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A STOCHASTIC GRONWALL INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. In this paper, we show a Gronwall type inequality for Itô integrals (Theorems 1.1 and 1.2) and give some applications. Our inequality gives a simple proof of the existence theorem for stochastic differential equation (Example 2.1) and also, the error estimate of Euler-Maruyama scheme follows immediately from our result (Example 2.2).

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1. A STOCHASTIC GRONWALL TYPE INEQUALITY

Let $w(t)$, $t \geq 0$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$, $t \geq 0$ be the natural filtration of $\mathcal{F}$. For a positive number T, $M_w^2[0,T]$ denotes the set of all separable nonanticipative functions $f(t)$ with respect to $\mathcal{F}_t$ defined on $[0,T]$ satisfying

$$E\left[\int_0^T f^2(t)\,dt\right] < \infty.$$

Theorem 1.1. *Assume that $\xi(t)$ and $\eta(t)$ belong to $M_w^2[0,T]$. If there exist functions $a(t)$ and $b(t)$ belonging to $M_w^2[0,T]$ such that*

$$|\xi(t)| \leq \left|\int_0^t a(s)\,ds + \int_0^t b(s)\,dw(s)\right| \tag{1.1}$$

and if there are nonnegative constants α_0, α_1, β_0 and β_1 such that

$$|a(t)| \leq \alpha_0|\eta(t)| + \alpha_1|\xi(t)|, \quad |b(t)| \leq \beta_0|\eta(t)| + \beta_1|\xi(t)| \tag{1.2}$$

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for $0 \le t \le T$*, then we have*

$$E\xi^2(t) \le 4\big(\alpha_0\sqrt{t}+\beta_0\big)^2 \exp\Big(4t\,(\alpha_1\sqrt{t}+\beta_1)^2\Big)\int_0^t E\eta^2(s)\,ds \tag{1.3}$$

for $0 \le t \le T$.

Proof. Since

$$E\left(\int_0^t b(s)\,dw(s)\right)^2 = E\int_0^t b^2(s)\,ds,$$

(1.1) implies, by Minkowsky and Schwarz inequalities,

$$\big(E\xi^2(t)\big)^{\frac{1}{2}} \le \left(t\int_0^t Ea^2(s)\,ds\right)^{\frac{1}{2}} + \left(\int_0^t Eb^2(s)\,ds\right)^{\frac{1}{2}}.$$

Direct computation gives, by (1.2),

$$\left(t\int_0^t Ea^2(s)\,ds\right)^{\frac{1}{2}} \le \sqrt{2t}\alpha_0\left(\int_0^t E\eta^2(s)\,ds\right)^{\frac{1}{2}} + \sqrt{2t}\alpha_1\left(\int_0^t E\xi^2(s)\,ds\right)^{\frac{1}{2}},$$

$$\left(\int_0^t Eb^2(s)\,ds\right)^{\frac{1}{2}} \le \sqrt{2}\beta_0\left(\int_0^t E\eta^2(s)\,ds\right)^{\frac{1}{2}} + \sqrt{2}\beta_1\left(\int_0^t E\xi^2(s)\,ds\right)^{\frac{1}{2}}.$$

Combining the above estimates, we obtain

$$E\xi^2(t) \le 4(\alpha_0\sqrt{t}+\beta_0)^2\int_0^t E\eta^2(s)\,ds + 4(\alpha_1\sqrt{t}+\beta_1)^2\int_0^t E\xi^2(s)\,ds \tag{1.4}$$

for $0 \le t \le T$.

Let us fix a nonnegative number $t_0 \le T$ arbitrarily. Then, for any $\delta > 0$, the last inequality (1.4) shows

$$\frac{d}{dt}\log\left(4(\alpha_0\sqrt{t_0}+\beta_0)^2\int_0^{t_0} E\eta^2(s)\,ds + 4(\alpha_1\sqrt{t_0}+\beta_1)^2\int_0^t E\xi^2(s)\,ds + \delta\right) \le 4(\alpha_1\sqrt{t_0}+\beta_1)^2$$

almost everywhere in $[0, t_0]$. Integrating this estimate from 0 to t_0 with respect to t, we get

$$\log\left(\frac{4(\alpha_0\sqrt{t_0}+\beta_0)^2\int_0^{t_0} E\eta^2(s)\,ds + 4(\alpha_1\sqrt{t_0}+\beta_1)^2\int_0^{t_0} E\xi^2(s)\,ds + \delta}{4(\alpha_0\sqrt{t_0}+\beta_0)^2\int_0^{t_0} E\eta^2(s)\,ds + \delta}\right) \le 4t_0(\alpha_1\sqrt{t_0}+\beta_1)^2.$$

Therefore, by (1.4), we have

$$E\xi^2(t_0) \le \exp\big(4t_0(\alpha_1\sqrt{t_0}+\beta_1)^2\big)\left(4(\alpha_0\sqrt{t_0}+\beta_0)^2\int_0^{t_0} E\eta^2(s)\,ds + \delta\right).$$

Now, letting $\delta \to 0$, we obtain (1.3). □

In case $\xi(t)$ is a step function, a weak assumption (1.6) will be enough to show the inequality (1.3), which would play an important role in the error analysis of the numerical solutions of stochastic differential equations.

Theorem 1.2. *Assume that $\xi(t)$ and $\eta(t)$ belong to $M_w^2[0,T]$ and $\xi(t)$ is a step function such that*

$$\xi(t) = \xi(t_n) \qquad \textit{when} \quad t_n \le t < t_{n+1} \tag{1.5}$$

for $n = 0,1,2,\dots,N-1$, where N is a positive integer and $\{t_n\}_{n=0}^N$ is a partition of the interval $[0,T]$ satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$. If there exist functions $a(t)$ and $b(t)$ belonging to $M_w^2[0,T]$ such that

$$|\xi(t_n)| \le \left| \int_0^{t_n} a(s)\,ds + \int_0^{t_n} b(s)\,dw(s) \right| \tag{1.6}$$

is valid for each $n = 0,1,2,\dots,N$ and if there are nonnegative constants α_0, α_1, β_0 and β_1 satisfying (1.2) for $0 \le t \le T$, then we have (1.3) for $0 \le t \le T$.

Proof. As in the proof of Theorem 1.1, we have

$$E\xi^2(t_n) \le 4(\alpha_0\sqrt{t_n} + \beta_0)^2 \int_0^{t_n} E\eta^2(s)\,ds + 4(\alpha_1\sqrt{t_n} + \beta_1)^2 \int_0^{t_n} E\xi^2(s)\,ds$$

for $n = 0,1,2,\dots,N$; this implies, by (1.5),

$$E\xi^2(t) \le 4(\alpha_0\sqrt{t} + \beta_0)^2 \int_0^{t} E\eta^2(s)\,ds + 4(\alpha_1\sqrt{t} + \beta_1)^2 \int_0^{t} E\xi^2(s)\,ds$$

for $0 \le t \le T$.

The remaining part of the proof is exactly same as that of Theorem 1.1. □

2. APPLICATIONS

Throughout this section, we assume that $\xi(t) \in M_w^2[0,T]$ is a solution of the stochastic differential equation

$$d\xi(t) = a(t,\xi(t))\,dt + b(t,\xi(t))\,dw(t), \quad 0 \le t \le T$$

satisfying the initial condition $\xi(0) = \xi_0$, where $a(t,x)$ and $b(t,x)$ are real-valued functions defined in $[0,T]$ such that

$$|a(t,x)|, |b(t,x)| \le K(1+|x|),$$
$$|a(t,x) - a(s,y)|, |b(t,x) - b(s,y)| \le L(|t-s| + |x-y|).$$

Here K and L are nonnegative constants.

Example 2.1. Theorem 1.1 gives a simple proof of the existence theorem for stochastic differential equations.

We use Picard's method. Let us consider a sequence $\{\xi_n(t)\}$ defined by $\xi_0(t) = \xi_0$ and

$$\xi_{n+1}(t) = \xi_0 + \int_0^t a(s,\xi_n(s))\,ds + \int_0^t b(s,\xi_n(s))\,dw(s)$$

for $n = 0,1,2,\dots$. Then, we easily have

$$\xi_{n+1}(t) - \xi_n(t) = \int_0^t \big(a(s,\xi_n(s)) - a(s,\xi_{n-1}(s))\big)\,ds \\ + \int_0^t \big(b(s,\xi_n(s)) - b(s,\xi_{n-1}(s))\big)\,dw(s)$$

and the Lipschitz continuity of $a(t,x)$ and $b(t,x)$ implies

$$\begin{aligned}\big|a(s,\xi_n(s))-a(s,\xi_{n-1}(s))\big| &\le L|\xi_n(s)-\xi_{n-1}(s)|,\\ \big|b(s,\xi_n(s))-b(s,\xi_{n-1}(s))\big| &\le L|\xi_n(s)-\xi_{n-1}(s)|.\end{aligned}$$

Hence, Theorem 1.1 with $\alpha_0=\beta_0=L$ and $\alpha_1=\beta_1=0$ shows

$$E|\xi_{n+1}(t)-\xi_n(t)|^2 \le 4L^2(\sqrt{t}+1)^2\int_0^t E|\xi_n(s)-\xi_{n-1}(s)|^2 ds$$

for $n=1,2,3,\dots$; the recursive use of this estimate gives

$$E\,|\xi_{n+1}(t)-\xi_n(t)|^2 \le \frac{\big(4L^2(\sqrt{t}+1)^2 t\big)^n}{n!}\sup_{0\le s\le t} E|\xi_1(s)-\xi_0(s)|^2.$$

Consequently, as is well-known, the convergence of $\{\xi_n(t)\}$ follows.

By virtue of Theorem 1.1 with $\alpha_0=\beta_0=0$ and $\alpha_1=\beta_1=L$, the uniqueness of the solution is clear.

Example 2.2. The error estimate of the Euler-Maruyama scheme

$$\xi_{n+1}=\xi_n+a(t_n,\xi_n)\Delta t+b(t_n,\xi_n)\Delta w_n,\quad n=0,1,2,\dots,N-1$$

follows immediately from Theorem 1.2, where N is a sufficiently large positive integer, $\Delta t = T/N$, $t_n=n\Delta t$ and $\Delta w_n=w(t_{n+1})-w(t_n)$ for $n=0,1,2,\dots,N-1$.

Since

$$\begin{aligned}\xi_{n+1}&=\xi_n+\int_{t_n}^{t_{n+1}} a(t_n,\xi_n)ds+\int_{t_n}^{t_{n+1}} b(t_n,\xi_n)dw(s),\\ \xi(t_{n+1})&=\xi(t_n)+\int_{t_n}^{t_{n+1}} a(s,\xi(s))ds+\int_{t_n}^{t_{n+1}} b(s,\xi(s))dw(s),\end{aligned}$$

we have

$$\begin{aligned}\xi_{n+1}-\xi(t_{n+1})=\xi_n-\xi(t_n)&+\int_{t_n}^{t_{n+1}}\big(a(t_n,\xi_n)-a(s,\xi(s))\big)\,ds\\ &+\int_{t_n}^{t_{n+1}}\big(b(t_n,\xi_n)-b(s,\xi(s))\big)\,dw(s).\end{aligned}$$

Now, for $n=0,1,2,\dots,N-1$, if we put

$$\begin{aligned}\varepsilon(s)&=\xi_n-\xi(t_n),\\ f(s)&=a(t_n,\xi_n)-a(s,\xi(s)),\\ g(s)&=b(t_n,\xi_n)-b(s,\xi(s))\end{aligned}$$

when $t_n\le s<t_{n+1}$ and

$$\begin{aligned}\varepsilon(t_N)&=\xi_N-\xi(t_N),\\ f(t_N)&=a(t_N,\xi_N)-a(t_N,\xi(t_N)),\\ g(t_N)&=b(t_N,\xi_N)-b(t_N,\xi(t_N)),\end{aligned}$$

then we obtain

$$\varepsilon(t_n)=\int_0^{t_n} f(s)\,ds+\int_0^{t_n} g(s)\,dw(s)$$

for $n = 0, 1, 2, \ldots, N$. The Lipschitz continuity of $a(t,x)$ and $b(t,x)$ shows

$$|f(s)| \leq L\big(\Delta t + |\xi(s) - \tilde{\xi}(s)| + |\varepsilon(s)|\big),$$
$$|g(s)| \leq L\big(\Delta t + |\xi(s) - \tilde{\xi}(s)| + |\varepsilon(s)|\big),$$

where $\tilde{\xi}(s) = \xi(t_n)$ when $t_n \leq s < t_{n+1}$ for $n = 0, 1, 2, \ldots, N-1$ and $\tilde{\xi}(t_N) = \xi(t_N)$. Hence, Theorem 1.2 with $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = L$ shows

$$E\varepsilon^2(t) \leq 4L^2(\sqrt{t}+1)^2 \exp\big(4L^2(\sqrt{t}+1)^2 t\big) \int_0^t E\big(\Delta t + |\xi(s) - \tilde{\xi}(s)|\big)^2 \, ds.$$

It follows from the fundamental property of Itô integrals that

$$E\,|\xi(s) - \tilde{\xi}(s)|^2 \leq C\Delta t,$$

where C is a nonnegative constant depending only on T, K and L. Combining the above estimates, we obtain

$$E\varepsilon^2(t) = O(\Delta t)$$

for any $0 \leq t \leq T$ when $\Delta t \to 0$; the error estimate of the Euler-Maruyama scheme is proved.

Our Gronwall type inequality works for other numerical solutions of stochastic differential equations.

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