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STOLARSKY MEANS OF SEVERAL VARIABLES

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Abstract

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Abstract

A generalization of the Stolarsky means to the case of several variables is presented. The new means are derived from the logarithmic mean of several variables studied in [9]. Basic properties and inequalities involving means under discussion are included. Limit theorems for these means with the underlying measure being the Dirichlet measure are established.

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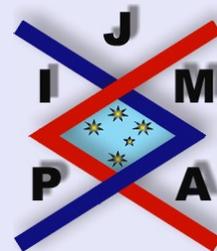
1. Introduction and Notation

In 1975 K.B. Stolarsky [16] introduced a two-parameter family of bivariate means named in mathematical literature as the Stolarsky means. Some authors call these means the extended means (see, e.g., [6, 7]) or the difference means (see [10]). For $r, s \in \mathbb{R}$ and two positive numbers x and y ($x \neq y$) they are defined as follows [16]

$$(1.1) \quad E_{r,s}(x, y) = \begin{cases} \left[\frac{s}{r} \frac{x^r - y^r}{x^s - y^s} \right]^{\frac{1}{r-s}}, & rs(r-s) \neq 0; \\ \exp\left(-\frac{1}{r} + \frac{x^r \ln x - y^r \ln y}{x^r - y^r}\right), & r = s \neq 0; \\ \left[\frac{x^r - y^r}{r(\ln x - \ln y)} \right]^{\frac{1}{r}}, & r \neq 0, s = 0; \\ \sqrt{xy}, & r = s = 0. \end{cases}$$

The mean $E_{r,s}(x, y)$ is symmetric in its parameters r and s and its variables x and y as well. Other properties of $E_{r,s}(x, y)$ include homogeneity of degree one in the variables x and y and monotonicity in r and s . It is known that $E_{r,s}$ increases with an increase in either r or s (see [6]). It is worth mentioning that the Stolarsky mean admits the following integral representation ([16])

$$(1.2) \quad \ln E_{r,s}(x, y) = \frac{1}{s-r} \int_r^s \ln I_t dt$$



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($r \neq s$), where $I_t \equiv I_t(x, y) = E_{t,t}(x, y)$ is the identric mean of order t . J. Pečarić and V. Šimić [15] have pointed out that

$$(1.3) \quad E_{r,s}(x, y) = \left[\int_0^1 (tx^s + (1-t)y^s)^{\frac{r-s}{s}} dt \right]^{\frac{1}{r-s}}$$

($s(r-s) \neq 0$). This representation shows that the Stolarsky means belong to a two-parameter family of means studied earlier by M.D. Tobey [18]. A comparison theorem for the Stolarsky means have been obtained by E.B. Leach and M.C. Sholander in [7] and independently by Zs. Páles in [13]. Other results for the means (1.1) include inequalities, limit theorems and more (see, e.g., [17, 4, 6, 10, 12]).

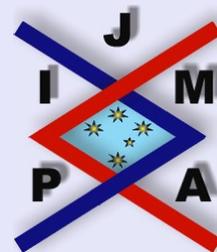
In the past several years researchers made an attempt to generalize Stolarsky means to several variables (see [6, 18, 15, 8]). Further generalizations include so-called functional Stolarsky means. For more details about the latter class of means the interested reader is referred to [14] and [11].

To facilitate presentation let us introduce more notation. In what follows, the symbol E_{n-1} will stand for the Euclidean simplex, which is defined by

$$E_{n-1} = \{(u_1, \dots, u_{n-1}) : u_i \geq 0, 1 \leq i \leq n-1, u_1 + \dots + u_{n-1} \leq 1\}.$$

Further, let $X = (x_1, \dots, x_n)$ be an n -tuple of positive numbers and let $X_{\min} = \min(X)$, $X_{\max} = \max(X)$. The following

$$(1.4) \quad L(X) = (n-1)! \int_{E_{n-1}} \prod_{i=1}^n x_i^{u_i} du = (n-1)! \int_{E_{n-1}} \exp(u \cdot Z) du$$



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is the special case of the logarithmic mean of X which has been introduced in [9]. Here $u = (u_1, \dots, u_{n-1}, 1 - u_1 - \dots - u_{n-1})$ where $(u_1, \dots, u_{n-1}) \in E_{n-1}$, $du = du_1 \dots du_{n-1}$, $Z = \ln(X) = (\ln x_1, \dots, \ln x_n)$, and $x \cdot y = x_1 y_1 + \dots + x_n y_n$ is the dot product of two vectors x and y . Recently J. Merikowski [8] has proposed the following generalization of the Stolarsky mean $E_{r,s}$ to several variables

$$(1.5) \quad E_{r,s}(X) = \left[\frac{L(X^r)}{L(X^s)} \right]^{\frac{1}{r-s}}$$

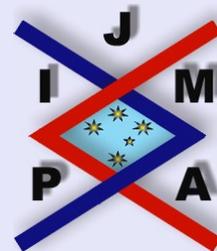
($r \neq s$), where $X^r = (x_1^r, \dots, x_n^r)$. In the paper cited above, the author did not prove that $E_{r,s}(X)$ is the mean of X , i.e., that

$$(1.6) \quad X_{\min} \leq E_{r,s}(X) \leq X_{\max}$$

holds true. If $n = 2$ and $rs(r - s) \neq 0$ or if $r \neq 0$ and $s = 0$, then (1.5) simplifies to (1.1) in the stated cases.

This paper deals with a two-parameter family of multivariate means whose prototype is given in (1.5). In order to define these means let us introduce more notation. By μ we will denote a probability measure on E_{n-1} . The logarithmic mean $\mathcal{L}(\mu; X)$ with the underlying measure μ is defined in [9] as follows

$$(1.7) \quad \mathcal{L}(\mu; X) = \int_{E_{n-1}} \prod_{i=1}^n x_i^{u_i} \mu(u) du = \int_{E_{n-1}} \exp(u \cdot Z) \mu(u) du.$$



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We define

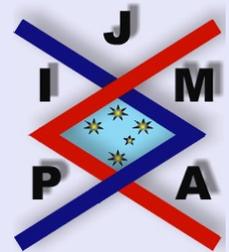
$$(1.8) \quad \mathcal{E}_{r,s}(\mu; X) = \begin{cases} \left[\frac{\mathcal{L}(\mu; X^r)}{\mathcal{L}(\mu; X^s)} \right]^{\frac{1}{r-s}}, & r \neq s \\ \exp \left[\frac{d}{dr} \ln \mathcal{L}(\mu; X^r) \right], & r = s. \end{cases}$$

Let us note that for $\mu(u) = (n-1)!$, the Lebesgue measure on E_{n-1} , the first part of (1.8) simplifies to (1.5).

In Section 2 we shall prove that $\mathcal{E}_{r,s}(\mu; X)$ is the mean value of X , i.e., it satisfies inequalities (1.6). Some elementary properties of this mean are also derived. Section 3 deals with the limit theorems for the new mean, with the probability measure being the Dirichlet measure. The latter is denoted by μ_b , where $b = (b_1, \dots, b_n) \in \mathbb{R}_+^n$, and is defined as [2]

$$(1.9) \quad \mu_b(u) = \frac{1}{B(b)} \prod_{i=1}^n u_i^{b_i-1},$$

where $B(\cdot)$ is the multivariate beta function, $(u_1, \dots, u_{n-1}) \in E_{n-1}$, and $u_n = 1 - u_1 - \dots - u_{n-1}$. In the Appendix we shall prove that under certain conditions imposed on the parameters r and s , the function $E_{r,s}^{r-s}(x, y)$ is strictly totally positive as a function of x and y .



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2. Elementary Properties of $\mathcal{E}_{r,s}(\mu; X)$

In order to prove that $\mathcal{E}_{r,s}(\mu; X)$ is a mean value we need the following version of the Mean-Value Theorem for integrals.

Proposition 2.1. *Let $\alpha := X_{\min} < X_{\max} =: \beta$ and let $f, g \in C([\alpha, \beta])$ with $g(t) \neq 0$ for all $t \in [\alpha, \beta]$. Then there exists $\xi \in (\alpha, \beta)$ such that*

$$(2.1) \quad \frac{\int_{E_{n-1}} f(u \cdot X) \mu(u) du}{\int_{E_{n-1}} g(u \cdot X) \mu(u) du} = \frac{f(\xi)}{g(\xi)}.$$

Proof. Let the numbers γ and δ and the function ϕ be defined in the following way

$$\gamma = \int_{E_{n-1}} g(u \cdot X) \mu(u) du, \quad \delta = \int_{E_{n-1}} f(u \cdot X) \mu(u) du,$$

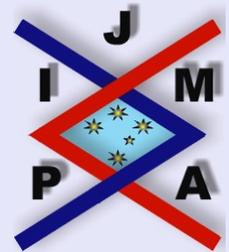
$$\phi(t) = \gamma f(t) - \delta g(t).$$

Letting $t = u \cdot X$ and, next, integrating both sides against the measure μ , we obtain

$$\int_{E_{n-1}} \phi(u \cdot X) \mu(u) du = 0.$$

On the other hand, application of the Mean-Value Theorem to the last integral gives

$$\phi(c \cdot X) \int_{E_{n-1}} \mu(u) du = 0,$$



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where $c = (c_1, \dots, c_{n-1}, c_n)$ with $(c_1, \dots, c_{n-1}) \in E_{n-1}$ and $c_n = 1 - c_1 - \dots - c_{n-1}$. Letting $\xi = c \cdot X$ and taking into account that

$$\int_{E_{n-1}} \mu(u) du = 1$$

we obtain $\phi(\xi) = 0$. This in conjunction with the definition of ϕ gives the desired result (2.1). The proof is complete. \square

The author is indebted to Professor Zsolt Páles for a useful suggestion regarding the proof of Proposition 2.1.

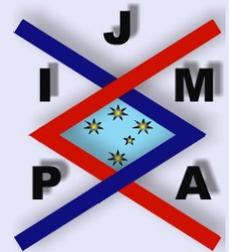
For later use let us introduce the symbol $\mathcal{E}_{r,s}^{(p)}(\mu; X)$ ($p \neq 0$), where

$$(2.2) \quad \mathcal{E}_{r,s}^{(p)}(\mu; X) = [\mathcal{E}_{r,s}(\mu; X^p)]^{\frac{1}{p}}.$$

We are in a position to prove the following.

Theorem 2.2. *Let $X \in \mathbb{R}_+^n$ and let $r, s \in \mathbb{R}$. Then*

- (i) $X_{\min} \leq \mathcal{E}_{r,s}(\mu; X) \leq X_{\max}$,
- (ii) $\mathcal{E}_{r,s}(\mu; \lambda X) = \lambda \mathcal{E}_{r,s}(\mu; X)$, $\lambda > 0$, ($\lambda X := (\lambda x_1, \dots, \lambda x_n)$),
- (iii) $\mathcal{E}_{r,s}(\mu; X)$ increases with an increase in either r and s ,
- (iv) $\ln \mathcal{E}_{r,s}(\mu; X) = \frac{1}{r-s} \int_s^r \ln \mathcal{E}_{t,t}(\mu; X) dt$, $r \neq s$,
- (v) $\mathcal{E}_{r,s}^{(p)}(\mu; X) = \mathcal{E}_{pr,ps}(\mu; X)$,



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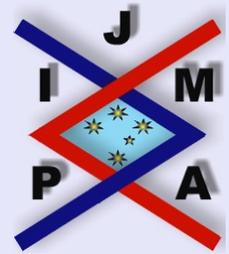


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$$(vi) \mathcal{E}_{r,s}(\mu; X)\mathcal{E}_{-r,-s}(\mu; X^{-1}) = 1, (X^{-1} := (1/x_1, \dots, 1/x_n)),$$

$$(vii) \mathcal{E}_{r,s}^{s-r}(\mu; X) = \mathcal{E}_{r,p}^{p-r}(\mu; X)\mathcal{E}_{p,s}^{s-p}(\mu; X).$$

Proof of (i). Assume first that $r \neq s$. Making use of (1.8) and (1.7) we obtain

$$\mathcal{E}_{r,s}(\mu; X) = \left[\frac{\int_{E_{n-1}} \exp[r(u \cdot Z)] \mu(u) du}{\int_{E_{n-1}} \exp[s(u \cdot Z)] \mu(u) du} \right]^{\frac{1}{r-s}}.$$

Application of (2.1) with $f(t) = \exp(rt)$ and $g(t) = \exp(st)$ gives

$$\mathcal{E}_{r,s}(\mu; X) = \left[\frac{\exp[r(c \cdot Z)]}{\exp[s(c \cdot Z)]} \right]^{\frac{1}{r-s}} = \exp(c \cdot Z),$$

where $c = (c_1, \dots, c_{n-1}, c_n)$ with $(c_1, \dots, c_{n-1}) \in E_{n-1}$ and $c_n = 1 - c_1 - \dots - c_{n-1}$. Since $c \cdot Z = c_1 \ln x_1 + \dots + c_n \ln x_n$, $\ln X_{\min} \leq c \cdot Z \leq \ln X_{\max}$. This in turn implies that $X_{\min} \leq \exp(c \cdot Z) \leq X_{\max}$. This completes the proof of (i) when $r \neq s$. Assume now that $r = s$. It follows from (1.8) and (1.7) that

$$\ln \mathcal{E}_{r,r}(\mu; X) = \left[\frac{\int_{E_{n-1}} (u \cdot Z) \exp[r(u \cdot Z)] \mu(u) du}{\int_{E_{n-1}} \exp[r(u \cdot Z)] \mu(u) du} \right].$$

Application of (2.1) to the right side with $f(t) = t \exp(rt)$ and $g(t) = \exp(rt)$ gives

$$\ln \mathcal{E}_{r,r}(\mu; X) = \left[\frac{(c \cdot Z) \exp[r(c \cdot Z)]}{\exp[r(c \cdot Z)]} \right] = c \cdot Z.$$

Since $\ln X_{\min} \leq c \cdot Z \leq \ln X_{\max}$, the assertion follows. This completes the proof of (i).

Proof of (ii). The following result

$$(2.3) \quad \mathcal{L}(\mu; (\lambda x)^r) = \lambda^r \mathcal{L}(\mu; X^r)$$

($\lambda > 0$) is established in [9, (2.6)]. Assume that $r \neq s$. Using (1.8) and (2.3) we obtain

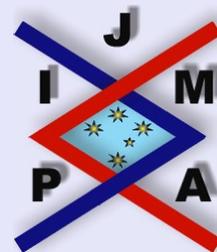
$$\mathcal{E}_{r,s}(\mu; \lambda x) = \left[\frac{\lambda^r \mathcal{L}(\mu; X^r)}{\lambda^s \mathcal{L}(\mu; X^s)} \right]^{\frac{1}{r-s}} = \lambda \mathcal{E}_{r,s}(\mu; X).$$

Consider now the case when $r = s \neq 0$. Making use of (1.8) and (2.3) we obtain

$$\begin{aligned} \mathcal{E}_{r,r}(\mu; \lambda X) &= \exp \left[\frac{d}{dr} \ln \mathcal{L}(\mu; (\lambda X)^r) \right] \\ &= \exp \left[\frac{d}{dr} \ln (\lambda^r \mathcal{L}(\mu; X^r)) \right] \\ &= \exp \left[\frac{d}{dr} (r \ln \lambda + \ln \mathcal{L}(\mu; X^r)) \right] = \lambda \mathcal{E}_{r,r}(\mu; X). \end{aligned}$$

When $r = s = 0$, an easy computation shows that

$$(2.4) \quad \mathcal{E}_{0,0}(\mu; X) = \prod_{i=1}^n x_i^{w_i} \equiv G(w; X),$$



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where

$$(2.5) \quad w_i = \int_{E_{n-1}} u_i \mu(u) du$$

($1 \leq i \leq n$) are called the natural weights or partial moments of the measure μ and $w = (w_1, \dots, w_n)$. Since $w_1 + \dots + w_n = 1$, $\mathcal{E}_{0,0}(\mu; \lambda X) = \lambda \mathcal{E}_{0,0}(\mu; X)$. The proof of (ii) is complete.

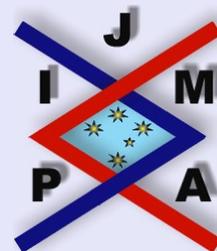
Proof of (iii). In order to establish the asserted property, let us note that the function $r \rightarrow \exp(rt)$ is logarithmically convex (log-convex) in r . This in conjunction with Theorem B.6 in [2], implies that a function $r \rightarrow \mathcal{L}(\mu; X^r)$ is also log-convex in r . It follows from (1.8) that

$$\ln \mathcal{E}_{r,s}(\mu; X) = \frac{\ln \mathcal{L}(\mu; X^r) - \ln \mathcal{L}(\mu; X^s)}{r - s}.$$

The right side is the divided difference of order one at r and s . Convexity of $\ln \mathcal{L}(\mu; X^r)$ in r implies that the divided difference increases with an increase in either r and s . This in turn implies that $\ln \mathcal{E}_{r,s}(\mu; X)$ has the same property. Hence the monotonicity property of the mean $\mathcal{E}_{r,s}$ in its parameters follows. Now let $r = s$. Then (1.8) yields

$$\ln \mathcal{E}_{r,r}(\mu; X) = \frac{d}{dr} [\ln \mathcal{L}(\mu; X^r)].$$

Since $\ln \mathcal{L}(\mu; X^r)$ is convex in r , its derivative with respect to r increases with an increase in r . This completes the proof of (iii).



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Proof of (iv). Let $r \neq s$. It follows from (1.8) that

$$\begin{aligned} \frac{1}{r-s} \int_s^r \ln \mathcal{E}_{t,t}(\mu; X) dt &= \frac{1}{r-s} \int_s^r \frac{d}{dt} [\ln \mathcal{L}(\mu; X^t)] dt \\ &= \frac{1}{r-s} [\ln \mathcal{L}(\mu; X^r) - \ln \mathcal{L}(\mu; X^s)] \\ &= \ln \mathcal{E}_{r,s}(\mu; X). \end{aligned}$$

Proof of (v). Let $r \neq s$. Using (2.2) and (1.8) we obtain

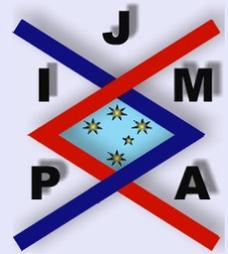
$$\mathcal{E}_{r,s}^{(p)}(\mu; X) = [\mathcal{E}_{r,s}(\mu; X^p)]^{\frac{1}{p}} = \left[\frac{\mathcal{L}(X^{pr})}{\mathcal{L}(X^{ps})} \right]^{\frac{1}{p(r-s)}} = \mathcal{E}_{pr,ps}(\mu; X).$$

Assume now that $r = s \neq 0$. Making use of (2.2), (1.8) and (1.7) we have

$$\begin{aligned} \mathcal{E}_{r,r}^{(p)}(\mu; X) &= \exp \left[\frac{1}{p} \frac{d}{dr} \ln \mathcal{L}(\mu; X^{pr}) \right] \\ &= \exp \left[\frac{1}{\mathcal{L}(\mu; X^{pr})} \int_{E_{n-1}} (u \cdot Z) \exp [pr(u \cdot Z)] \mu(u) du \right] \\ &= \mathcal{E}_{pr,pr}(\mu; X). \end{aligned}$$

The case when $r = s = 0$ is trivial because $\mathcal{E}_{0,0}(\mu; X)$ is the weighted geometric mean of X .

Proof of (vi). Here we use (v) with $p = -1$ to obtain $\mathcal{E}_{r,s}(\mu; X^{-1})^{-1} = \mathcal{E}_{-r,-s}(\mu; X)$. Letting $X := X^{-1}$ we obtain the desired result.



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Proof of (vii). There is nothing to prove when either $p = r$ or $p = s$ or $r = s$. In other cases we use (1.8) to obtain the asserted result. This completes the proof. \square

In the next theorem we give some inequalities involving the means under discussion.

Theorem 2.3. *Let $r, s \in \mathbb{R}$. Then the following inequalities*

$$(2.6) \quad \mathcal{E}_{r,r}(\mu; X) \leq \mathcal{E}_{r,s}(\mu; X) \leq \mathcal{E}_{s,s}(\mu; X)$$

are valid provided $r \leq s$. If $s > 0$, then

$$(2.7) \quad \mathcal{E}_{r-s,0}(\mu; X) \leq \mathcal{E}_{r,s}(\mu; X).$$

Inequality (2.7) is reversed if $s < 0$ and it becomes an equality if $s = 0$. Assume that $r, s > 0$ and let $p \leq q$. Then

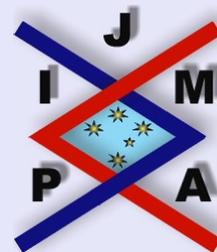
$$(2.8) \quad \mathcal{E}_{r,s}^{(p)}(\mu; X) \leq \mathcal{E}_{r,s}^{(q)}(\mu; X)$$

with the inequality reversed if $r, s < 0$.

Proof. Inequalities (2.6) and (2.7) follow immediately from Part (iii) of Theorem 2.2. For the proof of (2.8), let $r, s > 0$ and let $p \leq q$. Then $pr \leq qr$ and $ps \leq qs$. Applying Parts (v) and (iii) of Theorem 2.2, we obtain

$$\mathcal{E}_{r,s}^{(p)}(\mu; X) = \mathcal{E}_{pr,ps}(\mu; X) \leq \mathcal{E}_{qr,qs}(\mu; X) = \mathcal{E}_{r,s}^{(q)}(\mu; X).$$

When $r, s < 0$, the proof of (2.8) goes along the lines introduced above, hence it is omitted. The proof is complete. \square



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3. The Mean $\mathcal{E}_{r,s}(b; X)$

An important probability measure on E_{n-1} is the Dirichlet measure $\mu_b(u)$, $b \in \mathbb{R}_+^n$ (see (1.9)). Its role in the theory of special functions is well documented in Carlson's monograph [2]. When $\mu = \mu_b$, the mean under discussion will be denoted by $\mathcal{E}_{r,s}(b; X)$. The natural weights w_i (see (2.5)) of μ_b are given explicitly by

$$(3.1) \quad w_i = b_i/c$$

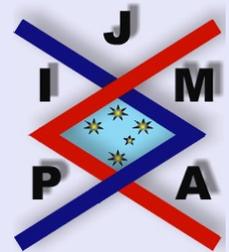
($1 \leq i \leq n$), where $c = b_1 + \dots + b_n$ (see [2, (5.6-2)]). For later use we define $w = (w_1, \dots, w_n)$. Recall that the weighted Dresher mean $D_{r,s}(p; X)$ of order $(r, s) \in \mathbb{R}^2$ of $X \in \mathbb{R}_+^n$ with weights $p = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ is defined as

$$(3.2) \quad D_{r,s}(p; X) = \begin{cases} \left[\frac{\sum_{i=1}^n p_i x_i^r}{\sum_{i=1}^n p_i x_i^s} \right]^{\frac{1}{r-s}}, & r \neq s \\ \exp \left[\frac{\sum_{i=1}^n p_i x_i^r \ln x_i}{\sum_{i=1}^n p_i x_i^r} \right], & r = s \end{cases}$$

(see, e.g., [1, Sec. 24]).

In this section we present two limit theorems for the mean $\mathcal{E}_{r,s}$ with the underlying measure being the Dirichlet measure. In order to facilitate presentation we need a concept of the Dirichlet average of a function. Following [2, Def. 5.2-1] let Ω be a convex set in \mathbb{C} and let $Y = (y_1, \dots, y_n) \in \Omega^n$, $n \geq 2$. Further, let f be a measurable function on Ω . Define

$$(3.3) \quad F(b; Y) = \int_{E_{n-1}} f(u \cdot Y) \mu_b(u) du.$$



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Then F is called the Dirichlet average of f with variables $Y = (y_1, \dots, y_n)$ and parameters $b = (b_1, \dots, b_n)$. We need the following result [2, Ex. 6.3-4]. Let Ω be an open circular disk in \mathbb{C} , and let f be holomorphic on Ω . Let $Y \in \Omega^n$, $c \in \mathbb{C}$, $c \neq 0, -1, \dots$, and $w_1 + \dots + w_n = 1$. Then

$$(3.4) \quad \lim_{c \rightarrow 0} F(cw; Y) = \sum_{i=1}^n w_i f(y_i),$$

where $cw = (cw_1, \dots, cw_n)$.

We are in a position to prove the following.

Theorem 3.1. *Let $w_1 > 0, \dots, w_n > 0$ with $w_1 + \dots + w_n = 1$. If $r, s \in \mathbb{R}$ and $X \in \mathbb{R}_+^n$, then*

$$\lim_{c \rightarrow 0^+} \mathcal{E}_{r,s}(cw; X) = D_{r,s}(w; X).$$

Proof. We use (1.7) and (3.3) to obtain $\mathcal{L}(cw; X) = F(cw; Z)$, where $Z = \ln X = (\ln x_1, \dots, \ln x_n)$. Making use of (3.4) with $f(t) = \exp(t)$ and $Y = \ln X$ we obtain

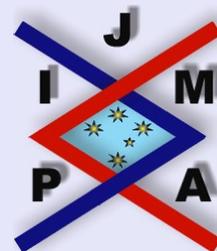
$$\lim_{c \rightarrow 0^+} \mathcal{L}(cw; X) = \sum_{i=1}^n w_i x_i.$$

Hence

$$(3.5) \quad \lim_{c \rightarrow 0^+} \mathcal{L}(cw; X^r) = \sum_{i=1}^n w_i x_i^r.$$

Assume that $r \neq s$. Application of (3.5) to (1.8) gives

$$\lim_{c \rightarrow 0^+} \mathcal{E}_{r,s}(cw; X) = \lim_{c \rightarrow 0^+} \left[\frac{\mathcal{L}(cw; X^r)}{\mathcal{L}(cw; X^s)} \right]^{\frac{1}{r-s}} = \left[\frac{\sum_{i=1}^n w_i x_i^r}{\sum_{i=1}^n w_i x_i^s} \right]^{\frac{1}{r-s}} = D_{r,s}(w; X).$$



Let $r = s$. Application of (3.4) with $f(t) = t \exp(rt)$ gives

$$\lim_{c \rightarrow 0^+} F(cw; Z) = \sum_{i=1}^n w_i z_i \exp(rz_i) = \sum_{i=1}^n w_i (\ln x_i) x_i^r.$$

This in conjunction with (3.5) and (1.8) gives

$$\begin{aligned} \lim_{c \rightarrow 0^+} \mathcal{E}_{r,r}(cw; X) &= \lim_{c \rightarrow 0^+} \exp \left[\frac{F(cw; Z)}{\mathcal{L}(cw; X^r)} \right] \\ &= \exp \left[\frac{\sum_{i=1}^n w_i x_i^r \ln x_i}{\sum_{i=1}^n w_i x_i^r} \right] \\ &= D_{r,r}(w; X). \end{aligned}$$

This completes the proof. □

Theorem 3.2. *Under the assumptions of Theorem 3.1 one has*

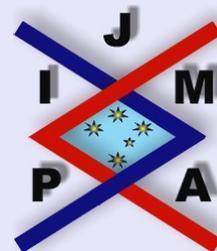
$$(3.6) \quad \lim_{c \rightarrow \infty} \mathcal{E}_{r,s}(cw; X) = G(w; X).$$

Proof. The following limit (see [9, (4.10)])

$$(3.7) \quad \lim_{c \rightarrow \infty} \mathcal{L}(cw; X) = G(w; X)$$

will be used in the sequel. We shall establish first (3.6) when $r \neq s$. It follows from (1.8) and (3.7) that

$$\lim_{c \rightarrow \infty} \mathcal{E}_{r,s}(cw; X) = \lim_{c \rightarrow \infty} \left[\frac{\mathcal{L}(cw; X^r)}{\mathcal{L}(cw; X^s)} \right]^{\frac{1}{r-s}} = [G(w; X)^{r-s}]^{\frac{1}{r-s}} = G(w; X).$$



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Assume that $r = s$. We shall prove first that

$$(3.8) \quad \lim_{c \rightarrow \infty} F(cw; Z) = [\ln G(w; X)] G(w; X)^r,$$

where F is the Dirichlet average of $f(t) = t \exp(rt)$. Averaging both sides of

$$f(t) = \sum_{m=0}^{\infty} \frac{r^m}{m!} t^{m+1}$$

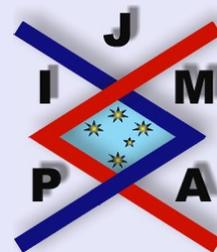
we obtain

$$(3.9) \quad F(cw; Z) = \sum_{m=0}^{\infty} \frac{r^m}{m!} R_{m+1}(cw; Z),$$

where R_{m+1} stands for the Dirichlet average of the power function t^{m+1} . We will show that the series in (3.9) converges uniformly in $0 < c < \infty$. This in turn implies further that as $c \rightarrow \infty$, we can proceed to the limit term by term. Making use of [2, 6.2-24)] we obtain

$$|R_{m+1}(cw; Z)| \leq |Z|^{m+1}, \quad m \in \mathbb{N},$$

where $|Z| = \max \{ |\ln x_i| : 1 \leq i \leq n \}$. By the Weierstrass M test the series in (3.9) converges uniformly in the stated domain. Taking limits on both sides



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of (3.9) we obtain with the aid of (3.4)

$$\begin{aligned}
 \lim_{c \rightarrow \infty} F(cw; Z) &= \sum_{m=0}^{\infty} \frac{r^m}{m!} \lim_{c \rightarrow \infty} R_{m+1}(cw; Z) \\
 &= \sum_{m=0}^{\infty} \frac{r^m}{m!} \left(\sum_{i=1}^n w_i z_i \right)^{m+1} \\
 &= [\ln G(w; X)] \sum_{m=0}^{\infty} \frac{r^m}{m!} [\ln G(w; X)]^m \\
 &= [\ln G(w; X)] \sum_{m=0}^{\infty} \frac{1}{m!} [\ln G(w; X)]^m \\
 &= [\ln G(w; X)] G(w; X)^r.
 \end{aligned}$$

This completes the proof of (3.8). To complete the proof of (3.6) we use (1.8), (3.7), and (3.8) to obtain

$$\begin{aligned}
 \lim_{c \rightarrow \infty} \ln \mathcal{E}_{r,r}(\mu; X) &= \lim_{c \rightarrow \infty} \frac{F(cw; Z)}{\mathcal{L}(cw; X^r)} \\
 &= \frac{[\ln G(w; X)] G(w; X)^r}{G(w; X)^r} = \ln G(w; X).
 \end{aligned}$$

Hence the assertion follows. □



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A. Appendix: Total Positivity of $E_{r,s}^{r-s}(x, y)$

A real-valued function $h(x, y)$ of two real variables is said to be strictly totally positive on its domain if every $n \times n$ determinant with elements $h(x_i, y_j)$, where $x_1 < x_2 < \dots < x_n$ and $y_1 < y_2 < \dots < y_n$ is strictly positive for every $n = 1, 2, \dots$ (see [5]).

The goal of this section is to prove that the function $E_{r,s}^{r-s}(x, y)$ is strictly totally positive as a function of x and y provided the parameters r and s satisfy a certain condition. For later use we recall the definition of the R -hypergeometric function $R_{-\alpha}(\beta, \beta'; x, y)$ of two variables $x, y > 0$ with parameters $\beta, \beta' > 0$

$$(A1) \quad R_{-\alpha}(\beta, \beta'; x, y) = \frac{\Gamma(\beta + \beta')}{\Gamma(\beta)\Gamma(\beta')} \int_0^1 u^{\beta-1}(1-u)^{\beta'-1} [ux + (1-u)y]^{-\alpha} du$$

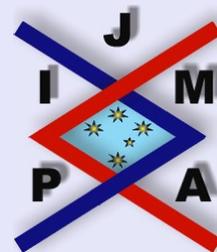
(see [2, (5.9-1)]).

Proposition A.1. *Let $x, y > 0$ and let $r, s \in \mathbb{R}$. If $|r| < |s|$, then $E_{r,s}^{r-s}(x, y)$ is strictly totally positive on \mathbb{R}_+^2 .*

Proof. Using (1.3) and (A1) we have

$$(A2) \quad E_{r,s}^{r-s}(x, y) = R_{\frac{r-s}{s}}(1, 1; x^s, y^s)$$

($s(r-s) \neq 0$). B. Carlson and J. Gustafson [3] have proven that $R_{-\alpha}(\beta, \beta'; x, y)$ is strictly totally positive in x and y provided $\beta, \beta' > 0$ and $0 < \alpha < \beta + \beta'$. Letting $\alpha = 1 - r/s$, $\beta = \beta' = 1$, $x := x^s$, $y := y^s$, and next, using (A2) we obtain the desired result. \square



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Corollary A.2. Let $0 < x_1 < x_2$, $0 < y_1 < y_2$ and let the real numbers r and s satisfy the inequality $|r| < |s|$. If $s > 0$, then

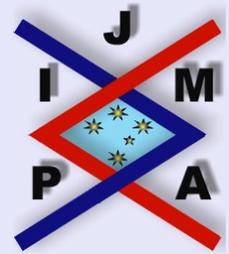
$$(A3) \quad E_{r,s}(x_1, y_1)E_{r,s}(x_2, y_2) < E_{r,s}(x_1, y_2)E_{r,s}(x_2, y_1).$$

Inequality (A3) is reversed if $s < 0$.

Proof. Let $a_{ij} = E_{r,s}^{r-s}(x_i, y_j)$ ($i, j = 1, 2$). It follows from Proposition A.1 that $\det([a_{ij}]) > 0$ provided $|r| < |s|$. This in turn implies

$$[E_{r,s}(x_1, y_1)E_{r,s}(x_2, y_2)]^{r-s} > [E_{r,s}(x_1, y_2)E_{r,s}(x_2, y_1)]^{r-s}.$$

Assume that $s > 0$. Then the inequality $|r| < s$ implies $r - s < 0$. Hence (A3) follows when $s > 0$. The case when $s < 0$ is treated in a similar way. \square



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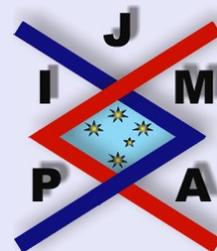
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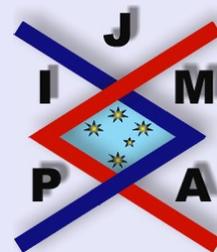
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