

# A BOUND FOR CERTAIN BIBASIC SUMS AND APPLICATIONS

MINGJIN WANG AND HONGSUN RUAN

Department of Applied Mathematics  
Jiangsu Polytechnic University  
Changzhou City 213164  
Jiangsu Province, P.R. China  
EMail: [wmj@jpu.edu.cn](mailto:wmj@jpu.edu.cn) [rhs@em.jpu.edu.cn](mailto:rhs@em.jpu.edu.cn)

- Received:* 28 June, 2007
- Accepted:* 15 May, 2009
- Communicated by:* S.S. Dragomir
- 2000 AMS Sub. Class.:* Primary 26D15; Secondary 33D15; 33D65.
- Key words:* Basic hypergeometric function;  $q$ -binomial coefficient; Euler formula;  $q$ -Gauss sum; Grüss inequality.
- Abstract:* In this paper, we use the terminating case of the Euler formula, the limiting case of the  $q$ -Gauss sum and the Grüss inequality to derive a bound for certain bibasic sums. Applications of the inequality are also given.
- Acknowledgements:* Supported by STF of Jiangsu Polytechnic University. The author would like to express his deep appreciation to the referee for helpful suggestions. In particular, the author thanks the referee for helping to improve the presentation of this paper.

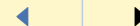
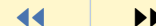


A Bound for Certain  
Bibasic Sums

Mingjin Wang and Hongsun Ruan  
vol. 10, iss. 2, art. 39, 2009

[Title Page](#)

[Contents](#)



Page 1 of 11

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

# Contents

1	Introduction and Some Known Results	3
2	A Bound for Bibasic Sums	5
3	Application of the Inequality	9



---

A Bound for Certain  
Bibasic Sums

Mingjin Wang and Hongsun Ruan

vol. 10, iss. 2, art. 39, 2009

---

Title Page

Contents



Page 2 of 11

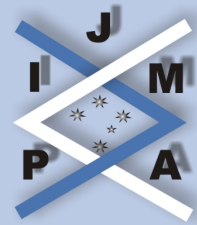
Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



Title Page

Contents



Page 3 of 11

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

© 2007 Victoria University. All rights reserved.

## 1. Introduction and Some Known Results

$q$ -Series, which are also called basic hypergeometric series, play a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials and physics, etc. Inequality techniques are useful tools in the study of  $q$ -series, see [1, 7, 8]. In [1], the authors gave some inequalities for hypergeometric functions. In this paper, we give a new inequality about  $q$ -series. First, we recall some definitions, notations and known results which will be used in this paper. Throughout this paper, it is supposed that  $0 < q < 1$ . The  $q$ -shifted factorials are defined as

$$(1.1) \quad (a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

We also adopt the following compact notation for multiple  $q$ -shifted factorials:

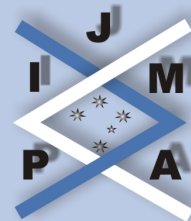
$$(1.2) \quad (a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n,$$

where  $n$  is an integer or  $\infty$ . The  $q$ -binomial coefficient is defined by

$$(1.3) \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

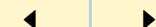
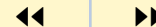
Although it is not obvious from (1.3), it is a well-known fact [2] that the  $q$ -binomial coefficient is a polynomial in  $q$  of degree  $k(n - k)$  with nonnegative integer coefficients. The  $q$ -Gauss sum [2, 3, 4]

$$(1.4) \quad \sum_{k=0}^{\infty} \frac{(a, b; q)_k}{(q, c; q)_k} \left( \frac{c}{ab} \right)^k = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty},$$



Title Page

Contents



Page 4 of 11

Go Back

Full Screen

Close

which has the limiting case [4]

$$(1.5) \quad \frac{1}{(x; q)_n} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)} \frac{x^k}{(x; q)_k}.$$

We also need the following Euler formula [2]

$$(1.6) \quad (x; q)_\infty = \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}k(k-1)} \frac{x^k}{(q; q)_k},$$

which has the terminating form [2, 4]

$$(1.7) \quad (x; q)_n = \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^k.$$

The following is well known in the literature as the Grüss inequality [6]:

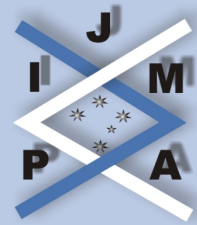
$$(1.8) \quad \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left( \frac{1}{b-a} \int_a^b f(x)dx \right) \cdot \left( \frac{1}{b-a} \int_a^b g(x)dx \right) \right| \leq \frac{(M-m)(N-n)}{4},$$

provided that  $f, g : [a, b] \rightarrow \mathbb{R}$  are integrable on  $[a, b]$  and  $m \leq f(x) \leq M$ ,  $n \leq g(x) \leq N$  for all  $x \in [a, b]$ , where  $m, M, n, N$  are given constants.

The discrete version of the Grüss inequality can be stated as: If  $a \leq a_i \leq A$  and  $b \leq b_i \leq B$  ( $i = 1, 2, \dots, n$ ), then we have

$$(1.9) \quad \left| \frac{1}{n} \sum_{i=1}^n a_i b_i - \left( \frac{1}{n} \sum_{i=1}^n a_i \right) \cdot \left( \frac{1}{n} \sum_{i=1}^n b_i \right) \right| \leq \frac{(A-a)(B-b)}{4},$$

where  $a, A, b, B$  are given real constants.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 5 of 11

Go Back

Full Screen

Close

## 2. A Bound for Bibasic Sums

In this section, by means of the terminating form of the Euler formula, the limiting case of the  $q$ -Gauss sum and the Grüss inequality, we derive a bound of the following bibasic sums

$$(2.1) \quad \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x; p)_k}.$$

For any real number  $x$ , let  $[x]$  denote the greatest integer less than or equal  $x$ . The main result of this paper is the following theorem, which gives an upper bound of (2.1). It is obvious that, under the conditions of the following theorem, the lower bound of (2.1) is zero.

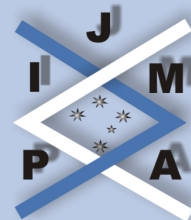
**Theorem 2.1.** *Let  $0 < p < 1$ ,  $0 < q < 1$ ,  $0 < x < 1$  and  $0 < y < 1$ , then for any positive integer  $n$  we have*

$$(2.2) \quad \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x; p)_k} \leq \frac{(-y; q)_n}{(n+1)(x; p)_n} + \frac{n+1}{4(x; p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \begin{bmatrix} n \\ k_0 \end{bmatrix}_q,$$

where  $k_0 = \lfloor \frac{n-1}{2} \rfloor$ .

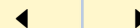
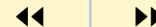
*Proof.* Let  $k_0 = \lfloor \frac{n-1}{2} \rfloor$ . Since,

$$\frac{\begin{bmatrix} n \\ k+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ k \end{bmatrix}_q} = \frac{(q; q)_n / (q; q)_{k+1} (q; q)_{n-k-1}}{(q; q)_n / (q; q)_k (q; q)_{n-k}} = \frac{1 - q^{n-k}}{1 - q^{k+1}},$$



Title Page

Contents



Page 6 of 11

Go Back

Full Screen

Close

we get

$$\begin{cases} \frac{\begin{bmatrix} n \\ k+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ k \end{bmatrix}_q} \geq 1, & \text{when } k \leq k_0, \\ \frac{\begin{bmatrix} n \\ k+1 \end{bmatrix}_q}{\begin{bmatrix} n \\ k \end{bmatrix}_q} < 1, & \text{when } k > k_0. \end{cases}$$

So we have

$$(2.3) \quad 1 \leq \begin{bmatrix} n \\ k \end{bmatrix}_q \leq \begin{bmatrix} n \\ k_0 \end{bmatrix}_q, \quad k = 0, 1, \dots, n.$$

Similarly,

$$(2.4) \quad 1 \leq \begin{bmatrix} n \\ k \end{bmatrix}_p \leq \begin{bmatrix} n \\ k_0 \end{bmatrix}_p, \quad k = 0, 1, \dots, n.$$

Under the conditions of the theorem, we also have

$$(2.5) \quad 0 < p^{k(k-1)} < 1, \quad 0 < q^{\frac{1}{2}k(k-1)} < 1,$$

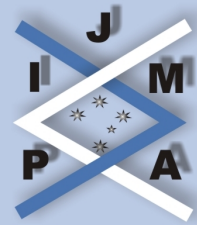
$$(2.6) \quad 0 < \frac{1}{(x;p)_k} = \frac{1}{1-x} \cdot \frac{1}{1-xp} \cdots \frac{1}{1-xp^{k-1}} \leq \frac{1}{(x;p)_n},$$

$$(2.7) \quad 0 < x^k < 1, \quad 0 < y^k < 1,$$

where  $k = 0, 1, \dots, n$ .

Let

$$(2.8) \quad \begin{cases} a_k = \begin{bmatrix} n \\ k \end{bmatrix}_p p^{k(k-1)} \frac{x^k}{(x;p)_k} \\ b_k = \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} y^k \end{cases}$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 7 of 11

Go Back

Full Screen

Close

in the discrete version of the Grüss inequality (1.9). Combining (2.3), (2.4), (2.5), (2.6) and (2.7) one gets

$$0 < a_k < \frac{1}{(x; p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \quad \text{and} \quad 0 < b_k < \begin{bmatrix} n \\ k_0 \end{bmatrix}_q.$$

Substituting  $a_k$  and  $b_k$  into the discrete version of the Grüss inequality (1.9), gives

$$(2.9) \quad \left| \frac{1}{n+1} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x; p)_k} \right. \\ \left. - \left\{ \frac{1}{n+1} \sum_{k=0}^n p^{k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix}_p \frac{x^k}{(x; p)_k} \right\} \left\{ \frac{1}{n+1} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} y^k \right\} \right| \\ \leq \frac{1}{4(x; p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \begin{bmatrix} n \\ k_0 \end{bmatrix}_q.$$

Using (1.5) and (1.7) one gets

$$(2.10) \quad \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_p p^{k(k-1)} \frac{x^k}{(x; p)_k} = \frac{1}{(x; p)_n},$$

$$(2.11) \quad \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} y^k = (-y; q)_n.$$

Substituting (2.10) and (2.11) into (2.9), we have

$$(2.12) \quad \left| \frac{1}{n+1} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x; p)_k} - \frac{(-y; q)_n}{(n+1)^2 (x; p)_n} \right|$$

$$\leq \frac{1}{4(x;p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \begin{bmatrix} n \\ k_0 \end{bmatrix}_q,$$

which is equivalent to

$$(2.13) \quad \frac{(-y;q)_n}{(n+1)(x;p)_n} - \frac{n+1}{4(x;p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \begin{bmatrix} n \\ k_0 \end{bmatrix}_q$$

$$\leq \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x;p)_k}$$

$$\leq \frac{(-y;q)_n}{(n+1)(x;p)_n} + \frac{n+1}{4(x;p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \begin{bmatrix} n \\ k_0 \end{bmatrix}_q.$$

The proof is thus completed.  $\square$

In the proof, we used the Euler formula and the limiting case of the  $q$ -Gauss sum. We wish to point out, that there may be other pairs of summation theorems which also lead to interesting results. As an application of the inequality, we can easily obtain the following one.

**Corollary 2.2.** *Under the conditions of Theorem 2.1, we have*

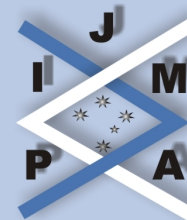
$$(2.14) \quad \sum_{k=0}^n \binom{n}{k}^2 \frac{x^k y^k}{(1-x)^k} \leq \frac{(1+y)^n}{(n+1)(1-x)^n} + \frac{n+1}{4(1-x)^n} \binom{n}{k_0}^2,$$

where  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ .

*Proof.* From [5], we know

$$\lim_{q \rightarrow 1} \begin{bmatrix} n \\ i \end{bmatrix}_q = \binom{n}{i}.$$

Letting  $p \rightarrow 1$  and  $q \rightarrow 1$  on both sides of inequality (2.2), we get (2.14).  $\square$



Title Page

Contents

◀◀ ▶▶

◀ ▶

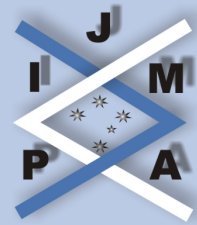
Page 8 of 11

Go Back

Full Screen

Close





Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 9 of 11

Go Back

Full Screen

Close

### 3. Application of the Inequality

Convergence is an important problem in the study of  $q$ -series. In this section, we use the inequality obtained in this paper to give a sufficient condition for the convergence of a bibasic series.

**Theorem 3.1.** *Suppose  $0 < p < 1$ ,  $0 < q < 1$ ,  $0 < x < 1$  and  $0 < y < 1$ . Let  $\{c_n\}$  be any sequence of numbers. If*

$$\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| < 1,$$

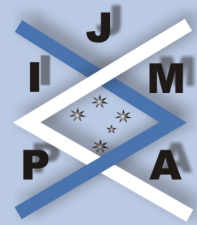
then the bibasic series

$$(3.1) \quad \sum_{n=0}^{\infty} \sum_{k=0}^n c_n \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x; p)_k}$$

converges absolutely.

*Proof.* Multiplying both sides of (2.2) by  $|c_n|$ , one gets

$$(3.2) \quad |c_n| \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_q p^{k(k-1)} q^{\frac{1}{2}k(k-1)} \frac{x^k y^k}{(x; p)_k} \\ \leq \frac{(-y; q)_n |c_n|}{(n+1)(x; p)_n} + \frac{(n+1)|c_n|}{4(x; p)_n} \begin{bmatrix} n \\ k_0 \end{bmatrix}_p \begin{bmatrix} n \\ k_0 \end{bmatrix}_q \\ \leq \frac{(-y; q)_n |c_n|}{(n+1)(x; p)_n} + \frac{(n+1)|c_n|}{4(x; p)_n (p; p)_{\infty} (q; q)_{\infty}}.$$



---

**A Bound for Certain  
Bibasic Sums**

Mingjin Wang and Hongsun Ruan

vol. 10, iss. 2, art. 39, 2009

---

Title Page

Contents



Page 10 of 11

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

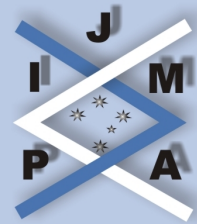
issn: 1443-5756

© 2007 Victoria University. All rights reserved.

The ratio test shows that both

$$\sum_{n=0}^{\infty} \frac{(-y; q)_n c_n}{(n+1)(x; p)_n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(n+1)c_n}{4(x; p)_n (p; p)_{\infty} (q; q)_{\infty}}$$

are absolutely convergent. Together with (3.2), this immediately yields that the series in (3.1) is absolutely convergent.  $\square$



Title Page

Contents



Page 11 of 11

Go Back

Full Screen

Close

## References

- [1] G.D. ANDERSON, R.W. BARNARD, K.C. VAMANAMURTHY AND M. VUORINEN, Inequalities for zero-balanced hypergeometric functions, *Trans. Amer. Math. Soc.*, **5**(347) (1995), 1713–1723.
- [2] G.E. ANDREWS, *The Theory of Partitions*, Encyclopedia of Mathematics and Applications, Vol. 2., Addison-Wesley Publishing Co., Reading/London/Amsterdam, 1976.
- [3] W.N. BAILEY, *Generalized Hypergeometric Series*, Cambridge Math. Tract No. 32, Cambridge Univ. Press, London and New York. 1960.
- [4] W.C. CHU, Gould-Hsu-Carlitz inverse and Rogers-Ramanujan identities, *Acta Mathematica Sinica*, **1**(33) (1990), 7–12.
- [5] G. GASPER AND M. RAHMAN, *Basic Hypergeometric Series*, Cambridge Univ. Press, MA, 1990.
- [6] G. GRÜSS, Über das Maximum des absoluten Betrages von  $\frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx\right) \left(\frac{1}{b-a} \int_a^b g(x)dx\right)$ , *Math. Z.*, **39** (1935), 215–226.
- [7] MINGJIN WANG, An inequality about  $q$ -series, *J. Inequal. Pure Appl. Math.*, **7**(4) (2006), Art. 136. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=756>].
- [8] MINGJIN WANG, An inequality and its  $q$ -analogue, *J. Inequal. Pure Appl. Math.*, **8**(2) (2007), Art. 50 [ONLINE: <http://jipam.vu.edu.au/article.php?sid=853>].