Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 1, Article 27, 2006

THE EXTENSION OF MAJORIZATION INEQUALITIES WITHIN THE FRAMEWORK OF RELATIVE CONVEXITY

CONSTANTIN P. NICULESCU AND FLORIN POPOVICI

UNIVERSITY OF CRAIOVA
DEPARTMENT OF MATHEMATICS
A. I. CUZA STREET 13
CRAIOVA 200585, ROMANIA.
cniculescu@central.ucv.ro

COLLEGE NICOLAE TITULESCU
BRAŞOV 500435, ROMANIA.
popovici.florin@yahoo.com

Received 12 July, 2005; accepted 07 December, 2005
Communicated by S.S. Dragomir

ABSTRACT. Some of the basic inequalities in majorization theory (Hardy-Littlewood-Pólya, Tomić-Weyl and Fuchs) are extended to the framework of relative convexity.

Key words and phrases: Relative convexity, Majorization, Abel summation formula.

2000 *Mathematics Subject Classification.* Primary 26A51, 26D15; Secondary 26D05.

Relative convexity is related to comparison of quasi-arithmetic means and goes back to B. Jessen. See [5], Theorem 92, p. 75. Later contributions came from G. T. Cargo [2], N. Elezović and J. Pečarić [3], M. Bessenyei and Z. Páles [1], C. P. Niculescu [10] and many others. The aim of this note is to prove the extension to this framework of all basic majorization inequalities, starting with the well known inequality of Hardy-Littlewood-Pólya. The classical text on majorization theory is still the monograph of A. W. Marshall and I. Olkin [7], but the results involved in what follows can be also found in [8] and [11].

Throughout this paper f and g will be two real-valued functions with the same domain of definition X. Moreover, g is assumed to be a nonconstant function.

Definition 1. We say that f is convex with respect to g $($abbreviated, $g \lhd f)$ if

$$\begin{vmatrix} 1 & g(x) & f(x) \\ 1 & g(y) & f(y) \\ 1 & g(z) & f(z) \end{vmatrix} \geq 0$$

ISSN (electronic): 1443-5756

The first author was partially supported by CNCSIS Grant 80/2005.

209-05

whenever $x, y, z \in X$ and $g(x) \leq g(y) \leq g(z)$.

When X is an interval, and g is continuous and strictly monotonic, this definition simply means that $f \circ g^{-1}$ is convex in the usual sense on the interval $Y = g(X)$. Our definition is strictly larger since we do not make any assumption on the monotonicity of g. For example,

$$f \lhd f^{\alpha} \text{ for all } f : X \to \mathbb{R}_{+} \text{ and all } \alpha \geq 1.$$

In particular, $\sin \lhd \sin^2$ on $[0, \pi]$, and $|x| \lhd x^2$ on $\mathbb{R}$.

Definition 1 allows us to bring together several classes of convex-like functions. In fact,

$$f \text{ is convex} \Leftrightarrow id \lhd f$$
$$f \text{ is log-convex} \Leftrightarrow id \lhd \log f$$
$$f \text{ is multiplicatively convex} \Leftrightarrow \log \lhd \log f.$$

Multiplicative convexity means that f acts on subintervals of $(0, \infty)$ and

$$f\left(x^{1-\lambda} y^{\lambda}\right) \leq f(x)^{1-\lambda} f(y)^{\lambda}$$

for all x and y in the domain of f and all $\lambda \in [0, 1]$. See [9], [11].

Lemma 1. *If $f, g : X \to \mathbb{R}$ are two functions such that $g \lhd f$, then*

$$g(x) = g(y) \textit{ implies } f(x) = f(y).$$

Proof. Since g is not constant, then there must be a $z \in X$ such that $g(x) = g(y) \neq g(z)$. The following two cases may occur:
Case 1: $g(x) = g(y) < g(z)$. This yields

$$0 \leq \begin{vmatrix} 1 & g(x) & f(x) \\ 1 & g(x) & f(y) \\ 1 & g(z) & f(z) \end{vmatrix} = \left(g(z) - g(x)\right)\left(f(x) - f(y)\right),$$

so that $f(x) \geq f(y)$. A similar argument gives us the reverse inequality, $f(x) \leq f(y)$.

Case 2: $g(z) < g(x) = g(y)$. This case can be treated in a similar way. $\square$

The analogue of Fuchs' majorization inequality [4] in the context of relative convexity will be established via a generalization of Galvani's Lemma.

Lemma 2. *If $g \lhd f$, then for every $a, u, v \in X$ with $g(u) \leq g(v)$ and $g(a) \notin \{g(u), g(v)\}$, we have*

$$\frac{f(u) - f(a)}{g(u) - g(a)} \leq \frac{f(v) - f(a)}{g(v) - g(a)}.$$

Proof. In fact, the following three cases may occur:
Case 1: $g(a) < g(u) \leq g(v)$. Then

$$\begin{aligned} 0 &\leq \begin{vmatrix} 1 & g(a) & f(a) \\ 1 & g(u) & f(u) \\ 1 & g(v) & f(v) \end{vmatrix} \\ &= \left(g(u) - g(a)\right)\left(f(v) - f(a)\right) - \left(g(v) - g(a)\right)\left(f(u) - f(a)\right) \end{aligned}$$

and the conclusion of Lemma 2 is clear.

Case 2: $g(u) \leq g(v) < g(a)$. This case can be treated in the same way.

Case 3: $g(u) < g(a) < g(v)$. According to the discussion above we have

$$\begin{aligned}\frac{f(u)-f(a)}{g(u)-g(a)} &= \frac{f(a)-f(u)}{g(a)-g(u)} \leq \frac{f(v)-f(u)}{g(v)-g(u)} \\ &= \frac{f(u)-f(v)}{g(u)-g(v)} \leq \frac{f(a)-f(v)}{g(a)-g(v)} = \frac{f(v)-f(a)}{g(v)-g(a)}\end{aligned}$$

and the proof is now complete. □

Theorem 3 (The generalization of the Hardy-Littlewood-Pólya inequality)**.** *Let $f, g : X \to \mathbb{R}$ be two functions such that $g \lhd f$ and consider points $x_1, \ldots, x_n, y_1, \ldots, y_n$ in X and real weights $p_1, \ldots, p_n$ such that:*

(i) $g(x_1) \geq \ldots \geq g(x_n)$ *and* $g(y_1) \geq \ldots \geq g(y_n)$;

(ii) $\sum_{k=1}^{r} p_k g(x_k) \leq \sum_{k=1}^{r} p_k g(y_k)$ *for all* $r = 1, \ldots, n$;

(iii) $\sum_{k=1}^{n} p_k g(x_k) = \sum_{k=1}^{n} p_k g(y_k)$.

Then

$$\sum_{k=1}^{n} p_k f(x_k) \leq \sum_{k=1}^{n} p_k f(y_k).$$

Proof. By mathematical induction. The case $n = 1$ is clear. Assuming the conclusion of Theorem 3 is valid for all families of length $n-1$, let us pass to the case of families of length n. If $g(x_k) = g(y_k)$ for some index k, then $f(x_k) = f(y_k)$ by Lemma 1, and we can apply our induction hypothesis. Thus we may restrict ourselves to the case where $g(x_k) \neq g(y_k)$ for all indices k. By Abel's summation formula, the difference

$$\sum_{k=1}^{n} p_k f(y_k) - \sum_{k=1}^{n} p_k f(x_k) \tag{1}$$

equals

$$\begin{aligned}&\frac{f(y_n)-f(x_n)}{g(y_n)-g(x_n)} \left(\sum_{i=1}^{n} p_i g(y_i) - \sum_{i=1}^{n} p_i g(x_i) \right) \\ &\qquad + \sum_{k=1}^{n-1} \left(\frac{f(y_k)-f(x_k)}{g(y_k)-g(x_k)} - \frac{f(y_{k+1})-f(x_{k+1})}{g(y_{k+1})-g(x_{k+1})} \right) \left(\sum_{i=1}^{k} p_i g(y_i) - \sum_{i=1}^{k} p_i g(x_i) \right)\end{aligned}$$

which, by (iii), reduces to

$$\sum_{k=1}^{n-1} \left(\frac{f(y_k)-f(x_k)}{g(y_k)-g(x_k)} - \frac{f(y_{k+1})-f(x_{k+1})}{g(y_{k+1})-g(x_{k+1})} \right) \left(\sum_{i=1}^{k} p_i g(y_i) - \sum_{i=1}^{k} p_i g(x_i) \right).$$

According to (ii), the proof will be complete if we show that

$$\frac{f(y_{k+1})-f(x_{k+1})}{g(y_{k+1})-g(x_{k+1})} \leq \frac{f(y_k)-f(x_k)}{g(y_k)-g(x_k)} \tag{2}$$

for all indices k.

In fact, if $g(x_k) = g(x_{k+1})$ or $g(y_k) = g(y_{k+1})$ for some index k, this follows from i) and Lemmas 1 and 2.

When $g(x_k) > g(x_{k+1})$ and $g(y_k) > g(y_{k+1})$, the following two cases may occur:

Case 1: $g(x_k) \neq g(y_{k+1})$. By a repeated application of Lemma 2 we get

$$\begin{aligned}\frac{f(y_{k+1})-f(x_{k+1})}{g(y_{k+1})-g(x_{k+1})} &= \frac{f(x_{k+1})-f(y_{k+1})}{g(x_{k+1})-g(y_{k+1})} \leq \frac{f(x_k)-f(y_{k+1})}{g(x_k)-g(y_{k+1})} \\ &= \frac{f(y_{k+1})-f(x_k)}{g(y_{k+1})-g(x_k)} \leq \frac{f(y_k)-f(x_k)}{g(y_k)-g(x_k)}.\end{aligned}$$

Case 2: $g(x_k) = g(y_{k+1})$. In this case, $g(x_{k+1}) < g(x_k) = g(y_{k+1}) < g(y_k)$, and Lemmas 1 and 2 leads us to

$$\begin{aligned}\frac{f(y_{k+1})-f(x_{k+1})}{g(y_{k+1})-g(x_{k+1})} &= \frac{f(x_k)-f(x_{k+1})}{g(x_k)-g(x_{k+1})} \\ &= \frac{f(x_{k+1})-f(x_k)}{g(x_{k+1})-g(x_k)} \leq \frac{f(y_k)-f(x_k)}{g(y_k)-g(x_k)}.\end{aligned}$$

Consequently, (1) is a sum of nonnegative terms, and the proof is complete. □

The classical Hardy-Littlewood-Pólya inequality corresponds to the case where g is the identity and $p_k = 1$ for all k. In this case, it is easily seen that the hypothesis i) can be replaced by a weaker condition,

$$g(x_1) \geq \ldots \geq g(x_n). \tag{i$'$}$$

When X is an interval, g is the identity map of X, and $p_1, \ldots, p_n$ are arbitrary weights, we recover the Fuchs inequality [4] (or [8, p. 165]).

An illustration of Theorem 3 is offered by the following simple example.

Example Suppose that $f : [0, \pi] \to \mathbb{R}$ is a function such that

$$(f(y) - f(z)) \sin x + (f(z) - f(x)) \sin y + (f(x) - f(y)) \sin z \geq 0 \tag{3}$$

for all x, y, z in $[0, \pi]$, with $\sin x \leq \sin y \leq \sin z$. Then

$$f\left(\frac{9\pi}{14}\right) - f\left(\frac{3\pi}{14}\right) + f\left(\frac{\pi}{14}\right) \leq f\left(\frac{\pi}{2}\right) - f\left(\frac{\pi}{6}\right) + f(0). \tag{4}$$

In fact, the condition (3) means precisely that $\sin \lhd f$*. The conclusion (4) is based on a little computation:*

$$\sin\frac{\pi}{2} > \sin\frac{\pi}{6} > \sin 0, \quad \sin\frac{9\pi}{14} > \sin\frac{3\pi}{14} > \sin\frac{\pi}{14},$$
$$\sin\frac{\pi}{2} > \sin\frac{9\pi}{14},$$
$$\sin\frac{\pi}{2} - \sin\frac{\pi}{6} > \sin\frac{9\pi}{14} - \sin\frac{3\pi}{14},$$
$$\sin\frac{\pi}{2} - \sin\frac{\pi}{6} + \sin 0 = \sin\frac{9\pi}{14} - \sin\frac{3\pi}{14} + \sin\frac{\pi}{14} = \frac{1}{2}.$$

The inequality (4) *is not obvious even when* $f(x) = \sin^2 x$.

In the same spirit we can extend the Tomić-Weyl theorem. This will be done for *synchronous* functions, that is, for functions $f, g : X \to \mathbb{R}$ such that

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for all x and y in X. For example, this happens when X is an interval and f and g have the same monotonicity. Another example is provided by the pair $f = h^\alpha$ and $g = h \geq 0$, for $\alpha \geq 1$; in this case, $g \lhd f$.

Theorem 4 (The extension of the Tomić-Weyl theorem)**.** *. Suppose that $f, g : X \to \mathbb{R}$ are two synchronous functions with $g \lhd f$. Consider points $x_1, \ldots, x_n, y_1, \ldots, y_n$ in X and real weights $p_1, \ldots, p_n$ such that:*

i) $g(x_1) \geq \ldots \geq g(x_n)$ *and* $g(y_1) \geq \ldots \geq g(y_n)$;

ii) $\sum_{k=1}^{r} p_k g(x_k) \leq \sum_{k=1}^{r} p_k g(y_k)$ *for all* $r = 1, \ldots, n$.

Then

$$\sum_{k=1}^{n} p_k f(x_k) \leq \sum_{k=1}^{n} p_k f(y_k).$$

Proof. Clearly, the statement of Theorem 4 is true for $n = 1$. Suppose that $n \geq 2$ and the statement is true for all families of length $n - 1$. If there exists a $k \in \{1, \ldots, n\}$ such that $g(x_k) = g(y_k)$, then the conclusion is a consequence of our induction hypothesis. If $g(x_k) \neq g(y_k)$ for all k, then we may compute the difference (1) as in the proof of Theorem 3, by using the Abel summation formula. By our hypothesis, all the terms in this formula are nonnegative, hence the difference (1) is nonnegative. $\square$

The integral version of the above results is more or less routine. For example, using Riemann sums, one can prove the following generalization of Theorem 4:

Theorem 5. *Suppose there are given a pair of synchronous functions $f, g : X \to \mathbb{R}$, with $g \lhd f$, a continuous weight $w : [a, b] \to \mathbb{R}$, and functions $\varphi, \psi : [a, b] \to X$ such that*

$$f \circ \varphi \text{ and } f \circ \psi \text{ are Riemann integrable and } g \circ \varphi \text{ and } g \circ \psi \text{ are nonincreasing}$$

and

$$\int_a^x g(\varphi(t))w(t)dt \leq \int_a^x g(\psi(t))w(t)dt \quad \text{for all } x \in [a, b].$$

Then

$$\int_a^b f(\varphi(t))w(t)dt \leq \int_a^b f(\psi(t))w(t)dt.$$

With some extra work one can adapt these results to the context of Lebesgue integrability and symmetric-decreasing rearrangements. Notice that a less general integral form of the Hardy-Littlewood-Pólya inequality appears in [7], Ch. 1, Section D. See [5] and [6] for a thorough presentation of the topics of symmetric-decreasing rearrangements.

Finally, let us note that a more general concept of relative convexity, with respect to a pair of functions, is available in the literature. Given a pair (ω_1, ω_2) of continuous functions on an interval I such that

$$\begin{vmatrix} \omega_1(x) & \omega_1(y) \\ \omega_2(x) & \omega_2(y) \end{vmatrix} \neq 0 \quad \text{for all } x < y, \tag{5}$$

a function $f : I \to \mathbb{R}$ is said to be (ω_1, ω_2)-*convex* (in the sense of Pólya) if

$$\begin{vmatrix} f(x) & f(y) & f(z) \\ \omega_1(x) & \omega_1(y) & \omega_1(z) \\ \omega_2(x) & \omega_2(y) & \omega_2(z) \end{vmatrix} \geq 0$$

for all $x < y < z$ in I. It is proved that the (ω_1, ω_2)-convexity implies the continuity of f at the interior points of I, as well as the integrability on compact subintervals of I.

If I is an open interval, $\omega_1 > 0$ and the determinant in the formula (5) is positive, then f is (ω_1, ω_2)-convex if and only if the function $\frac{f}{\omega_1} \circ \left(\frac{\omega_2}{\omega_1}\right)^{-1}$ is convex in the usual sense (equivalently, if and only if $\omega_2/\omega_1 \lhd f/\omega_1$).

Historically, the concept of (ω_1, ω_2)-convexity can be traced back to G. Pólya. See [12] and the comments to Theorem 123, p. 98, in [5]. Recently, M. Bessenyei and Z. Páles [1] have obtained a series of interesting results in this context, which opens the problem of a full generalization of the Theorems 3 and 4 to the context of relative convexity in the sense of Pólya. But this will be considered elsewhere.

REFERENCES

[1] M. BESSENYEI AND Z. PÁLES, Hadamard-type inequalities for generalized convex functions, *Math. Inequal. Appl.*, **6** (2003), 379–392.

[2] G.T. CARGO, Comparable means and generalized convexity, *J. Math. Anal. Appl.*, **12** (1965), 387–392.

[3] N. ELEZOVIĆ AND J. PEČARIĆ, Differential and integral F-means and applications to digamma function, *Math. Inequal. Appl.*, **3** (2000), 189–196.

[4] L. FUCHS, A new proof of an inequality of Hardy, Littlewood and Pólya, *Mat. Tidsskr. B.*, 1947, pp. 53–54.

[5] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge Mathematical Library, 2nd Ed., 1952, Reprinted 1988.

[6] E.H. LIEB AND M. LOSS, *Analysis*, 2nd Edition, Amer. Math. Soc., Providence, R. I., 2001.

[7] A.W. MARSHALL AND I. OLKIN, *Inequalities: Theory of Majorization and its Applications*, Academic Press, 1979.

[8] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, Berlin and New York, 1970.

[9] C.P. NICULESCU, Convexity according to the geometric mean, *Math. Inequal. Appl.*, **3** (2000), 155–167.

[10] C.P. NICULESCU, Convexity according to means, *Math. Inequal. Appl.*, **6** (2003), 571–579.

[11] C.P. NICULESCU AND L.-E. PERSSON, *Convex Functions and their Applications. A Contemporary Approach*, CMS Books in Mathematics **23**, Springer, New York, 2006.

[12] G. PÓLYA, On the mean-value theorem corresponding to a given linear homogeneous differential equation, *Trans. Amer. Math. Soc.*, **24** (1922), 312–324.