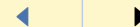
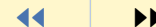




[Title Page](#)

[Contents](#)



Page 1 of 18

[Go Back](#)

[Full Screen](#)

[Close](#)

ITERATION SEMIGROUPS WITH GENERALIZED CONVEX, CONCAVE AND AFFINE ELEMENTS

DOROTA KRASSOWSKA

Faculty of Mathematics, Computer Science and Econometrics

University of Zielona Góra

ul. Licealna 9

PL-65-417 Zielona Góra, POLAND

E-Mail: d.krassowska@wmie.uz.zgora.pl

Received: 31 July, 2008

Accepted: 22 March, 2009

Communicated by: [A. Gilányi](#)

2000 AMS Sub. Class.: Primary 39B12, 26A18; Secondary 54H15.

Key words: Iteration semigroup, Convex (concave, affine)-functions with respect to some functions.

Abstract: Given continuous functions M and N of two variables, it is shown that if in a continuous iteration semigroup with only (M, N) -convex or (M, N) -concave elements there are two (M, N) -affine elements, then $M = N$ and every element of the semigroup is M -affine. Moreover, all functions in the semigroup either are M -convex or M -concave.

Contents

1	Introduction	3
2	Preliminaries	4
3	Results	10



**Semigroups with Generalized
Convex Elements**

Dorota Krassowska

vol. 10, iss. 3, art. 76, 2009

Title Page

Contents



Page 2 of 18

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

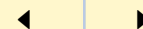
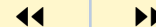
1. Introduction

In this paper we use the definition of (M, N) -convex, (M, N) -concave and (M, N) -affine functions, introduced earlier by G. Aumann [1]. For a given M in $(0, \infty) \times (0, \infty)$ J. Matkowski [5] considered a continuous multiplicative iteration group of homeomorphisms $f^t : (0, \infty) \rightarrow (0, \infty)$, consisting of M -convex or M -concave elements. In the present paper we generalize some results of Matkowski considering the problem proposed in [5]. Let M and N be arbitrary continuous functions. We prove that, if in a continuous iteration semigroup with only (M, N) -convex or (M, N) -concave elements there are two (M, N) -affine functions, then every element of the semigroup is M -affine. Moreover, we show that if in a semigroup there exist f^{t_0} , which is (M, N) -affine, and two iterates with indices greater than t_0 , one (M, N) -convex and the second (M, N) -concave, then the thesis is the same (all elements in a semigroup are M -affine). We end the paper with theorems describing the regularity of semigroups containing generalized convex and concave elements.



Title Page

Contents



Page 3 of 18

Go Back

Full Screen

Close

2. Preliminaries

Let $I, J \subset \mathbb{R}$ be open intervals and let $M : I^2 \rightarrow I$, $N : J^2 \rightarrow J$ be arbitrary functions.

A function $f : I \rightarrow J$ is said to be (M, N) -convex, if

$$f(M(x, y)) \leq N(f(x), f(y)), \quad x, y \in I;$$

(M, N) -concave, if

$$f(M(x, y)) \geq N(f(x), f(y)), \quad x, y \in I;$$

(M, N) -affine, if it is both (M, N) -convex and (M, N) -concave.

In the case when $M = N$, the respective functions are called M -convex, M -concave, and M -affine, respectively.

We start with three remarks which can easily be verified.

Remark 1. If a function f is increasing and (M, N) -convex, then for all M_1 and N_1 satisfying $M_1 \leq M$ and $N_1 \geq N$ it is (M_1, N_1) -convex. Analogously, if a function f is decreasing and (M, N) -concave, then for all M_1 and N_1 satisfying $M_1 \leq M$ and $N_1 \leq N$ it is (M_1, N_1) -concave.

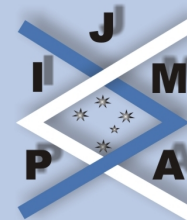
Remark 2. Let $f : I \rightarrow J$ be strictly increasing and onto J . If f is (M, N) -convex then its inverse function f^{-1} is (N, M) -concave.

If $f : I \rightarrow J$ is strictly decreasing, onto and (M, N) -convex, then its inverse function is (N, M) -convex.

If $f : I \rightarrow J$ is (M, N) -affine, then its inverse function is (N, M) -affine.

Remark 3. Let $I, J, K \subset \mathbb{R}$ be open intervals and $M : I^2 \rightarrow I$, $N : J^2 \rightarrow J$, $P : K^2 \rightarrow K$ be arbitrary functions.

If $g : I \rightarrow K$ is (M, P) -affine and $f : K \rightarrow J$ is (P, N) -affine, then $f \circ g$ is (M, N) -affine.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 4 of 18

Go Back

Full Screen

Close



Title Page

Contents



Page 5 of 18

Go Back

Full Screen

Close

Under some additional conditions on f and g , the converse implication also holds true. Namely, we have the following:

Lemma 2.1. *Suppose that $g : I \rightarrow K$ is onto and (M, P) -convex and $f : K \rightarrow J$ is strictly increasing and (P, N) -convex. If $f \circ g$ is (M, N) -affine, then g is (M, P) -affine and f is (P, N) -affine.*

Proof. Let $f \circ g$ be (M, N) -affine. Assume, to the contrary, that f is not (P, N) -affine. Then $u_0, v_0 \in K$ would exist such that

$$f(P(u_0, v_0)) < N(f(u_0), f(v_0)).$$

Since g is onto K , there are $x_0, y_0 \in I$ such that $g(x_0) = u_0$ and $g(y_0) = v_0$. Hence, by the monotonicity of f and the (M, P) -convexity of g ,

$$\begin{aligned} f \circ g(M(x_0, y_0)) &\leq f(P(g(x_0), g(y_0))) \\ &= f(P(u_0, v_0)) \\ &< N(f(u_0), f(v_0)) \\ &= N(f \circ g(x_0), f \circ g(y_0)), \end{aligned}$$

which contradicts the assumption that $f \circ g$ is (M, N) -affine.

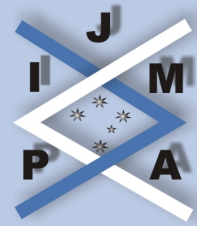
Similarly, if g were not (M, P) -affine then we would have

$$g(M(x_0, y_0)) < P(g(x_0), g(y_0))$$

for some $x_0, y_0 \in I$. By the monotonicity and the (P, N) -convexity of f we would obtain

$$f(g(M(x_0, y_0))) < f(P(g(x_0), g(y_0))) \leq N(f(g(x_0)), f(g(y_0))),$$

which contradicts the (M, N) -affinity of $f \circ g$. This contradiction completes the proof. \square



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 6 of 18

Go Back

Full Screen

Close

In a similar way one can show the following:

Remark 4. Suppose that $g : I \rightarrow K$ is onto and (M, P) -concave and $f : K \rightarrow J$ is strictly increasing and (P, N) -concave. If $f \circ g$ is (M, N) -affine, then g is (M, P) -affine and f is (P, N) -affine.

Remark 5. Observe that, without any loss of generality, considering the (M, N) -affinity, the (M, N) -convexity or the (M, N) -concavity of a function f we can assume that $I = J = (0, \infty)$.

Indeed, let $\varphi : (0, \infty) \rightarrow I$ and $\psi : J \rightarrow (0, \infty)$ be one-to-one and onto. Put $M_\varphi(s, t) := \varphi^{-1}(M(\varphi(s), \varphi(t)))$ and $N_\psi(u, v) := \psi(N(\psi^{-1}(u), \psi^{-1}(v)))$. A function $f : I \rightarrow J$ satisfies the equation

$$f(M(x, y)) = N(f(x), f(y)), \quad x, y \in I,$$

if and only if the function $f^* := \psi \circ f \circ \varphi : (0, \infty) \rightarrow (0, \infty)$ satisfies

$$f^*(M_\varphi(s, t)) = N_\psi(f^*(s), f^*(t)), \quad s, t \in (0, \infty).$$

Moreover, if ψ is strictly increasing, then f is (M, N) -convex ((M, N) -concave) if and only if f^* is (M_φ, N_ψ) -convex ((M_φ, N_ψ) -concave); if ψ is strictly decreasing, then f is (M, N) -convex ((M, N) -concave) if and only if f^* is (M_φ, N_ψ) -concave ((M_φ, N_ψ) -convex).

In what follows, we assume that $I = J$.

In the proof of the main result we need the following

Lemma 2.2. *Suppose that a non-decreasing function $f : I \rightarrow I$ is M -convex (or M -concave) and one-to-one or onto. If, for a positive integer m , the m -th iterate of f is M -affine, then f is M -affine.*

Proof. Assume that f is M -convex. Using, in turn, the convexity, the monotonicity, and again the convexity of f , we get, for $x, y \in I$,

$$f^2(M(x, y)) = f(f(M(x, y))) \leq f(M(f(x), f(y))) \leq M(f^2(x), f^2(y)),$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 7 of 18

Go Back

Full Screen

Close

and further, by induction, for all $x, y \in I$ and $n \in \mathbb{N}$,

$$f^n(M(x, y)) = f(f^{n-1}(M(x, y))) \leq f(M(f^{n-1}(x), f^{n-1}(y))) \leq M(f^n(x), f^n(y)).$$

Hence, since f^m is M -affine for an $m \in \mathbb{N}$, i.e.

$$f^m(M(x, y)) = M(f^m(x), f^m(y)), \quad x, y \in I,$$

we obtain, for all $x, y \in I$,

$$(2.1) \quad f^m(M(x, y)) = f(M(f^{m-1}(x), f^{m-1}(y))) = M(f^m(x), f^m(y)).$$

Now, if f is one-to-one, from the first of these equalities we get

$$f^{m-1}(M(x, y)) = M(f^{m-1}(x), f^{m-1}(y)), \quad x, y \in I,$$

which means that f^{m-1} is an M -affine function. Repeating this procedure $m - 2$ times we obtain the M -affinity of f . Now assume that f is onto I . If $m = 1$ there is nothing to prove. Assume that $m \geq 2$. Since f^{m-1} is also onto I , for arbitrary $u, v \in I$ there exist $x, y \in I$ such that $u = f^{m-1}(x)$ and $v = f^{m-1}(y)$. Now, from the second equality in (2.1), we get

$$f(M(u, v)) = M(f(u), f(v)), \quad u, v \in I,$$

that is, f is M -affine.

As the same argument can be used in the case when f is M -concave, the proof is finished. \square

Let us introduce the notions of an iteration group and an iteration semigroup.

A family $\{f^t : t \in \mathbb{R}\}$ of homeomorphisms of an interval I is said to be an *iteration group (of function f)*, if $f^s \circ f^t = f^{s+t}$ for all $s, t \in \mathbb{R}$ (and $f^1 = f$). An



Title Page

Contents



Page 8 of 18

Go Back

Full Screen

Close

iteration group is called *continuous* if for every $x \in I$ the function $t \rightarrow f^t(x)$ is continuous.

Note that f^t is increasing for every $t \in \mathbb{R}$.

A one parameter family $\{f^t : t \geq 0\}$ of continuous one-to-one functions $f^t : I \rightarrow I$ such that $f^t \circ f^s = f^{t+s}$, $t, s \geq 0$ is said to be *an iteration semigroup*. If for every $x \in I$ the mapping $t \rightarrow f^t(x)$ is continuous then an iteration semigroup is said to be *continuous*.

More information on iteration groups and semigroups can be found, for example, in [3], [4], [8] and [10].

Remark 6 (see [10, Remark 4.1]). If $I \subset \mathbb{R}$ is an open interval and there exists at least one element of an iteration semigroup $\{f^t : t \geq 0\}$ without a fixed point and it is not surjective, then this semigroup is continuous.

Remark 7. Every iteration semigroup can be uniquely extended to the relative iteration group (cf. Zdun [9]). Namely, for a given iteration semigroup $\{f^t : t \geq 0\}$ define

$$F^t := \begin{cases} f^t, & t \geq 0, \\ (f^{-t})^{-1}, & t < 0, \end{cases}$$

where $\text{Dom } F^t = I$ and $\text{Dom } F^{-t} = f^t[I]$ for $t > 0$. It is easy to observe that $\{F^t : t \in \mathbb{R}\}$ is a continuous group, i.e. $F^t \circ F^s(x) = F^{t+s}(x)$ for all values of x for which this formula holds. Moreover, if at least one of f^t is a homeomorphism, then $\{F^t : t \in \mathbb{R}\}$ is an iteration group.

In this paper we consider iteration semigroups consisting of (M, N) -convex and (M, N) -concave elements or semigroups consisting of M -convex and M -concave elements. Iteration groups consisting of convex functions were studied earlier, among others, by A. Smajdor [6], [7] and M.C. Zdun [10].

Remark 8. Let $\{f^t : t \geq 0\}$ be a continuous iteration semigroup. If there exists a sequence $(f^{t_n})_{n \in \mathbb{N}}$ of (M, N) -convex functions such that $\lim_{n \rightarrow +\infty} t_n = 0$, then

$M \leq N$. Similarly, if in a continuous semigroup $\{f^t : t \geq 0\}$ there exists a sequence $(f^{t_n})_{n \in \mathbb{N}}$ of (M, N) -concave elements such that $\lim_{n \rightarrow +\infty} t_n = 0$, then $M \geq N$.

Indeed, the continuity of the semigroup implies that f^0 , as the limit of a sequence of (M, N) -convex or (M, N) -concave functions, is (M, N) -convex or (M, N) -concave, respectively. Since $f^0 = id$, it follows that $M \leq N$ or $M \geq N$, respectively.



**Semigroups with Generalized
Convex Elements**

Dorota Krassowska

vol. 10, iss. 3, art. 76, 2009

Title Page

Contents



Page 9 of 18

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 10 of 18

Go Back

Full Screen

Close

3. Results

We start with an example of an iteration semigroup consisting of (M, N) -concave elements, where $M \neq N$.

Example 3.1. Let $I = (0, \infty)$. For every $t \geq 0$ put $f^t(x) = x^{4^t}$ and let $M(x, y) = x + y$, $N(x, y) = \frac{x+y}{2}$. Since the inequality

$$(3.1) \quad (x + y)^{4^t} \geq \frac{x^{4^t} + y^{4^t}}{2}$$

holds for all $t, x, y > 0$, there are (M, N) -concave elements in the semigroup $\{f^t : t \geq 0\}$. One can use standard calculus methods to prove (3.1).

In [5], Matkowski considered continuous multiplicative iteration groups of homeomorphisms $f^t : (0, \infty) \rightarrow (0, \infty)$ such that, for every $t > 0$ the function f^t is M -convex or M -concave, where M is continuous on $(0, \infty) \times (0, \infty)$. The main result of [5] says that if in such a group there are two elements f^r and f^s , $r < 1 < s$, which are both M -convex or both M -concave, then all elements of the group are M -affine. While discussing the possibility of a generalization of this result it was shown that an analogous theorem with (M, N) -convex or (M, N) -concave functions, where $M \neq N$, is not valid.

Our first result establishes conditions under which the desirable thesis holds.

Theorem 3.1. *Let $M, N : I^2 \rightarrow I$ be continuous functions. Suppose that a continuous iteration semigroup $\{f^t : t \geq 0\}$ is such that f^t is (M, N) -convex or (M, N) -concave for every $t > 0$.*

If there exist $r > s > 0$ such that f^r and f^s are (M, N) -affine, then every element of this semigroup is M -affine and $M = N$ on the set $f^s[I] \times f^s[I]$.

Proof. Let f^r and f^s be (M, N) -affine. By Remark 2, the function $(f^s)^{-1}$ is (N, M) -affine. It is easy to see that $h := (f^s)^{-1} \circ f^r = f^{r-s}$ is M -affine. Moreover, by the



Title Page

Contents



Page 11 of 18

Go Back

Full Screen

Close

(M, N) -affinity of f^s ,

$$(3.2) \quad N(x, y) = f^s(M((f^s)^{-1}(x), (f^s)^{-1}(y))), \quad x, y \in f^s[I].$$

The (M, N) -convexity or the (M, N) -concavity of f^u for every $u > 0$, and (3.2) imply that, for every $u > 0$, the function f^u satisfies the inequality

$$f^u(M(x, y)) \leq N(f^u(x), f^u(y)) = f^s(M((f^s)^{-1}(f^u(x)), (f^s)^{-1}(f^u(y))))$$

or the inequality

$$f^u(M(x, y)) \geq N(f^u(x), f^u(y)) = (f^s)(M((f^s)^{-1}(f^u(x)), (f^s)^{-1}(f^u(y)))),$$

for every x and y such that $f^u(x), f^u(y) \in f^s[I]$. Since for $u \geq s$ the inclusion $f^u(x) \in f^s[I]$ holds for every $x \in I$, we hence get, for all $u \geq s$, $x, y \in I$

$$(3.3) \quad \begin{aligned} f^{u-s}(M(x, y)) &= (f^s)^{-1} \circ f^u(M(x, y)) \\ &\leq M((f^s)^{-1} \circ f^u(x), (f^s)^{-1} \circ f^u(y)) \\ &= M(f^{u-s}(x), f^{u-s}(y)), \end{aligned}$$

or

$$(3.4) \quad \begin{aligned} f^{u-s}(M(x, y)) &= (f^s)^{-1} \circ f^u(M(x, y)) \\ &\geq M((f^s)^{-1} \circ f^u(x), (f^s)^{-1} \circ f^u(y)) \\ &= M(f^{u-s}(x), f^{u-s}(y)), \end{aligned}$$

i.e. for every $t := u - s \geq 0$ and all $x, y \in I$,

$$f^t(M(x, y)) \leq M(f^t(x), f^t(y))$$

or

$$f^t(M(x, y)) \geq M(f^t(x), f^t(y)),$$



Title Page

Contents



Page 12 of 18

Go Back

Full Screen

Close

which means that every element of the semigroup with iterative index $t \geq 0$ is M -convex or M -concave. Define $h^t := \{f^{t(r-s)} : t \geq 0\}$. Since $h^{1/m} = f^{(r-s)/m}$ for $m \in \mathbb{N}$, it is M -convex or M -concave as an element of the semigroup. On the other hand, $h^{1/m}$ is the m -th iterative root of $h = h^1$ which is M -affine. Hence, by Lemma 2.2, the function $h^{1/m}$ is M -affine. It follows that, for all positive integers m, n , the function $h^{n/m}$ is M -affine. Thus the set $\{h^t : t \in \mathbb{Q}^+\}$ consists of M -affine functions. The continuity of the iteration semigroup and the continuity of M imply that, for every $t \geq 0$, the function h^t is M -affine and, consequently, f^t , for all $t \geq 0$, are M -affine. To end the proof take f^s which is both (M, N) -affine and M -affine. Then, for all $x, y \in I$,

$$f^s(M(x, y)) = N(f^s(x), f^s(y))$$

and

$$f^s(M(x, y)) = M(f^s(x), f^s(y)),$$

whence

$$N(f^s(x), f^s(y)) = M(f^s(x), f^s(y)), \quad x, y \in I.$$

Since f^s is onto $f^s[I]$, $M(x, y) = N(x, y)$ for $x, y \in f^s[I]$. The proof is completed. \square

Remark 9. Let us note that if in an iteration group for some $t_0 \in \mathbb{R}$ the function f^{t_0} is M -convex, then the function $(f^{t_0})^{-1}$ is M -concave.

Now we present two results which generalize Matkowski's Theorem 1 ([5]).

Theorem 3.2. *Let $M : I^2 \rightarrow I$ be continuous. Suppose that an iteration semigroup $\{f^t : t \geq 0\}$ is continuous. If there exist $r, s > 0$ such that $\frac{r}{s} \notin \mathbb{Q}$, $f^r < id$, $f^s < id$ and f^r is M -convex and f^s is M -concave, then every element of the semigroup is M -affine.*



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 13 of 18

Go Back

Full Screen

Close

Proof. Take the relative iteration group $\{F^t : t \in \mathbb{R}\}$ defined as in Remark 7. Assume that f^r is M -convex and f^s is M -concave. Put $g := f^r$ and $h := f^{-s}$. It is obvious that, for each pair (m, n) of positive integers, the functions g^m and h^n are M -convex.

Let $\mathcal{N}(x) := \{(m, n) \in \mathbb{N} \times \mathbb{N} : h^n(x) \in g^m[I]\}$ and $D(x) := \{rm - sn : (m, n) \in \mathcal{N}(x)\}$. Note that if $x < y$, then $\mathcal{N}(x) \subset \mathcal{N}(y)$. Moreover, for every $x \in I$, the set $D(x)$ is dense in \mathbb{R} (see [2]).

Let $x \in I$ be fixed. Take an arbitrary $t \in \mathbb{R}$. By the density of the set $D(x)$, there exists a sequence (m_k, n_k) with terms from $\mathcal{N}(x)$ such that $t = \lim_{k \rightarrow +\infty} (m_k r - n_k s)$. Moreover,

$$F^t(x) = \lim_{k \rightarrow +\infty} f^{-n_k s} \circ f^{m_k r}(x) = \lim_{k \rightarrow +\infty} h^{n_k} \circ g^{m_k}(x).$$

Hence, for every $t \in \mathbb{R}$, the function F^t is M -convex, as it is the limit of a sequence of M -convex functions.

Now let $t > 0$ be fixed. Since F^t and F^{-t} are both M -convex and $F^{-t} \circ F^t = id$, by Lemma 2.1, F^t is M -affine. Consequently, f^t is M -affine for every $t \geq 0$. \square

Theorem 3.3. *Let $M : I^2 \rightarrow I$ be continuous. Suppose that $\{f^t : t \geq 0\}$ is a continuous iteration semigroup such that f^t is M -convex or M -concave for every $t > 0$. If there exist $r, s > 0$ such that $f^r < id$ is M -convex and $f^s < id$ is M -concave, then f^t is M -affine for every $t > 0$.*

Proof. If $\frac{r}{s} \notin \mathbb{Q}$, then the thesis follows from the previous theorem. Suppose that $\frac{r}{s} \in \mathbb{Q}$. Then there exist $m, n \in \mathbb{N}$ such that $nr = ms$. Thus $(f^r)^n = (f^s)^m$. Put $H := (f^r)^n$. Since $(f^r)^n$ is M -convex and $(f^s)^m$ is M -concave, H is M -affine. By Lemma 2.2, the function f^r is M -affine. Let $n \in \mathbb{N}$ be fixed. As

$$\underbrace{f^{r/n} \circ f^{r/n} \circ \dots \circ f^{r/n}}_{n \text{ times}} = f^r,$$



Title Page

Contents



Page 14 of 18

Go Back

Full Screen

Close

by Lemma 2.2, the function $f^{r/n}$ is M -affine. Thus for all $n, m \in \mathbb{N}$, the functions $f^{\frac{m}{n}r} = (f^{r/n})^m$ are M -affine. Let us fix $t > 0$ and take a sequence $(w_n)_{n \in \mathbb{N}}$ of positive rational numbers such that $f^t = \lim_{n \rightarrow \infty} f^{w_n r}$. The continuity of M , the continuity of the semigroup and the formula for f^t imply that f^t is M -affine. \square

From Theorems 3.2 and 3.3 we obtain the additive version of Matkowski's result [5] which reads as follows.

Corollary 3.4. *Let $M : I^2 \rightarrow I$ be continuous and suppose that $\{f^t : t \geq 0\}$ is a continuous iteration semigroup of homeomorphisms $f^t : I \rightarrow I$ such that:*

- (i) f^t is M -convex or M -concave for every $t > 0$;
- (ii) there exist $r, s > 0$ such that f^r is M -convex and f^s is M -concave.

Then f^t is M -affine for every $t \geq 0$.

Now we prove the following

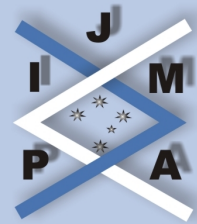
Theorem 3.5. *Let $M : I^2 \rightarrow I$ be a continuous function. If every element of a continuous iteration semigroup $\{f^t : t \geq 0\}$ is M -convex or M -concave and there exists an $s \neq 0$ such that f^s is M -affine, then f^t is M -affine for every $t \geq 0$.*

Proof. Assume that every element of the iteration semigroup is M -convex and $g := f^s$ is M -affine. By Lemma 2.2, for an $m \in \mathbb{N}$ the function $g^{1/m}$ is M -affine. Now the same argument as in the proof of Theorem 3.1 can be repeated. \square

Coming back to a group with (M, N) -convex or (M, N) -concave elements, we present:

Theorem 3.6. *Let $M, N : I^2 \rightarrow I$ be continuous functions. Suppose that an iteration semigroup $\{f^t : t \geq 0\}$ is continuous and such that, for every $t > 0$, the function f^t is (M, N) -convex or (M, N) -concave.*

Assume moreover that:



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 15 of 18

[Go Back](#)

[Full Screen](#)

[Close](#)

(i) there exists $t_0 > 0$ such that f^{t_0} is (M, N) -affine;

(ii) there exist $r, s > t_0$ such that f^r is (M, N) -convex and f^s is (M, N) -concave.

Then, for every $t \geq 0$, the function f^t is M -affine and $M = N$ on $f^{t_0}[I] \times f^{t_0}[I]$.

Proof. By (i) we obtain equality (3.2) with f^{t_0} instead of f^s . This equality and the (M, N) -convexity of f^r give

$$f^r(M(x, y) \leq N(f^r(x), f^r(y))) = f^{t_0}(M((f^{t_0})^{-1}(f^r(x)), (f^{t_0})^{-1}(f^r(y))))$$

for all $x, y \in I$. The monotonicity of the function $(f^{t_0})^{-1}$ implies that

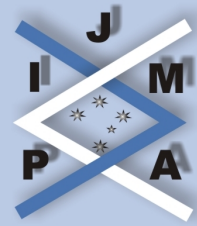
$$(f^{t_0})^{-1}(f^r(M(x, y))) \leq M((f^{t_0})^{-1}(f^r(x)), (f^{t_0})^{-1}(f^r(y))), \quad x, y \in I,$$

that is, the function f^{r-t_0} is M -convex. Similarly, f^{s-t_0} is M -concave. Moreover, repeating the procedure used in the proof of Theorem 3.1, we have (3.3) or (3.4) with t_0 instead of s for every $u \geq t_0$. Hence for every $t \geq 0$, the function f^t is M -convex or M -concave. Since the semigroup satisfies all the assumptions of Theorem 3.3, we obtain the first part of the thesis. To prove the second part, it is enough to take $f = f^{t_0}$, that is, simultaneously (M, N) -affine and M -affine, and apply the argument used at the end of the proof of Theorem 3.1. \square

In the context of the above proof a natural question arises. Is it true that every (M, N) -convex function has to be M -convex? The following example shows that the answer is negative.

Example 3.2. Let $I = (0, \infty)$, $M(x, y) = x + y$, $N(x, y) = \sqrt{xy}$ and put $f^t(x) = \frac{x}{tx+1}$ for every $t > 0$. It is easy to check that $\{f^t : t \geq 0\}$ is a semigroup. The function f^t is (M, N) -concave and M -convex for every $t > 0$.

The proof needs only some standard calculations.



Title Page

Contents



Page 16 of 18

Go Back

Full Screen

Close

We now present theorems which establish the regularity of the semigroup we deal with. Namely,

Theorem 3.7. *Suppose that $\{f^t : t \geq 0\}$ is a continuous iteration semigroup. If f^t is M -convex or M -concave for every $t > 0$, then in this semigroup either for every $t > 0$ element f^t is M -convex or, contrarily, for every $t > 0$ element f^t is M -concave.*

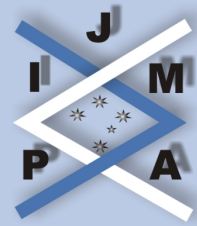
Proof. Let $A = \{t > 0 : f^t(M(x, y)) \leq M(f^t(x), f^t(y)), x, y \in I\}$ and $B = \{t > 0 : f^t(M(x, y)) \geq M(f^t(x), f^t(y)), x, y \in I\}$. The sets A and B are relatively closed subsets of $(0, \infty)$. Moreover, $A \cup B = (0, \infty)$. Let us consider two cases:

(i) $A \cap B = \emptyset$. Then the connectivity of the set $(0, \infty)$ implies that $A = \emptyset$ or $B = \emptyset$;
(ii) $A \cap B \neq \emptyset$. Then there exists $u \in A \cap B$, $u \neq 0$, so f^u is M -affine. Hence all the assumptions of Theorem 3.5 are satisfied and the semigroup consists only of M -affine elements, so the thesis is fulfilled. \square

However, for a semigroup with (M, N) -convex or (M, N) -concave elements, we have the following weaker result:

Theorem 3.8. *Suppose that $\{f^t : t \geq 0\}$ is a continuous iteration semigroup. If f^t is (M, N) -convex or (M, N) -concave for every $t > 0$, then there exists $t_0 \geq 0$ such that in this semigroup either for every $t \geq t_0$ the element f^t is (M, N) -convex and for every $0 \leq t \leq t_0$ the element f^t is (M, N) -concave or, contrarily, for every $t \geq t_0$ the element f^t is (M, N) -concave and for every $0 \leq t \leq t_0$ the element f^t is (M, N) -convex.*

Proof. Let $A = \{t > 0 : f^t(M(x, y)) \leq N(f^t(x), f^t(y)), x, y \in I\}$ and $B = \{t > 0 : f^t(M(x, y)) \geq N(f^t(x), f^t(y)), x, y \in I\}$. The sets A and B are relatively closed subsets of $(0, \infty)$. Moreover, $A \cup B = (0, \infty)$. Now we consider three cases:
(i) $A \cap B = \emptyset$. Then the connectivity of the set $(0, \infty)$ implies that $A = \emptyset$ or $B = \emptyset$;



Title Page

Contents



Page 17 of 18

Go Back

Full Screen

Close

(ii) $A \cap B \neq \emptyset$ and there exist at least two elements in this set. All the assumptions of Theorem 3.1 are satisfied and the semigroup consists only of (M, N) -affine elements, of course $t_0 = 0$;

(iii) $A \cap B$ is a singleton. Denote $A \cap B = \{u\}$. The function f^u is (M, N) -affine. Hence all the assumptions of Theorem 3.6 are satisfied and the semigroup contains only (M, N) -affine elements. The thesis is thus fulfilled. Of course, f^{t_0} is (M, N) -affine. \square

Applying Theorem 3.8, we obtain the following

Corollary 3.9. *Let us assume that a continuous iteration semigroup $\{f^t : t \geq 0\}$ consists only of (M, N) -convex or (M, N) -concave functions and there are $r, s > 0$ such that f^r and f^s are both (M, N) -affine. Then either $M \leq N$ or $N \leq M$. If $M \leq N$ and for at least one point $(x_0, y_0) \in I^2$ the strict inequality*

$$(3.5) \quad M(x_0, y_0) < N(x_0, y_0)$$

holds, then for every $t > 0$, the functions f^t are (M, N) -convex.

Proof. Assume, on the contrary, that there exists $t_0 > 0$ such that f^{t_0} is (M, N) -concave. By Theorem 3.8, for every $t > 0$, the function f^t is (M, N) -concave. Hence $f^0 = id$ is (M, N) -concave since it is the limit of an (M, N) -concave function. Thus

$$M(x, y) \geq N(x, y) \quad x, y \in I,$$

which contradicts the assumed inequality (3.5). \square

In all theorems, according to Remark 6, if at least one function in a semigroup is without a fixed point and not surjective, then the assumption of the continuity of the semigroup can be omitted.

References

- [1] G. AUMANN, Konvexe Funktionen und die Induktion bei Ungleichungen, *Bayer. Akad. Wiss. Math. Natur. Kl. S.-B.*, (1933b), 403–415.
- [2] D. KRASSOWSKA AND M.C. ZDUN, On limit sets of iterates of commuting mappings, *Aequationes Math.*, in print.
- [3] M. KUCZMA, *Functional Equations in a Single Variable*, Monograf Mat 46, PWN, Warszawa 1968.
- [4] M. KUCZMA, B. CHOCZEWSKI AND R. GER, *Iterative Functional Equations*, Encyclopedia of mathematics and its applications, v. 32, Cambridge University Press 1990.
- [5] J. MATKOWSKI, Iteration groups with generalized convex and concave elements, *Grazer Math. Ber. ISSN 1016-7692*, Bericht **Nr.334** (1997), 199–216.
- [6] A. SMAJDOR, On convex iteration groups, *Bull. Acad. Polon. Sci, Sér Sci. Math. Phys.*, **15 (5)** (1967), 325–328.
- [7] A. SMAJDOR, Note the existence of convex iteration groups, *Fund. Math.*, **87** (1975), 213–218.
- [8] Gy. TARGOŃSKI, *Topics in Iteration Theory*, Vandenhoeck and Ruprecht, Göttingen, 1981.
- [9] M.C. ZDUN, Some remarks on iteration semigroups, *Prace matematyczne, t.7, Prace naukowe Uniwersytetu Śląskiego*, **158**, (1977), 65–69.
- [10] M.C. ZDUN, *Continuous and differentiable iteration semigroups*, Prace Naukowe Uniwersytetu Śląskiego w Katowicach, **308**, Katowice, 1979.



Semigroups with Generalized
Convex Elements

Dorota Krassowska

vol. 10, iss. 3, art. 76, 2009

Title Page

Contents



Page 18 of 18

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756