^{pr})}{\mathcal{L}(X^{ps})}\right]^{\frac{1}{p(r-s)}} = \mathcal{E}_{pr,ps}(\mu; X).$$

Assume now that $r = s \neq 0$. Making use of (2.2), (1.8) and (1.7) we have

$$\begin{aligned}\mathcal{E}^{(p)}_{r,r}(\mu; X) &= \exp\left[\frac{1}{p}\frac{d}{dr}\ln \mathcal{L}(\mu; X^{pr})\right] \\ &= \exp\left[\frac{1}{\mathcal{L}(\mu; X^{pr})}\int_{E_{n-1}} (u \cdot Z)\exp\left[pr(u \cdot Z)\right]\mu(u)\,du\right] \\ &= \mathcal{E}_{pr,pr}(\mu; X).\end{aligned}$$

The case when $r = s = 0$ is trivial because $\mathcal{E}_{0,0}(\mu; X)$ is the weighted geometric mean of X.

Proof of (vi). Here we use (v) with $p = -1$ to obtain $\mathcal{E}_{r,s}(\mu; X^{-1})^{-1} = \mathcal{E}_{-r,-s}(\mu; X)$. Letting $X := X^{-1}$ we obtain the desired result.

Proof of (vii). There is nothing to prove when either $p = r$ or $p = s$ or $r = s$. In other cases we use (1.8) to obtain the asserted result. This completes the proof. □

In the next theorem we give some inequalities involving the means under discussion.

Theorem 2.3. *Let $r, s \in \mathbb{R}$. Then the following inequalities*

$$\mathcal{E}_{r,r}(\mu; X) \leq \mathcal{E}_{r,s}(\mu; X) \leq \mathcal{E}_{s,s}(\mu; X) \tag{2.6}$$

are valid provided $r \leq s$. If $s > 0$, then

$$\mathcal{E}_{r-s,0}(\mu; X) \leq \mathcal{E}_{r,s}(\mu; X). \tag{2.7}$$

Inequality (2.7) *is reversed if $s < 0$ and it becomes an equality if $s = 0$. Assume that $r, s > 0$ and let $p \leq q$. Then*

$$\mathcal{E}^{(p)}_{r,s}(\mu; X) \leq \mathcal{E}^{(q)}_{r,s}(\mu; X) \tag{2.8}$$

with the inequality reversed if $r, s < 0$.

Proof. Inequalities (2.6) and (2.7) follow immediately from Part (iii) of Theorem 2.2. For the proof of (2.8), let $r, s > 0$ and let $p \leq q$. Then $pr \leq qr$ and $ps \leq qs$. Applying Parts (v) and (iii) of Theorem 2.2, we obtain

$$\mathcal{E}^{(p)}_{r,s}(\mu; X) = \mathcal{E}_{pr,ps}(\mu; X) \leq \mathcal{E}_{qr,qs}(\mu; X) = \mathcal{E}^{(q)}_{r,s}(\mu; X).$$

When $r, s < 0$, the proof of (2.8) goes along the lines introduced above, hence it is omitted. The proof is complete. □

3. THE MEAN $\mathcal{E}_{r,s}(b; X)$

An important probability measure on E_{n-1} is the Dirichlet measure $\mu_b(u)$, $b \in \mathbb{R}^n_+$ (see (1.9)). Its role in the theory of special functions is well documented in Carlson's monograph [2]. When $\mu = \mu_b$, the mean under discussion will be denoted by $\mathcal{E}_{r,s}(b; X)$. The natural weights w_i (see (2.5)) of μ_b are given explicitly by

$$w_i = b_i/c \tag{3.1}$$

$(1 \le i \le n)$, where $c = b_1 + \cdots + b_n$ (see [2, (5.6-2)]). For later use we define $w = (w_1, \ldots, w_n)$. Recall that the weighted Dresher mean $D_{r,s}(p; X)$ of order $(r, s) \in \mathbb{R}^2$ of $X \in \mathbb{R}^n_+$ with weights $p = (p_1, \ldots, p_n) \in \mathbb{R}^n_+$ is defined as

$$D_{r,s}(p; X) = \begin{cases} \left[\dfrac{\sum_{i=1}^n p_i x_i^r}{\sum_{i=1}^n p_i x_i^s}\right]^{\frac{1}{r-s}}, & r \neq s \\ \exp\left[\dfrac{\sum_{i=1}^n p_i x_i^r \ln x_i}{\sum_{i=1}^n p_i x_i^r}\right], & r = s \end{cases} \tag{3.2}$$

(see, e.g., [1, Sec. 24]).

In this section we present two limit theorems for the mean $\mathcal{E}_{r,s}$ with the underlying measure being the Dirichlet measure. In order to facilitate presentation we need a concept of the Dirichlet average of a function. Following [2, Def. 5.2-1] let Ω be a convex set in $\mathbb{C}$ and let $Y = (y_1, \ldots, y_n) \in \Omega^n$, $n \ge 2$. Further, let f be a measurable function on Ω. Define

$$F(b; Y) = \int_{E_{n-1}} f(u \cdot Y)\mu_b(u)\,du. \tag{3.3}$$

Then F is called the Dirichlet average of f with variables $Y = (y_1, \ldots, y_n)$ and parameters $b = (b_1, \ldots, b_n)$. We need the following result [2, Ex. 6.3-4]. Let Ω be an open circular disk in $\mathbb{C}$, and let f be holomorphic on Ω. Let $Y \in \Omega^n$, $c \in \mathbb{C}$, $c \neq 0, -1, \ldots$, and $w_1 + \cdots + w_n = 1$. Then

$$\lim_{c \to 0} F(cw; Y) = \sum_{i=1}^n w_i f(y_i), \tag{3.4}$$

where $cw = (cw_1, \ldots, cw_n)$.

We are in a position to prove the following.

Theorem 3.1. *Let $w_1 > 0, \ldots, w_n > 0$ with $w_1 + \cdots + w_n = 1$. If $r, s \in \mathbb{R}$ and $X \in \mathbb{R}^n_+$, then*

$$\lim_{c \to 0^+} \mathcal{E}_{r,s}(cw; X) = D_{r,s}(w; X).$$

Proof. We use (1.7) and (3.3) to obtain $\mathcal{L}(cw; X) = F(cw; Z)$, where $Z = \ln X = (\ln x_1, \ldots, \ln x_n)$. Making use of (3.4) with $f(t) = \exp(t)$ and $Y = \ln X$ we obtain

$$\lim_{c \to 0^+} \mathcal{L}(cw; X) = \sum_{i=1}^n w_i x_i.$$

Hence

$$\lim_{c \to 0^+} \mathcal{L}(cw; X^r) = \sum_{i=1}^n w_i x_i^r. \tag{3.5}$$

Assume that $r \neq s$. Application of (3.5) to (1.8) gives

$$\lim_{c \to 0^+} \mathcal{E}_{r,s}(cw; X) = \lim_{c \to 0^+} \left[\frac{\mathcal{L}(cw; X^r)}{\mathcal{L}(cw; X^s)}\right]^{\frac{1}{r-s}} = \left[\frac{\sum_{i=1}^n w_i x_i^r}{\sum_{i=1}^n w_i x_i^s}\right]^{\frac{1}{r-s}} = D_{r,s}(w; X).$$

Let $r = s$. Application of (3.4) with $f(t) = t\exp(rt)$ gives

$$\lim_{c \to 0^+} F(cw; Z) = \sum_{i=1}^n w_i z_i \exp(r z_i) = \sum_{i=1}^n w_i (\ln x_i) x_i^r.$$

This in conjunction with (3.5) and (1.8) gives

$$\lim_{c\to 0^+}\mathcal{E}_{r,r}(cw;X) = \lim_{c\to 0^+}\exp\left[\frac{F(cw;Z)}{\mathcal{L}(cw;X^r)}\right] = \exp\left[\frac{\sum_{i=1}^n w_i x_i^r \ln x_i}{\sum_{i=1}^n w_i x_i^r}\right] = D_{r,r}(w;X).$$

This completes the proof. □

Theorem 3.2. *Under the assumptions of Theorem 3.1 one has*

$$\lim_{c\to\infty}\mathcal{E}_{r,s}(cw;X) = G(w;X). \tag{3.6}$$

Proof. The following limit (see [9, (4.10)])

$$\lim_{c\to\infty}\mathcal{L}(cw;X) = G(w;X) \tag{3.7}$$

will be used in the sequel. We shall establish first (3.6) when $r \neq s$. It follows from (1.8) and (3.7) that

$$\lim_{c\to\infty}\mathcal{E}_{r,s}(cw;X) = \lim_{c\to\infty}\left[\frac{\mathcal{L}(cw;X^r)}{\mathcal{L}(cw;X^s)}\right]^{\frac{1}{r-s}} = \left[G(w;X)^{r-s}\right]^{\frac{1}{r-s}} = G(w;X).$$

Assume that $r = s$. We shall prove first that

$$\lim_{c\to\infty}F(cw;Z) = \left[\ln G(w;X)\right]G(w;X)^r, \tag{3.8}$$

where F is the Dirichlet average of $f(t) = t\exp(rt)$. Averaging both sides of

$$f(t) = \sum_{m=0}^{\infty}\frac{r^m}{m!}t^{m+1}$$

we obtain

$$F(cw;Z) = \sum_{m=0}^{\infty}\frac{r^m}{m!}R_{m+1}(cw;Z), \tag{3.9}$$

where R_{m+1} stands for the Dirichlet average of the power function t^{m+1}. We will show that the series in (3.9) converges uniformly in $0 < c < \infty$. This in turn implies further that as $c \to \infty$, we can proceed to the limit term by term. Making use of [2, 6.2-24)] we obtain

$$|R_{m+1}(cw;Z)| \le |Z|^{m+1}, \quad m \in \mathbb{N},$$

where $|Z| = \max\left\{|\ln x_i| : 1 \le i \le n\right\}$. By the Weierstrass M test the series in (3.9) converges uniformly in the stated domain. Taking limits on both sides of (3.9) we obtain with the aid of (3.4)

$$\begin{aligned}\lim_{c\to\infty}F(cw;Z) &= \sum_{m=0}^{\infty}\frac{r^m}{m!}\lim_{c\to\infty}R_{m+1}(cw;Z)\\ &= \sum_{m=0}^{\infty}\frac{r^m}{m!}\left(\sum_{i=1}^{n}w_i z_i\right)^{m+1}\\ &= \left[\ln G(w;X)\right]\sum_{m=0}^{\infty}\frac{r^m}{m!}\left[\ln G(w;X)\right]^m\\ &= \left[\ln G(w;X)\right]\sum_{m=0}^{\infty}\frac{1}{m!}\left[\ln G(w;X)^r\right]^m\\ &= \left[\ln G(w;X)\right]G(w;X)^r.\end{aligned}$$

This completes the proof of (3.8). To complete the proof of (3.6) we use (1.8), (3.7), and (3.8) to obtain

$$\lim_{c\to\infty} \ln \mathcal{E}_{r,r}(\mu; X) = \lim_{c\to\infty} \frac{F(cw; Z)}{\mathcal{L}(cw; X^r)} = \frac{\big[\ln G(w; X)\big] G(w; X)^r}{G(w; X)^r} = \ln G(w; X).$$

Hence the assertion follows. □

APPENDIX A. TOTAL POSITIVITY OF $E^{r-s}_{r,s}(x, y)$

A real-valued function $h(x, y)$ of two real variables is said to be strictly totally positive on its domain if every $n \times n$ determinant with elements $h(x_i, y_j)$, where $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ is strictly positive for every $n = 1, 2, \ldots$ (see [5]).

The goal of this section is to prove that the function $E^{r-s}_{r,s}(x, y)$ is strictly totally positive as a function of x and y provided the parameters r and s satisfy a certain condition. For later use we recall the definition of the R-hypergeometric function $R_{-\alpha}(\beta, \beta'; x, y)$ of two variables $x, y > 0$ with parameters $\beta, \beta' > 0$

$$\text{(A1)} \qquad R_{-\alpha}(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 u^{\beta-1}(1-u)^{\beta'-1}\big[ux + (1-u)y\big]^{-\alpha}\, du$$

(see [2, (5.9-1)]).

Proposition A.1. *Let $x, y > 0$ and let $r, s \in \mathbb{R}$. If $|r| < |s|$, then $E^{r-s}_{r,s}(x, y)$ is strictly totally positive on $\mathbb{R}^2_+$.*

Proof. Using (1.3) and (A1) we have

$$\text{(A2)} \qquad E^{r-s}_{r,s}(x, y) = R_{\frac{r-s}{s}}(1, 1; x^s, y^s)$$

($s(r-s) \neq 0$). B. Carlson and J. Gustafson [3] have proven that $R_{-\alpha}(\beta, \beta'; x, y)$ is strictly totally positive in x and y provided $\beta, \beta' > 0$ and $0 < \alpha < \beta + \beta'$. Letting $\alpha = 1 - r/s$, $\beta = \beta' = 1$, $x := x^s$, $y := y^s$, and next, using (A2) we obtain the desired result. □

Corollary A.2. *Let $0 < x_1 < x_2$, $0 < y_1 < y_2$ and let the real numbers r and s satisfy the inequality $|r| < |s|$. If $s > 0$, then*

$$\text{(A3)} \qquad E_{r,s}(x_1, y_1)E_{r,s}(x_2, y_2) < E_{r,s}(x_1, y_2)E_{r,s}(x_2, y_1).$$

Inequality (A3) is reversed if $s < 0$.

Proof. Let $a_{ij} = E^{r-s}_{r,s}(x_i, y_j)$ $(i, j = 1, 2)$. It follows from Proposition A.1 that $\det\big([a_{ij}]\big) > 0$ provided $|r| < |s|$. This in turn implies

$$\big[E_{r,s}(x_1, y_1)E_{r,s}(x_2, y_2)\big]^{r-s} > \big[E_{r,s}(x_1, y_2)E_{r,s}(x_2, y_1)\big]^{r-s}.$$

Assume that $s > 0$. Then the inequality $|r| < s$ implies $r - s < 0$. Hence (A3) follows when $s > 0$. The case when $s < 0$ is treated in a similar way. □

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