journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 9 (2008), Issue 3, Article 76, 11 pp.

RATIO VECTORS OF POLYNOMIAL–LIKE FUNCTIONS

ALAN HORWITZ
PENN STATE UNIVERSITY
25 YEARSLEY MILL RD.
MEDIA, PA 19063
alh4@psu.edu

Received 26 August, 2006; accepted 29 July, 2008
Communicated by S.S. Dragomir

ABSTRACT. Let $p(x)$ be a hyperbolic polynomial–like function of the form $p(x) = (x - r_1)^{m_1} \cdots (x - r_N)^{m_N}$, where $m_1, \ldots, m_N$ are given positive real numbers and $r_1 < r_2 < \cdots < r_N$. Let $x_1 < x_2 < \cdots < x_{N-1}$ be the $N - 1$ critical points of p lying in $I_k = (r_k, r_{k+1}), k = 1, 2, \ldots, N - 1$. Define the ratios $\sigma_k = \frac{x_k - r_k}{r_{k+1} - r_k}, k = 1, 2, \ldots, N - 1$. We prove that $\frac{m_k}{m_k + \cdots + m_N} < \sigma_k < \frac{m_1 + \cdots + m_k}{m_1 + \cdots + m_{k+1}}$. These bounds generalize the bounds given by earlier authors for strictly hyperbolic polynomials of degree n. For $N = 3$, we find necessary and sufficient conditions for (σ_1, σ_2) to be a ratio vector. We also find necessary and sufficient conditions on m_1, m_2, m_3 which imply that $\sigma_1 < \sigma_2$. For $N = 4$, we also give necessary and sufficient conditions for $(\sigma_1, \sigma_2, \sigma_3)$ to be a ratio vector and we simplify some of the proofs given in an earlier paper of the author on ratio vectors of fourth degree polynomials. Finally we discuss the monotonicity of the ratios when $N = 4$.

Key words and phrases: Polynomial, Real roots, Ratio vector, Critical points.

2000 *Mathematics Subject Classification.* 26C10.

1. INTRODUCTION AND MAIN RESULTS

If $p(x)$ is a polynomial of degree $n \geq 2$ with n distinct real roots $r_1 < r_2 < \cdots < r_n$ and critical points $x_1 < x_2 < \cdots < x_{n-1}$, let

$$\sigma_k = \frac{x_k - r_k}{r_{k+1} - r_k}, \qquad k = 1, 2, \ldots, n - 1.$$

$(\sigma_1, \ldots, \sigma_{n-1})$ is called the *ratio vector* of p, and σ_k is called the kth ratio. Ratio vectors were first discussed in [4] and in [1], where the inequalities

$$\frac{1}{n - k + 1} < \sigma_k < \frac{k}{k + 1}, \qquad k = 1, 2, \ldots, n - 1$$

were derived. For $n = 3$ it was shown in [1] that σ_1 and σ_2 satisfy the polynomial equation $3(1 - \sigma_1)\sigma_2 - 1 = 0$. In addition, necessary and sufficient conditions were given in [5] for (σ_1, σ_2) to be a ratio vector. For $n = 4$, a polynomial, Q, in three variables was given in [5] with the property that $Q(\sigma_1, \sigma_2, \sigma_3) = 0$ for any ratio vector $(\sigma_1, \sigma_2, \sigma_3)$. It was also shown that

223-06

the ratios are monotonic–that is, $\sigma_1 < \sigma_2 < \sigma_3$ for any ratio vector $(\sigma_1, \sigma_2, \sigma_3)$. For $n = 3$, $\frac{1}{3} < \sigma_1 < \frac{1}{2}$ and $\frac{1}{2} < \sigma_2 < \frac{2}{3}$, and thus it follows immediately that $\sigma_1 < \sigma_2$. The monotonicity of the ratios does not hold in general for $n \geq 5$ (see [5]). Further results on ratio vectors for $n = 4$ were proved by the author in [6]. In particular, necessary and sufficient conditions were given for $(\sigma_1, \sigma_2, \sigma_3)$ to be a ratio vector. For a discussion of complex ratio vectors for the case $n = 3$, see [7].

We now want to extend the notion of ratio vector to hyperbolic polynomial-like functions (HPLF) of the form

$$p(x) = (x - r_1)^{m_1} \cdots (x - r_N)^{m_N},$$

where $m_1, \ldots, m_N$ are given positive real numbers with $\sum_{k=1}^{N} m_k = n$ and $r_1, \ldots, r_N$ are real numbers with $r_1 < r_2 < \cdots < r_N$. See [8] and the references therein for much more about HPLFs. We extend some of the results and simplify some of the proofs in [5] and in [6], and we prove some new results as well. In particular, we derive more general bounds on the σ_k (Theorem 1.2). Even for $N = 3$ or $N = 4$, the monotonicity of the ratios does not hold in general for all positive real numbers $m_1, \ldots, m_N$. We provide examples below and we also derive necessary and sufficient conditions on m_1, m_2, m_3 for $\sigma_1 < \sigma_2$ (Theorem 1.4). In order to define the ratios for HPLFs, we need the following lemma.

Lemma 1.1. *p' has exactly one root, $x_k \in I_k = (r_k, r_{k+1})$, $k = 1, 2, \ldots, N - 1$.*

Proof. By Rolle's Theorem, p' has at least one root in I_k for each $k = 1, 2, \ldots, N - 1$. Now $\frac{p'}{p} = \sum_{k=1}^{N} \frac{m_k}{x - r_k}$, which has at most $N - 1$ real roots since $\left\{\frac{1}{x - r_k}\right\}_{k=1,\ldots,N}$ is a Chebyshev system. □

Now we define the $N - 1$ ratios

$$\sigma_k = \frac{x_k - r_k}{r_{k+1} - r_k}, \qquad k = 1, 2, \ldots, N - 1. \tag{1.1}$$

$(\sigma_1, \ldots, \sigma_{N-1})$ is called the *ratio vector* of p.

We now state our first main result, inequalities for the ratios defined in (1.1).

Theorem 1.2. *If $\sigma_1, \ldots, \sigma_{N-1}$ are defined by (1.1), then*

$$\frac{m_k}{m_k + \cdots + m_N} < \sigma_k < \frac{m_1 + \cdots + m_k}{m_1 + \cdots + m_{k+1}} \tag{1.2}$$

Remark 1. Well after this paper was written and while this paper was being considered for publication, the paper of Melman [9] appeared. Theorem 2 of [9] is essentially Theorem 1.2 of this paper for the case when the m_k are all nonnegative integers.

Most of the rest of our results are for the special cases when $N = 3$ or $N = 4$. For $N = 3$ we give necessary and sufficient conditions on m_1, m_2, m_3 for (σ_1, σ_2) to be a ratio vector. The following theorem generalizes ([5],Theorem 1). Note that $n = m_1 + m_2 + m_3$.

Theorem 1.3. *Let $p(x) = (x - r_1)^{m_1}(x - r_2)^{m_2}(x - r_3)^{m_3}$. Then (σ_1, σ_2) is a ratio vector if and only if $\frac{m_1}{n} < \sigma_1 < \frac{m_1}{m_1 + m_2}$, $\frac{m_2}{m_2 + m_3} < \sigma_2 < \frac{m_1 + m_2}{n}$, and $\sigma_2 = \frac{m_2}{n(1 - \sigma_1)}$.*

We now state some results about the monotonicity of the ratios when $N = 3$. For $m_1 = m_2 = m_3 = 1$, Theorem 1.2 yields $\frac{1}{3} < \sigma_1 < \frac{1}{2}$ and $\frac{1}{2} < \sigma_2 < \frac{2}{3}$, and thus it follows immediately that $\sigma_1 < \sigma_2$. $\sigma_1 \leq \sigma_2$ does not hold in general for all positive real numbers(or even positive integers) m_1, m_2, and m_3. For example, if $m_1 = 2$, $m_2 = 1$, $m_3 = 3$, then it is not hard to show that $\sigma_2 < \sigma_1$ for all $r_1 < r_2 < r_3$(see the example in § 2 below). Also, if $m_1 = 4$, $m_2 = 3$, and $m_3 = 6$, then $\sigma_1 < \sigma_2$ for certain $r_1 < r_2 < r_3$, while $\sigma_2 < \sigma_1$ for other $r_1 < r_2 < r_3$. For

$$p(x) = x^4(x-1)^3\left(x + \frac{1}{2} - \frac{1}{2}\sqrt{13}\right)^6, \qquad \sigma_1 = \sigma_2 = \frac{1}{2} - \frac{1}{26}\sqrt{13}.$$

One can easily derive **sufficient** conditions on m_1, m_2, m_3 which imply that $\sigma_1 < \sigma_2$ for all $r_1 < r_2 < r_3$. For example, if $m_1 m_3 < m_2^2$, then $\frac{m_1}{m_1+m_2} < \frac{m_2}{m_2+m_3}$, which implies that $\sigma_1 < \sigma_2$ by (1.2) with $N = 3$(see (2.6) in § 2). Also, if $m_1 + m_3 < 3m_2$, then $n < 4m_2$, which implies that

$$\sigma_2 = \frac{m_2}{n(1-\sigma_1)} > \frac{1}{4(1-\sigma_1)} \geq \sigma_1$$

since $4x(1-x) \leq 1$ for all real x. We shall now derive necessary and sufficient conditions on m_1, m_2, m_3 for $\sigma_1 < \sigma_2$.

Theorem 1.4. *$\sigma_1 < \sigma_2$ for all $r_1 < r_2 < r_3$ if and only if $m_2^2 + m_1(m_2 - m_3) > 0$ and one of the following holds:*

$$m_2^2 + m_2(m_1 + m_3) - 2m_1m_3 \geq 0 \quad \textit{and} \quad m_2^2 + m_3(m_2 - m_1) > 0 \tag{1.3}$$

or

$$m_2^2 + m_2(m_1 + m_3) - 2m_1m_3 < 0 \quad \textit{and} \quad 3m_2 - m_1 - m_3 > 0. \tag{1.4}$$

As noted above, if $m_1 = m_2 = m_3 = 1$, then $\sigma_1 < \sigma_2$. The following corollary is a slight generalization of that and follows immediately from Theorem 1.4.

Corollary 1.5. *Suppose that $m_1 = m_2 = m_3 = m > 0$. Then $\sigma_1 < \sigma_2$ for all $r_1 < r_2 < r_3$.*

For $N = 4$ we now give necessary and sufficient conditions on m_1, m_2, m_3, m_4 for $(\sigma_1, \sigma_2, \sigma_3)$ to be a ratio vector. Note that $n = m_1 + m_2 + m_3 + m_4$. To simplify the notation, we use $\sigma_1 = u, \sigma_2 = v$,and $\sigma_3 = w$ for the ratios. The following theorem generalizes ([6],Theorem 3).

Theorem 1.6. *Let*

$$D \equiv D(u,v,w) = \begin{vmatrix} n(w-v) - m_3 & n(1-w) - m_4 \\ n(u-1)v(1-w) & n(u-1)vw + m_2 \end{vmatrix},$$

$$D_1 \equiv D_1(u,v,w) = (nu - m_1)(m_2 - nvw(1-u)),$$
$$D_2 \equiv D_2(u,v,w) = (nu - m_1)nv(1-u)(1-w),$$

and

$$R \equiv R(u,v,w)$$
$$= \frac{nv(1-w)D_1^2 + (nvw - m_1 - m_2)D_1D_2 + (n(1-u)(w-v-1) + m_2 + m_4)D_1D + (nw(u-1) + m_2 + m_3)D_2D}{(nu-m_1)(m_2 - nv(1-u))},$$

which is a polynomial in u, v, and w of degree 7. Then $(u,v,w) \in \Re^3$ is a ratio vector of

$$p(x) = (x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3}(x-r_4)^{m_4}$$

if and only if $0 < D_1(u,v,w) < D_2(u,v,w)$, $D(u,v,w) > 0$, and $R(u,v,w) = 0$.

We now state a sufficiency result about the monotonicity of the ratios when $N = 4$. We do not derive necessary and sufficient conditions in general on m_1, m_2, m_3, m_4 for $\sigma_1 < \sigma_2 < \sigma_3$.

Theorem 1.7. *Suppose that $m_1 + m_4 \leq \min\{3m_2 - m_3, 3m_3 - m_2\}$. Then $\sigma_1 < \sigma_2 < \sigma_3$.*

As with $N = 3$, we have the following generalization of the case when $m_1 = m_2 = m_3 = m_4 = 1$, which follows immediately from Theorem 1.7

Corollary 1.8. *Suppose that $m_1 = m_2 = m_3 = m_4 = m > 0$. Then $\sigma_1 < \sigma_2 < \sigma_3$.*

2. PROOFS

We shall derive a system of nonlinear equations in the $\{r_k\}$ and $\{\sigma_k\}$ using (1.1). Let

$$p(x) = (x - r_1)^{m_1} \cdots (x - r_N)^{m_N},$$

where $m_1, \ldots, m_N$ are given positive real numbers with $\sum_{k=1}^{N} m_k = n$ and $r_1, \ldots, r_N$ are real numbers with $r_1 < r_2 < \cdots < r_N$. By the product rule,

$$p'(x) = (x - r_1)^{m_1 - 1} \cdots (x - r_N)^{m_N - 1} \sum_{j=1}^{N} \left(m_j \prod_{i=1, i \neq j}^{N} (x - r_i) \right).$$

Since

$$p'(x) = n(x - r_1)^{m_1 - 1} \cdots (x - r_N)^{m_N - 1} \times \prod_{k=1}^{N-1} (x - x_k)$$

as well, we have

$$n \prod_{k=1}^{N-1} (x - x_k) = \sum_{j=1}^{N} \left(m_j \prod_{i=1, i \neq j}^{N} (x - r_i) \right). \tag{2.1}$$

Let $e_k \equiv e_k(r_1, \ldots, r_N)$ denote the kth elementary symmetric function of the r_j, $j = 1, 2, \ldots, N$, starting with $e_0(r_1, \ldots, r_N) = 1$, $e_1(r_1, \ldots, r_N) = r_1 + \cdots + r_N$, and so on. Let

$$e_{k,j}(r_1, \ldots, r_N) = e_k(r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_N),$$

that is, $e_{k,j}(r_1, \ldots, r_N)$ equals $e_k(r_1, \ldots, r_N)$ with r_j removed, $j = 1, \ldots, N$. Since $p(x + c)$ and $p(x)$ have the same ratio vectors for any constant c, we may assume that

$$r_2 = 0.$$

Equating coefficients in (2.1) using the elementary symmetric functions yields

$$ne_k(x_1, \ldots, x_{N-1}) = \sum_{j=1}^{N} m_j e_{k,j}(r_1, 0, r_3, \ldots, r_N), \qquad k = 1, 2, \ldots, N - 1.$$

Since

$$e_{k,j}(r_1, 0, r_3, \ldots, r_N) = \begin{cases} e_{k,j}(r_1, r_3, \ldots, r_N) & \text{if } j \neq 2 \text{ and } k \leq N - 2; \\ e_k(r_1, r_3, \ldots, r_N) & \text{if } j = 2; \\ 0 & \text{if } j \neq 2 \text{ and } k = N - 1, \end{cases}$$

we have

$$ne_k(x_1, \ldots, x_{N-1}) = m_2 e_k(r_1, r_3, \ldots, r_N) + \sum_{j=1, j \neq 2}^{N} m_j e_{k,j}(r_1, r_3, \ldots, r_N), \quad k = 1, \ldots, N - 2$$

$$nx_1 \cdots x_{N-1} = m_2 r_1 r_3 \cdots r_N. \tag{2.2}$$

Solving (1.1) for x_k yields

$$x_k = \Delta_k \sigma_k + r_k, \qquad k = 1, 2, \ldots, N - 1, \tag{2.3}$$

where $\Delta_k = r_{k+1} - r_k$. Substituting (2.3) into (2.2) gives the following equivalent system of equations involving the roots and the ratios.

$$(2.4)\quad ne_k((1-\sigma_1)r_1, r_3\sigma_2, \Delta_3\sigma_3 + r_3, \dots, \Delta_{N-1}\sigma_{N-1} + r_{N-1}) \\ = m_2 e_k(r_1, r_3, \dots, r_N) + \sum_{j=1, j\neq 2}^{N} m_j e_{k,j}(r_1, r_3, \dots, r_N), \quad k = 1, \dots, N-2,$$

$$n(1-\sigma_1)r_1 r_3 \sigma_2 (\Delta_3\sigma_3 + r_3)\cdots(\Delta_{N-1}\sigma_{N-1} + r_{N-1}) = m_2 r_1 r_3 \cdots r_N.$$

Critical in proving the inequalities $\frac{1}{n-k+1} < \sigma_k < \frac{k}{k+1}$ was the root–dragging theorem (see [2]). First we generalize the root–dragging theorem. The proof is very similar to the proof in [2] where $m_1 = \cdots = m_N = 1$. For completeness, we provide the details here.

Lemma 2.1. *Let $x_1 < x_2 < \cdots < x_{N-1}$ be the $N-1$ critical points of p lying in $I_k = (r_k, r_{k+1})$, $k = 1, 2, \dots, N-1$. Let $q(x) = (x - r_1')^{m_1} \cdots (x - r_N')^{m_N}$, where $r_k' > r_k$, $k = 1, 2, \dots, N-1$ and let $x_1' < x_2' < \cdots < x_{N-1}'$ be the $N-1$ critical points of q lying in $J_k = (r_k', r_{k+1}')$, $k = 1, 2, \dots, N-1$. Then $x_k' > x_k$, $k = 1, 2, \dots, N-1$.*

Proof. Suppose that for some i, $x_i' \leq x_i$. Now

$$p'(x_i) = 0 \Rightarrow \sum_{k=1}^{N} \frac{m_k}{x_i - r_k} = 0 \quad \text{and} \quad q'(x_i') = 0 \Rightarrow \sum_{k=1}^{N} \frac{m_k}{x_i' - r_k'} = 0.$$

$r_k' > r_k$ and $x_i' \leq x_i$ implies that

$$(2.5)\qquad x_i' - r_k' < x_i - r_k, \quad k = 1, 2, \dots, N-1.$$

Since both sides of (2.5) have the same sign,

$$\frac{m_k}{x_i' - r_k'} > \frac{m_k}{x_i - r_k}, \quad k = 1, 2, \dots, N-1,$$

which contradicts the fact that $\sum_{k=1}^{N} \frac{m_k}{x_i - r_k}$ and $\sum_{k=1}^{N} \frac{m_k}{x_i' - r_k'}$ are both zero. □

Proof of Theorem 1.2. To obtain an upper bound for σ_k we use Lemma 2.1. Arguing as in [1], we can move the critical point $x_k \in (r_k, r_{k+1})$ as far to the right as possible by letting $r_1, \dots, r_{k-1} \to r_k$ and $r_{k+2}, \dots, r_N \to \infty$. Let $s = m_1 + \cdots + m_k$, $t = m_{k+2} + \cdots + m_N$, and let

$$q_b(x) = (x - r_k)^s (x - r_{k+1})^{m_{k+1}} (x - b)^t.$$

Then

$$\begin{aligned} q_b'(x) &= (x - r_k)^s \left[(x - r_{k+1})^{m_{k+1}} t (x-b)^{t-1} + m_{k+1}(x - r_{k+1})^{m_{k+1}-1}(x-b)^t\right] \\ &\qquad + s(x - r_k)^{s-1}(x - r_{k+1})^{m_{k+1}}(x-b)^t \\ &= (x - r_{k+1})^{m_{k+1}-1}(x - r_k)^{s-1}(x-b)^{t-1} \\ &\qquad \times \left[t(x - r_{k+1})(x - r_k) + m_{k+1}(x - r_k)(x - b) + s(x - r_{k+1})(x-b)\right]. \end{aligned}$$

x_k is the smallest root of the quadratic polynomial

$$\begin{aligned} &t(x - r_{k+1})(x - r_k) + m_{k+1}(x - r_k)(x - b) + s(x - r_{k+1})(x - b) \\ &\quad = (m_{k+1} + t + s)\, x^2 + (-t r_{k+1} - t r_k - m_{k+1} r_k - m_{k+1} b - s r_{k+1} - s b)\, x \\ &\qquad + t r_{k+1} r_k + s r_{k+1} b + m_{k+1} r_k b. \end{aligned}$$

As $b \to \infty$, x_k increases and approaches the root of $(-m_{k+1} - s)x + sr_{k+1} + m_{k+1}r_k$. Thus

$$x_k \uparrow \frac{sr_{k+1} + m_{k+1}r_k}{m_{k+1} + s} \Rightarrow \sigma_k \uparrow \left(\frac{sr_{k+1} + m_{k+1}r_k}{m_{k+1} + s} - r_k\right) / (r_{k+1} - r_k)$$
$$= \frac{sr_{k+1} + m_{k+1}r_k - r_k(m_{k+1} + s)}{(m_{k+1} + s)(r_{k+1} - r_k)}$$
$$= \frac{s}{m_{k+1} + s} = \frac{m_1 + \cdots + m_k}{m_1 + \cdots + m_{k+1}}.$$

Similarly, to obtain a lower bound for σ_k, move the critical point $x_k \in (r_k, r_{k+1})$ as far to the left as possible by letting $r_{k+2}, \ldots, r_N \to r_{k+1}$ and $r_1, \ldots, r_{k-1} \to -\infty$. By considering

$$q_b(x) = (x - r_k)^{m_k}(x - r_{k+1})^s(x + b)^t,$$

where $s = m_{k+1} + \cdots + m_N$ and $t = m_1 + \cdots + m_{k-1}$, one obtains $\sigma_k \downarrow \frac{m_k}{m_k + \cdots + m_N}$. □

Proof of Theorem 1.3. Let $n = m_1 + m_2 + m_3$. To prove the necessity part, from Theorem 1.2 with $N = 3$ we have

$$\frac{m_1}{n} < \sigma_1 < \frac{m_1}{m_1 + m_2}, \qquad \frac{m_2}{m_2 + m_3} < \sigma_2 < \frac{m_1 + m_2}{n}. \tag{2.6}$$

With $N = 3$, (2.4) becomes

$$\begin{aligned} n\left((1 - \sigma_1)r_1 + r_3\sigma_2\right) &= m_2\left(r_1 + r_3\right) + m_1r_3 + m_3r_1, \\ n(1 - \sigma_1)r_1(r_3\sigma_2) &= m_2r_1r_3. \end{aligned} \tag{2.7}$$

Since $r_1 \neq 0 \neq r_3$, the second equation in (2.4) immediately implies that $n(1 - \sigma_1)\sigma_2 = m_2$.

To prove sufficiency, suppose that (σ_1, σ_2) is any ordered pair of real numbers with $\frac{m_1}{n} < \sigma_1 < \frac{m_1}{m_1+m_2}$ and $\sigma_2 = \frac{m_2}{n(1-\sigma_1)}$. Let $r = \frac{n\sigma_1 - m_1}{m_1 + m_2 - n\sigma_2}$ and let $p(x) = (x + 1)^{m_1}x^{m_2}(x - r)^{m_3}$. Note that $r > 0$ since $n\sigma_1 - m_1 > 0$ and

$$m_1 + m_2 - n\sigma_2 = m_1 + m_2 - n\frac{m_2}{n(1 - \sigma_1)}$$
$$= \frac{\sigma_1\left(m_1 + m_2\right) - m_1}{-1 + \sigma_1}$$
$$= \frac{m_1 - \sigma_1\left(m_1 + m_2\right)}{1 - \sigma_1} > 0.$$

A simple computation shows that the critical points of p in $(-1, 0)$ and in $(0, r)$, respectively, are $x_1 = \sigma_1 - 1$ and

$$x_2 = -\frac{m_2}{m_1 + m_2 + m_3}\frac{\sigma_1(m_2 + m_3) + (\sigma_1 - 1)\,m_1}{(m_1 + m_2)\,\sigma_1 - m_1}.$$

Thus the ratios of p are $x_1 + 1 = \sigma_1$ and $\frac{x_2}{r} = \frac{m_2}{n(1-\sigma_1)} = \sigma_2$. That finishes the proof of Theorem 1.3. □

Proof of Theorem 1.4. Since $p(cx)$ and $p(x)$ have the same ratios when $c > 0$, in addition to $r_1 = 0$, we may also assume that $r_2 = 1$. Thus $p(x) = x^{m_1}(x-1)^{m_2}(x-r)^{m_3}$, $r > 1$. A simple computation shows that

$$\sigma_1 = \frac{1}{2n}\left((n - m_3)r - n - m_2 - \sqrt{A(r)}\right) + 1,$$
$$\sigma_2 = \frac{\frac{1}{2n}\left((n - m_3)r - n - m_2 + \sqrt{A(r)}\right)}{r - 1},$$

where

$$A(r) = (m_1 + m_2)^2 r^2 + 2(m_2 m_3 - m_1 n) r + (m_1 + m_3)^2.$$

Let

$$f(r) = (n - m_3)\, r^2 + (-n + 2m_3 - m_2)\, r + 2m_2.$$

Note that $f(1) = m_2 + m_3 > 0$, $f'(1) = m_1 + m_3 > 0$, and $f''(r) = 2m_2 + 2m_1 > 0$, which implies that $f(r) > 0$ when $r > 1$. Now

$$\begin{aligned}\sigma_2 - \sigma_1 &= \frac{\frac{1}{2n}\left((n - m_3)r - n - m_2 + \sqrt{A(r)}\right)}{r - 1} \\ &\qquad - \frac{1}{2n}\left((n - m_3)r - n - m_2 - \sqrt{A(r)}\right) - 1 \\ &= \frac{r\sqrt{A(r)} - f(r)}{2n\,(r - 1)}.\end{aligned}$$

$\sigma_2 - \sigma_1 > 0$ when

$$\begin{aligned} r > 1 &\iff \sqrt{A} r > f(r) \\ &\iff Ar^2 > \left((n - m_3)\, r^2 + (-n + 2m_3 - m_2)\, r + 2m_2\right)^2 \\ &\iff 4\,(r - 1)\left(\left(m_2^2 + m_1 m_2 - m_1 m_3\right) r^2 + \left(m_2 m_3 - m_1 m_2 - m_2^2\right) r + m_2^2\right) > 0 \\ &\iff h(r) > 0 \end{aligned}$$

when $r > 1$, where

$$h(r) = \left(m_2^2 + m_1(m_2 - m_3)\right) r^2 + m_2\,(m_3 - m_2 - m_1)\, r + m_2^2.$$

We want to determine necessary and sufficient conditions on m_1, m_2, m_3 which imply that $h(r) > 0$ for all $r > 1$. A necessary condition is clearly

$$m_2^2 + m_1(m_2 - m_3) > 0, \tag{2.8}$$

so we assume that (2.8) holds. Let

$$r_0 = \frac{1}{2} m_2 \frac{m_1 + m_2 - m_3}{m_2^2 + m_1(m_2 - m_3)}$$

be the unique root of h'. If $r_0 \le 1$, then it is necessary and sufficient to have $h\,(1) > 0$. If $r_0 > 1$, then it is necessary and sufficient to have $h\,(r_0) > 0$. Now

$$\begin{aligned} r_0 \le 1 &\iff 2\left(m_2^2 + m_1(m_2 - m_3)\right) \ge m_2\,(m_2 + m_1 - m_3) \\ &\iff m_2^2 + m_2\,(m_1 + m_3) - 2m_1 m_3 \ge 0, \end{aligned}$$

and

$$h(1) > 0 \iff m_2^2 + m_3(m_2 - m_1) > 0.$$

That proves (1.3). If

$$m_2^2 + m_2\,(m_1 + m_3) - 2m_1 m_3 < 0,$$

then $r_0 > 1$. It is then necessary and sufficient that

$$h\,(r_0) = \frac{1}{4} m_2^2\,(m_1 + m_2 + m_3)\,\frac{3m_2 - m_1 - m_3}{m_2^2 + m_1(m_2 - m_3)} > 0 \iff 3m_2 - m_1 - m_3 > 0.$$

That proves (1.4). □

One can also easily derive necessary and sufficient conditions on m_1, m_2, m_3 for $\sigma_2 < \sigma_1$. We simply cite an example here that shows that this is possible.

Example 2.1. Let $m_1 = 2$, $m_2 = 1$, $m_3 = 3$. As noted above, we may assume that $p(x) = x^2(x-1)(x-r)^3$, $r > 1$. Then a simple computation shows that

$$\sigma_1 = \frac{5}{12} + \frac{1}{4}r - \frac{1}{12}\sqrt{25 - 18r + 9r^2} \qquad \text{and} \qquad \sigma_2 = \frac{3r - 7 + \sqrt{25 - 18r + 9r^2}}{12}.$$

Simplifying yields

$$\sigma_2 - \sigma_1 = \frac{-3r^2 + r - 2 + r\sqrt{25 - 18r + 9r^2}}{12(r-1)}.$$

$\sigma_2 - \sigma_1 < 0$,

$$\begin{aligned} r > 1 \iff & r\sqrt{25 - 18r + 9r^2} < 3r^2 - r + 2 \\ \iff & \left(3r^2 - r + 2\right)^2 - r^2\left(25 - 18r + 9r^2\right) > 0 \\ \iff & 4\left(r-1\right)\left(3r^2 - 1\right) > 0. \end{aligned}$$

Hence $\sigma_2 < \sigma_1$ for all $r > 1$.

Remark 2. For the example above, if we choose $r = 2$, then the roots are **equispaced**, but $\sigma_2 < \sigma_1$. Contrast this with ([5, Theorem 6]), where it was shown that for any $N \geq 3$, if $m_1 = \cdots = m_N = 1$ and the roots are *equispaced*, then the ratios of p are **increasing**.

We now discuss the case $N = 4$, so that $n = m_1 + m_2 + m_3 + m_4$. Theorem 1.2 then yields

$$\begin{aligned} \frac{m_1}{n} &< u < \frac{m_1}{m_1 + m_2}, \\ \frac{m_2}{m_2 + m_3 + m_4} &< v < \frac{m_1 + m_2}{m_1 + m_2 + m_3}, \\ \frac{m_3}{m_3 + m_4} &< w < \frac{m_1 + m_2 + m_3}{n}. \end{aligned} \tag{2.9}$$

In [6] necessary and sufficient conditions were given for $(\sigma_1, \sigma_2, \sigma_3)$ to be a ratio vector when $m_1 = m_2 = m_3 = 1$. We now give a simpler proof than that given in [6] which also generalizes to any positive real numbers m_1, m_2, and m_3. The proof here for $N = 4$ does not require the use of Groebner bases as in [6].

Proof of Theorem 1.6. ($\Longleftarrow$ Suppose first that (u, v, w) is a ratio vector of

$$p(x) = (x - r_1)^{m_1}(x - r_2)^{m_2}(x - r_3)^{m_3}(x - r_4)^{m_4}.$$

Since $p(x + c)$ and $p(x)$ have the same ratio vectors for any constant c, we may assume that $r_2 = 0$, and thus the equations (2.4) hold with $N = 4$. In addition, since $p(cx)$ and $p(x)$ have the same ratio vectors for any constant $c > 0$, we may also assume that $r_1 = -1$. Let $r_3 = r$ and $r_4 = s$, so that $0 < r < s$. Then (2.4) becomes

$$(n(w - v) - m_3)\, r + (n(1 - w) - m_4)s = nu - m_1, \tag{2.10}$$

$$\begin{aligned} nv(1 - w)r^2 + (nvw - m_1 - m_2)\, rs + (n\left(1 - u\right)\left(w - v - 1\right) + m_2 + m_4)\, r \\ + (nw(u - 1) + m_2 + m_3)\, s = 0, \end{aligned} \tag{2.11}$$

$$nv\left(u - 1\right)\left(1 - w\right) r + \left(nvw\left(u - 1\right) + m_2\right) s = 0. \tag{2.12}$$

In particular, (2.10) – (2.12) must be consistent. Eliminating r and s from (2.10) and (2.12) yields

$$(nv\,(u-1)\,(1-w)\,(n(1-w)-m_4)-(n(w-v)-m_3)\,(nvw\,(u-1)+m_2))\,s \\ = (nu-m_1)\,nv\,(u-1)\,(1-w)\,,$$

or

$$D(u,v,w)s = (nu-m_1)\,nv\,(1-u)\,(1-w)\,.$$

Note that $nu-m_1>0, 1-u>0, v>0$, and $1-w>0$ by (2.9). Thus $D(u,v,w)>0$ and by Cramer's Rule,

(2.13) $$r=\frac{D_1(u,v,w)}{D(u,v,w)},\qquad s=\frac{D_2(u,v,w)}{D(u,v,w)},$$

where

$$D_1(u,v,w)=\begin{vmatrix} nu-m_1 & n(1-w)-m_4 \\ 0 & nvw\,(u-1)+m_2 \end{vmatrix} = (nu-m_1)\,(m_2-nvw\,(1-u))\,,$$

and

$$D_2(u,v,w)=\begin{vmatrix} n(w-v)-m_3 & nu-m_1 \\ nv\,(u-1)\,(1-w) & 0 \end{vmatrix} = (nu-m_1)\,nv\,(1-u)\,(1-w)\,.$$

(2.13) and $D(u,v,w)>0$ imply that $D_1(u,v,w)>0$, and $r<s$ implies that $D_1(u,v,w)<D_2(u,v,w)$. Now substitute the expressions for r and s in (2.13) into (2.11). Clearing denominators gives

(2.14) $$nv(1-w)D_1^2+(nvw-m_1-m_2)\,D_1D_2 \\ +(n\,(1-u)\,(w-v-1)+m_2+m_4)\,D_1D \\ +(nw(u-1)+m_2+m_3)\,D_2D=0.$$

Factoring the LHS of (2.14) yields

$$(nu-m_1)\,(nv(1-u)-m_2)\,R(u,v,w)=0.$$

Also, (2.12) and $r<s$ implies that

(2.15) $$\frac{m_2}{n}-vw\,(1-u)<v\,(1-u)\,(1-w) \\ \Rightarrow \frac{m_2}{n}<vw\,(1-u)+v\,(1-u)\,(1-w)=v\,(1-u) \\ \Rightarrow v(1-u)>\frac{m_2}{n}.$$

Thus $m_2-nv(1-u)\neq 0$, which implies that $R(u,v,w)=0$.

($\Longrightarrow$ Now suppose that u,v, and w are real numbers with $0<D_1(u,v,w)<D_2(u,v,w)$, $D(u,v,w)>0$, and $R(u,v,w)=0$. Let $r=\frac{D_1(u,v,w)}{D(u,v,w)}$ and $s=\frac{D_2(u,v,w)}{D(u,v,w)}$. Then $0<r<s$ and it follows as above that (r,s,u,v,w) satisfies (2.10) – (2.12). Let $x_1=u-1$, $x_2=rv$, and $x_3=(s-r)w+r$. Then (2.2) must hold since (2.2) and (2.4) are an equivalent system of equations. Let

$$p(x)=(x+1)^{m_1}x^{m_2}(x-r)^{m_3}(x-s)^{m_4}.$$

Working backwards, it is easy to see that (2.1) must hold and hence x_1,x_2, and x_3 must be the critical points of p. Since $u=\frac{x_1-(-1)}{0-(-1)}$, $v=\frac{x_2-0}{r-0}$, and $w=\frac{x_3-r}{s-r}$, (u,v,w) is a ratio vector of p. □

Remark 3. As noted in [6] for the case when $m_1 = m_2 = m_3 = m_4 = 1$, the proof above shows that if (u, v, w) is a ratio vector, then there are *unique* real numbers $0 < r < s$ such that the polynomial

$$p(x) = (x+1)^{m_1} x^{m_2} (x-r)^{m_3} (x-s)^{m_4}$$

has (u, v, w) as a ratio vector. For general N we make the following conjecture.

Conjecture 2.2. *Let*

$$p(x) = (x+1)^{m_1} x^{m_2} (x-r_3)^{m_3} \cdots (x-r_N)^{m_N},$$
$$q(x) = (x+1)^{m_1} x^{m_2} (x-s_3)^{m_3} \cdots (x-s_N)^{m_N},$$

where $0 < r_3 < \cdots < r_N$ *and* $0 < s_3 < \cdots < s_N$*. Suppose that* p *and* q *have the same ratio vectors. Then* $p = q$*.*

As with $N = 3$, it was shown in [5] that $m_1 = m_2 = m_3 = m_4 = 1$ implies that $\sigma_1 < \sigma_2 < \sigma_3$. Not suprisingly, this does not hold for general positive real numbers $m_1, m_2, m_3,$ and m_4. For example, if $p(x) = (x+1)^{3/2} x (x-4)^{\sqrt{2}} (x-6)^2$, then $\sigma_1 > \sigma_3 > \sigma_2$.

Proof of Theorem 1.7. $m_1 + m_4 \le 3m_2 - m_3 \Rightarrow n \le 4m_2$. By (2.15) in the proof of Theorem 1.6, $v(1-u) > \frac{1}{4}$. Thus $\frac{v}{u} > \frac{1}{4u(1-u)} \ge 1$ since $u(1-u) \le 1$. By letting $r_1 = r < r_2 = -1 < r_3 = 0 < r_4 = s$, one can derive equations similar to (2.2) with $N = 4$. The third equation becomes

$$(m_3 - nw(1-u)(1-v))\, r + nwu(1-v) = 0 \Rightarrow r = \frac{nwu(1-v)}{nw(1-v)(1-u) - m_3}.$$

$$\begin{aligned} r < -1 &\Rightarrow \frac{1}{r} > -1 \\ &\Rightarrow \frac{nw(1-v)(1-u) - m_3}{nwu(1-v)} > -1 \\ &\Rightarrow nw(1-v)(1-u) - m_3 > -nwu(1-v) \\ &\Rightarrow nw(1-v)(1-u) + nwu(1-v) > m_3 \\ &\Rightarrow nw(1-v) > m_3 \\ &\Rightarrow \frac{w}{v} > \frac{m_3}{nv(1-v)}. \end{aligned}$$

Now $m_1 + m_4 \le 3m_3 - m_2 \Rightarrow n \le 4m_3$. Thus $\frac{w}{v} > \frac{1}{4v(1-v)} \ge 1$. □

REFERENCES

[1] P. ANDREWS, Where not to find the critical points of a polynomial-variation on a Putnam theme, *Amer. Math. Monthly,* **102** (1995), 155–158.

[2] B. ANDERSON, Polynomial root dragging, *Amer. Math. Monthly*, **100** (1993), 864–866.

[3] D. COX, J. LITTLE, AND D. O'SHEA, *Ideals, Varieties, and Algorithms*, Second Ed., Springer-Verlag, New York, 1997.

[4] G. PEYSER, On the roots of the derivative of a polynomial with real roots, *Amer. Math. Monthly,* **74** (1967), 1102–1104.

[5] A. HORWITZ, On the ratio vectors of polynomials, *J. Math. Anal. Appl.*, **205** (1997), 568–576.

[6] A. HORWITZ, Ratio vectors of fourth degree polynomials, *J. Math. Anal. Appl.*, **313** (2006), 132–141.

[7] A. HORWITZ, Complex ratio vectors of cubic polynomials, *IJPAM*, **33**(1) (2006), 49–62.

[8] V. KOSTOV, On hyperbolic polynomial-like functions and their derivatives, *Proc. Roy. Soc. Edinburgh Sect. A*, **137** (2007), 819–845.

[9] A. MELMAN, Bounds on the zeros of the derivative of a polynomial with all real zeros, *Amer. Math. Monthly*, **115** (2008), 145–147.