Journal of Inequalities in Pure and Applied Mathematics

ORIENTED SITE PERCOLATION, PHASE TRANSITIONS AND PROBABILITY BOUNDS

C.E.M. PEARCE AND F.K. FLETCHER

School of Mathematical Sciences
The University of Adelaide
Adelaide SA 5005, Australia.
EMail: cpearce@maths.adelaide.edu.au

Maritime Operations Division
DSTO, PO Box 1500
Edinburgh SA 5111, Australia.
EMail: Fiona.Fletcher@defence.dsto.gov.au

ISSN (electronic): 1443-5756
252-05

volume 6, issue 5, article 135,
2005.

Received 25 August, 2005;
accepted 01 September, 2005.

Communicated by: S.S. Dragomir

Abstract

We show that one half is a lower bound for the critical probability of an oriented site percolation process of Grimmett and Hiemer. This value improves the known lower bound of one third. We employ an Ansatz which we use also for a related oriented site percolation problem considered by Bishir. Monte Carlo simulation indicates a critical value of close to 0.535, so the bound appears to be fairly tight.

2000 Mathematics Subject Classification: 60K35, 82B43.
Key words: Oriented site percolation, Critical probability, Phase transition, Positive term power series.

This paper is based on the talk given by the first author within the "International Conference of Mathematical Inequalities and their Applications, I", December 06-08, 2004, Victoria University, Melbourne, Australia [http://rgmia.vu.edu.au/conference]

Contents

1. Introduction

Percolation theory investigates questions related to the deterministic flow of fluid through a random medium consisting of a lattice of sites (vertices, atoms) with adjacent sites connected by edges (bonds). In the bond percolation process, each edge is open (with probability p) or closed (with probability $1-p$). In the site percolation process, each site is open (with probability p) or closed (with probability $1-p$). In either process "fluid" is envisaged as entering the lattice at the origin. In the site process, any site connected to the origin by a chain of consecutive adjacent open sites is said to be wetted. Similarly in the bond process, any edge joined to the origin through a connected sequence of open edges is termed wetted. Percolation occurs when an infinite number of sites (resp. edges) are wetted. Mixed site and bond percolation processes are also possible, sites and bonds being open with respective probabilities p_s and p_b. Fluid will flow between two sites if and only if both are open and an open bond exists between them.

Each formulation admits oriented versions. Here bonds between pairs of sites have an associated orientation and fluid may flow only in the direction of that orientation. For a discussion of oriented percolation see [7].

A phenomenon associated with percolation processes is that of phase transitions: for small p percolation does not occur while if p is above a critical probability threshold p_c there is a positive probability $\theta(p)$ of percolation. Thus

$$p_c = \sup\{p : \theta(p) = 0\}.$$

The function θ is nondecreasing in p. A conceptual graph of $\theta(p)$ is shown in Figure 1 (see [13, 14, 20]).

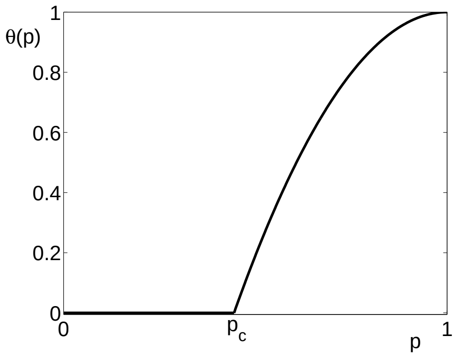

Figure 1: *The behaviour of the percolation probability* $\theta(p)$ *with* p

Key problems in percolation theory include ascertaining the critical probability p_c and characterising the system in the subcritical and supercritical phases and its behaviour for p close to p_c. Summaries are given in [13, 14, 17, 19]. For a one–dimensional percolation process, $p_c = 1$. For a hypercubic lattice $\mathbb{L}^d$ of dimension $d \geq 2$ we have $0 < p_c(\mathbb{L}^d) < 1$ (see [13, 14]). To distinguish the critical probabilities for site and bond processes we denote the former by p_{cs} and the latter by p_{cb}.

The study of percolation processes has grown enormously following the work of Broadbent [5] and Broadbent and Hammersley [6]. The following exact results have been determined for p_{cb} in the two–dimensional lattices shown in Figure 2.

Kesten [18]: for (a), $p_{cb} = 1/2$.

Wierman [25]: for (b), $p_{cb} = 2\sin(\pi/18)$.
Wierman [25]: for (c), $p_{cb} = 1 - 2\sin(\pi/18)$.
Wierman [26]: for (d), p_{cb} is the unique root in $(0, 1)$ of $1-p-6p^2+6p^3-p^5 = 0$.

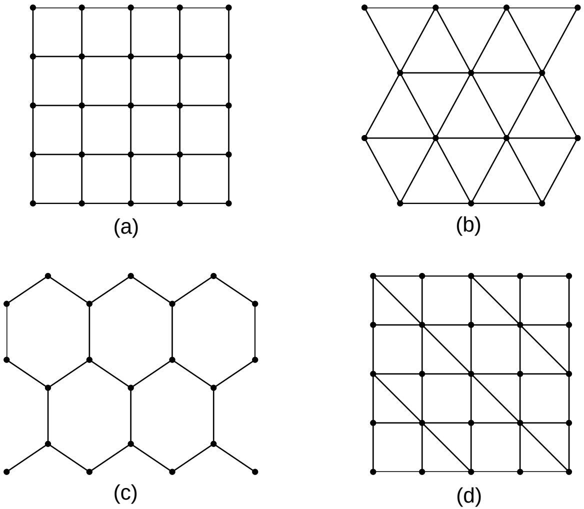

Figure 2: *Illustration of generic portions of the graphs for which p_{cb} is known: (a) square lattice, (b) triangular lattice, (c) hexagonal lattice and (d) bow-tie lattice.*

By contrast there are few exact results for site percolation or oriented percolation. The results above were derived using dual graphs, a technique generally inapplicable to oriented percolation (though see [27]). For site percolation the relevant structural idea is that of *matching* in place of duality (see

[14, Ch. 3]). Some results of Monte Carlo simulation for site percolation are given in [10, 11]. With most percolation problems effort has concentrated on finding lower and upper bounds for the critical probability, see for example [1, 4, 22, 28, 29, 30]. The result

$$p_{cb} < p_{cs} \tag{1.1}$$

was originally shown for a general class of graph structures by Hammersley [16]. Later proofs have centred on a lemma of Oxley and Welsh [24].

In Section 2 we introduce two oriented lattices, $\vec{\mathbb{L}}^2$ and $\vec{\mathbb{L}}^2_{alt}$, on which site percolations exhibit phase transitions. In Section 3 we provide a useful Ansatz. In Section 4 we make use of this in amplifying a derivation by Bishir [3] of a lower bound for $p_{cs}\left(\vec{\mathbb{L}}^2\right)$. Finally, in Section 5, we give our main result, an improved lower bound for $p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right)$.

2. The Oriented Lattices $\vec{\mathbb{L}}^2$ and $\vec{\mathbb{L}}^2_{alt}$

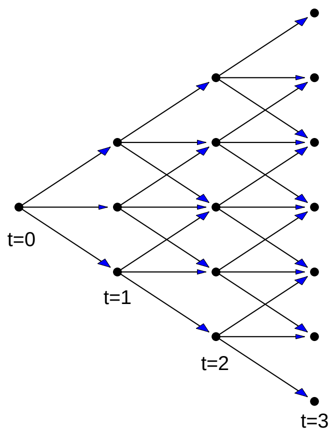

Figure 3: *Possible state transitions in the first three time steps on* $\vec{\mathbb{L}}^2_{alt}$.

The graph structure illustrated in Figure 3 was first considered in an oriented bond percolation context by Grimmett and Hiemer [15]. We follow their notation $\vec{\mathbb{L}}^2_{alt}$. We write $\vec{\mathbb{L}}^2$ for the two–dimensional lattice $\mathbb{L}^2$ with bonds oriented in the positive x and y directions. The set of sites that may be reached at time n from the origin is then the set of sites $\{(x, y)\}$ on the diagonal $x + y = n$ (see Figure 4(a)). Figure 4(b) shows this graph rotated through $\pi/4$.

Consider the graph formed by removing all sites (x, y) with $x + y$ odd. This consists of bonds directed from each site (x, y) with $x + y$ even to $(x + 1, y - 1)$

and $(x+1, y+1)$ and so is simply the graph $\vec{\mathbb{L}}^2$, showing that $\vec{\mathbb{L}}^2_{alt} \supset \vec{\mathbb{L}}^2$.

Durrett [7], Liggett [21], Ballister, Bollobas and Stacey [1] use the graph $\vec{\mathbb{L}}^2$ in an oriented bond or site percolation model. In particular, Liggett [21] considers percolation on the graph $\vec{\mathbb{L}}^2$, where the probability of a site being present at time t is dependent on whether it has 0, 1 or 2 neighbours at time $t-1$. Denote by A_n the set of sites open at time n, that is, sites with $x+y=n$. The probability of a site (x, y) being open at time $n+1$ is then given by

$$\mathbb{P}\{(x,y) \in A_{n+1} | A_n\} = \begin{cases} q & \text{if } |A_n \cap \{(x, y-1), (x-1, y)\}| = 2 \\ p & \text{if } |A_n \cap \{(x, y-1), (x-1, y)\}| = 1 \\ 0 & \text{otherwise} \end{cases} .$$

This general formulation allows for site percolation, bond percolation and mixed percolation processes on the graph. We say that (A_n) survives or dies out according to whether $P(A_n \neq \emptyset \, \forall n)$ is positive or zero (for nonempty finite initial states). Liggett proved that
(a) if $q < 2(1-p)$, then (A_n) dies out;
(b) if $\frac{1}{2} < p \leq 1$ and $q \geq 4p(1-p)$, then (A_n) survives.

For site percolation on $\vec{\mathbb{L}}^2$, the probability of each site being open is independent of the number of adjacent bonds and sites, so $p = q$. Result (b) then gives that (A_n) survives for $p \geq 3/4$, so that $p_{cs}\left(\vec{\mathbb{L}}^2\right)$ satisfies

$$p_{cs}\left(\vec{\mathbb{L}}^2\right) \leq \frac{3}{4}. \tag{2.1}$$

This leads to the following.

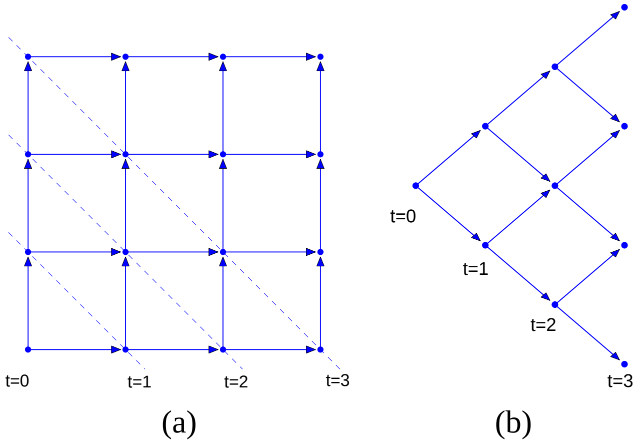

Figure 4: *The graph* $\vec{\mathbb{L}}^2$ *(a) oriented as the square lattice and (b) rotated* 45^o *so that the* x*-axis represents time*

Theorem 2.1. *The site percolation process on* $\vec{\mathbb{L}}^2_{alt}$ *undergoes a phase transition, with*

$$\frac{1}{3} \leq p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right) \leq p_{cs}\left(\vec{\mathbb{L}}^2\right) \leq \frac{3}{4}.$$

Proof. Let $N(n)$ be the total number of open n–step paths in the site process on $\vec{\mathbb{L}}^2_{alt}$. From the orientation of the graph, these will be self–avoiding. Then $N(n) \leq 3^n$, the total number of n-step paths on $\vec{\mathbb{L}}^2_{alt}$, so

$$\mathbb{P}(N(n) \geq 1) \leq \mathbb{E}(N(n)) \leq 3^n p^n.$$

Since $3^n p^n \to 0$ when $p < 1/3$, we have

$$\lim_{n\to\infty} \mathbb{P}(N(n) \geq 1) = 0 \quad \text{for} \quad p < \frac{1}{3}.$$

This gives $p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right) \geq 1/3$.

Since $\vec{\mathbb{L}}^2_{alt} \supset \vec{\mathbb{L}}^2$, we have $p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right) \leq p_{cs}\left(\vec{\mathbb{L}}^2\right)$. The remainder of the enunciation follows from (2.1). □

The above derivation of $p_{cs}\left(\vec{\mathbb{L}}^2\right) \leq 3/4$ was given by Liggett [21] in 1995. Earlier rigorous upper bounds are 0.819 (Liggett [8] 1992), 0.762 (Balister *et al.* [1] 1993) and 0.7491 (Balister *et al.* [2] 1994). The last paper corrected a misprint in [1]. The tighter bounds required substantial computer calculation. A nonrigorous estimate 0.7055 was given by Onody and Neves [23] in 1992. These values may be compared with the lower bound 2/3 found by Bishir and discussed in Section 4. Although derived as far back as 1963, this does not appear to have been improved subsequently. Thus (a) of Liggett also gives $p_{cs}\left(\vec{\mathbb{L}}^2\right) \geq 2/3$.

The derivation of the first inequality in Theorem 2.1 is due to Grimmett [14]. In fact by considering instead the corresponding bond percolation and invoking (1.1), this result can be strengthened minimally to $p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right) > 1/3$. In Section 5 we improve the lower bound for $p_c\left(\vec{\mathbb{L}}^2_{alt}\right)$ from one third to one half.

3. Ansatz

As a prelude to deriving an improved lower bound for $p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right)$ and filling out Bishir's derivation of a lower bound for $p_{cs}\left(\vec{\mathbb{L}}^2\right)$, we introduce a useful lemma.

Lemma 3.1. *Suppose R_1, R_2 are proper real polynomials in z, with R_2 of degree $m \geq 1$ and R_1 of degree less than or equal to m, and that*

$$h(z) = \frac{R_1(z)}{(1-z)R_2(z)}$$

has a partial fractions decomposition

$$h(z) = \frac{A_1}{1-z} + \sum_{i=2}^{m+1} \frac{A_i}{1-z/z_i}$$

with

$$z_{m+1} > z_m > \ldots > z_2 > 1$$

and the A's satisfying

$$\sum_{j=1}^{i} A_j > 0 \quad \textit{for} \quad i = 1, 2, \ldots, m+1.$$

If

$$h(z) := \sum_{n=0}^{\infty} h_n z^n,$$

then $(h_n)_{n=0}^{\infty}$ is positive and bounded above.

Proof. From the given conditions we have for $n \geq 0$ that

$$\begin{aligned} h_n &= A_1 + \sum_{i=2}^{m+1} \frac{A_i}{z_i^n} \\ &\geq \frac{A_1 + A_2}{z_2^n} + \sum_{i=3}^{m+1} \frac{A_i}{z_i^n} \\ &\geq \quad \dots \quad \dots \\ &\geq \frac{A_1 + A_2 + \dots + A_{m+1}}{z_m^n} \\ &> 0, \end{aligned}$$

supplying positivity. Boundedness follows from

$$h_n \to A_1 \quad \text{as } n \to \infty.$$

□

4. Bishir's Lower Bound

In this section a result of Bishir [3] is presented and proved. The result provides a lower bound for the critical probability for oriented site percolation on the graph $\vec{\mathbb{L}}^2$. The convergence arguments presented by Bishir [3] are incomplete. We present a more complete argument utilising the lemma.

Theorem 4.1. *The critical probability $p_{cs}\left(\vec{\mathbb{L}}^2\right)$ satisfies*

$$p_{cs}\left(\vec{\mathbb{L}}^2\right) \geq \frac{2}{3}.$$

Proof. Consider a modification of the percolation process wherein sites are open with probability p but where, if any two sites are wetted at time t, then all intervening sites are deemed to be wetted. Let $\gamma(p)$ be the probability that an infinite number of sites will be wetted in the modified process and p_{cs}^{γ} the corresponding critical probability. Then $\gamma(p) \geq \theta(p)$, since more sites are wetted in the modified process. Accordingly $p_{cs}^{\gamma} \leq p_{cs}\left(\vec{\mathbb{L}}^2\right)$. It thus suffices to show that $p_{cs}^{\gamma} = 2/3$.

The modified process is a Markov chain whose state at time t is the number n of consecutive wetted sites. As for the original process, if there are no sites wetted at some time then no sites can be wetted at any later time, so state 0 is absorbing. The transition probability $p_{i,j}$ takes the form

$$p_{i,j} = \begin{cases} \delta_{0,j} & \text{for } i = 0 \\ q^{i+1} & \text{for } i \geq 1 \text{ and } j = 0 \\ (i+1)pq^i & \text{for } i \geq 1 \text{ and } j = 1 \\ (i+2-j)p^2q^{i+1-j} & \text{for } i \geq 1 \text{ and } j = 2, \dots, i+1 \\ 0 & \text{for } i \geq 1 \text{ and } j > i+1. \end{cases} \tag{4.1}$$

Let b_n be the probability that the process is never in state 0, given that it started in state n. We note that (b_n) must be nondecreasing. Since the percolation process has initial state 1, then $\gamma(p) = b_1$. Set $B = (b_1, b_2, \dots)^T$.

Suppose the states of the modified process are partitioned as $[0|1, 2, \dots]$, inducing a partition

$$P = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}$$

of its transition matrix. It is well known (see, for example, [9, p. 364]) that B is the maximal solution to

$$B = QB \tag{4.2}$$

satisfying

$$0 \le b_n \le 1. \tag{4.3}$$

From (4.2)

$$B(z) := \sum_{n=1}^{\infty} b_n z^n = (z, z^2, z^3, \dots)B = (z, z^2, z^3, \dots)QB. \tag{4.4}$$

Since (b_n) is nondecreasing, (4.3) gives that $B(z)$ has radius of convergence unity unless $b_n \equiv 0$, when the radius of convergence is infinity. From (4.1) we have

$$(z, z^2, z^3, \dots)Q = \left(\frac{p}{(1-qz)^2} - p, \frac{p^2 z}{(1-qz)^2}, \frac{p^2 z^2}{(1-qz)^2}, \dots \right),$$

where $q = 1 - p$.

Substitution into (4.4) gives

$$B(z) = \frac{p^2}{z(1-qz)^2} B(z) + \left(\frac{p-p^2}{(1-qz)^2} - p \right) b_1$$
$$= \frac{pz(q-(1-qz)^2)}{z(1-qz)^2 - p^2} b_1$$
$$= \frac{pzg(z)}{1-z} b_1,$$

where

$$g(z) = \frac{(1-qz)^2 - q}{(p-qz)^2 - q^2 z}. \tag{4.5}$$

Since $B(z)$ is convergent on the open unit disk, the series $g(z) := \sum_{n=0}^{\infty} g_n z^n$ must also have a radius of convergence of at least unity.

When $p = 0$, absorption occurs at the first step, so that $b_n = 0$ for $n > 0$. When $p = 1$, the process always survives provided it does not start in state 0, so that $b_n = 1$ for $n > 0$. For $0 < p < 1$, the denominator of the right–hand side of (4.5) has two zeros given by

$$z_2 = \frac{1 + p - \sqrt{(1+p)^2 - 4p^2}}{2q},$$
$$z_3 = \frac{1 + p + \sqrt{(1+p)^2 - 4p^2}}{2q}.$$

The factorisation $(1+p)^2 - 4p^2 = (1+3p)(1-p) > 0$ for all $0 < p < 1$ ensures that z_2 and z_3 are real and positive. Also $z_3 > 1$ for all $0 < p < 1$. It may be seen by taking the derivative of z_2 with respect to p that z_2 is increasing for $0 < p < 1$.

First suppose $0 < p < 2/3$. In this case $0 < z_2 < 1$, so $g(z)$ has a pole inside the unit disk unless the numerator in (4.5) vanishes for $z = z_2$. The latter is readily seen to be impossible for $p > 0$. For $B(z)$ to converge inside that disk we require $b_1 = 0$, which implies that $b_n = 0$ for all $n \geq 1$.

Next suppose $2/3 < p < 1$. In this case

$$z_3 > z_2 > 1. \tag{4.6}$$

The function

$$h(z) := \frac{g(z)}{1-z}$$

has partial fraction decomposition

$$\frac{g(z)}{1-z} = \frac{A_1}{1-z} + \frac{A_2}{1-z/z_2} + \frac{A_3}{1-z/z_3},$$

where

$$A_1 = \frac{p^2 - q}{(p-q)^2 - q^2},$$
$$A_2 = \frac{(1-qz_2)^2 - q}{(1-z_2)p^2(1-z_2/z_3)},$$
$$A_3 = \frac{(1-qz_3)^2 - q}{(1-z_3)p^2(1-z_3/z_2)}.$$

We have $A_1 > 0$ for $2/3 < p < 1$. Further,

$$A_1 + A_2 + A_3 = g(0) = \frac{1}{p} > 0.$$

To derive $A_1 + A_2 > 0$, it suffices to demonstrate that $A_3 < 0$. By (4.6) the denominator of A_3 must be positive. Substitution of z_3 into the numerator gives

$$(1 - qz_3)^2 - q = \frac{-q}{2}(q + \sqrt{4q - 3q^2}) < 0,$$

yielding the desired result $A_3 < 0$.

Thus $h(z)$ satisfies the conditions of the lemma, so that $(h_n)_{n=0}^{\infty}$ is positive and bounded above. Since $B(z) = pzb_1h(z)$, the sequence (b_n) is also positive and bounded above unless $b_1 = 0$, when $b_n \equiv 0$.

The value $b = \lim_{n\to\infty} b_n$ may be obtained from Abel's theorem as

$$b = \lim_{z\to 1^-} (1 - z)B(z) = \frac{p^2 - q}{1 - 3q} b_1.$$

When $b_1 > 0$, the maximal solution to (4.2) satisfying (4.3) has $b = 1$, so that $b_1 = (1 - 3q)/(p^2 - q)$ and

$$B(z) = \frac{1 - 3q}{p^2 - q} pzg(z).$$

Finally suppose $p = 2/3$. In this case $z_2 = 1$, so $B(z)$ has a pole of order two at $z = 1$ unless $b_n \equiv 0$. Suppose, if possible, that $b_n \to b > 0$ as $n \to \infty$. By Abel's theorem

$$b = \lim_{z\to 1^-} (1 - z)B(z) = \infty,$$

contradicting $b \leq 1$. Thus we must have $b_n \equiv 0$ for $p = 2/3$.

Accordingly the probability of obtaining an infinite number of wetted sites starting from a single site is

$$\gamma(p) = \begin{cases} 0 & \text{for } p \leq \dfrac{2}{3} \\ \dfrac{1-3q}{p^2-q} & \text{for } p > \dfrac{2}{3} \end{cases} .$$

Thus $p_{cs}^{\gamma} = \sup\{p : \gamma(p) = 0\} = 2/3$, completing the proof. $\square$

5. A Lower Bound for $p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right)$

The approach of the previous section may be employed to develop a lower bound for $p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right)$. In this section, we use this technique to derive a bound that is a substantial improvement on that of Theorem 2.1.

Theorem 5.1. *The critical probability* $p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right)$ *satisfies* $p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right) \geq 1/2$.

Proof. We introduce a modified process on the graph $\vec{\mathbb{L}}^2_{alt}$ with the same structure as the original oriented site percolation problem except in that if any two sites are wetted at time t, then all sites between them at time t are deemed wetted, so the wetted sites at time t are consecutive. Denote the probability of wetting an infinite number of sites for this new process by $\eta(p)$. The percolation threshold p^η_c for this process is

$$p^\eta_{cs} = \sup\{p : \eta(p) = 0\}.$$

The percolation probability for the modified process will be at least as large as that for the original oriented site percolation process, since sites not wetted at time t in the latter may be in the former. These sites may in turn lead to other sites being wetted at the next time step. Thus

$$\theta(p) \leq \eta(p) \quad \text{and} \quad p_{cs}\left(\vec{\mathbb{L}}^2_{alt}\right) \geq p^\eta_{cs}$$

and it suffices to demonstrate that $p^\eta_{cs} = 1/2$.

The state of the process at any time is the number of sites wetted at that time. By construction these sites are contiguous. The modified process is a Markov chain whose states are the nonnegative integers.

When no sites are wetted at some time k, then none are wetted subsequently, so 0 is an absorbing state. The transition probabilities for the chain are

$$p_{i,j} = \begin{cases} \delta_{0,j} & \text{for } i = 0 \\ q^{i+2} & \text{for } i \geq 1 \text{ and } j = 0 \\ (i+2)pq^{i+1} & \text{for } i \geq 1 \text{ and } j = 1 \\ (i+3-j)p^2q^{i+2-j} & \text{for } i \geq 1 \text{ and } j = 2, \dots, i+2 \\ 0 & \text{for } i \geq 1 \text{ and } j > i+2, \end{cases}$$

where $q = 1 - p$. We define b_n, B, Q as in Theorem 4.1. With initial state 1, we have $\eta(p) = b_1$. As before (4.2)–(4.4) hold.

We set

$$Q_j(z) = \sum_{i=1}^{\infty} z^i p_{i,j} \quad (j \geq 1).$$

This is well–defined for $|z| < 1$, since $0 \leq p_{i,j} \leq 1$. We derive

$$Q_1(z) = \sum_{i=1}^{\infty} z^i (i+2)pq^{i+1} = pq^2 z \frac{3 - 2qz}{(1-qz)^2},$$

$$Q_2(z) = \sum_{i=1}^{\infty} z^i (i+1)p^2q^i = \frac{p^2}{(1-qz)^2} - p^2,$$

and for $j \geq 3$

$$\begin{aligned} Q_j(z) &= \sum_{i=j-2}^{\infty} z^i(i+3-j)p^2q^{i+2-j} \\ &= \sum_{k=0}^{\infty}(k+1)z^{k+j-2}p^2q^k \\ &= \frac{p^2z^{j-2}}{(1-qz)^2}. \end{aligned}$$

Hence for $|z| < 1$

$$\begin{aligned} (z, z^2, z^3, \dots)Q &= (Q_1(z), Q_2(z), Q_3(z), \dots) \\ &= \frac{p^2}{z(1-qz)^2}\,(1, z, z^2, \dots) \\ &\quad + \left(pq^2z\frac{3-2qz}{(1-qz)^2} - \frac{p^2}{z(1-qz)^2}, -p^2, 0, 0, \dots\right). \end{aligned}$$

By (4.3), the power series

$$B(z) := \sum_{n=1}^{\infty} b_n z^n$$

converges absolutely for $|z| < 1$. From (4.4) we derive

$$B(z) = \frac{p^2}{z^2(1-qz)^2}B(z) + \left[pq^2z\frac{3-2qz}{(1-qz)^2} - \frac{p^2}{z(1-qz)^2}\right]b_1 - p^2b_2$$

for $|z| < 1$, so that

$$\left[z^2(1-qz)^2 - p^2\right] B(z) = pzN(z) \quad \text{for} \quad |z| < 1, \tag{5.1}$$

where

$$N(z) = \left[q^2 z^2 (3-2qz) - p\right] b_1 - pz(1-qz)^2 b_2.$$

To show that $p_{cs}^{\eta} = 1/2$, we now establish that a necessary and sufficient condition for the b_n to be not all zero is that $q < 1/2$. When this holds, $b_n > 0$ for all $n \geq 1$ and the radius of convergence of $B(z)$ is unity.

A factorisation of the left–hand side of (5.1) yields

$$[z(1-qz)+p](1-z)(qz-p)B(z) = pzN(z) \quad (|z| < 1). \tag{5.2}$$

The zeros on the left–hand side of this expression occur at $z_1 = 1$, $z_2 = p/q$ and at the roots of $z(1-qz)+p = 0$.

The cases $p = 0$ and $p = 1$ are trivial: if $p = 0$, the process dies out at the first step with probability 1; if $p = 1$, the process grows strictly monotonically with probability 1.

Suppose first $0 < p < 1/2$, so that $1/2 < q < 1$ and $z_2 = p/q < 1$. The left–hand side of (5.2) vanishes for $z = z_2$, so that $N(z_2) = 0$. Substitution of $z = z_2$ into $N(z)$ gives

$$\begin{aligned} N(z_2) &= [p^2(3-2p) - p]b_1 - p^2 q b_2 \\ &= p[(1-p)(2p-1)b_1 - pqb_2] \\ &< 0 \end{aligned}$$

unless $b_1 = b_2 = 0$. In the latter event, $N(z) \equiv 0$, so that $B(z)$ vanishes for each z in the unit circle, entailing $b_n = 0$ for each $n \geq 1$.

If $p = q = 1/2$, then $z_2 = 1$ and $N(1) < 0$, so $B(z)$ has a pole of order two at $z = 1$ unless $b_n \equiv 0$. Suppose if possible that $b_n \to b > 0$ as $n \to \infty$. Then by Abel's theorem,

$$b = \lim_{z \to 1} (1 - z) B(z) = \infty \qquad \text{as } n \to \infty,$$

contradicting $b \leq 1$. Hence we must have $b_n \equiv 0$ for $q = 1/2$.

This establishes necessity. For sufficiency, suppose that $1/2 < p < 1$ so that $0 < q < 1/2$. In this case, $z_2 = p/q > 1$, so that $qz - p$ is non-vanishing inside the unit disk. The quadratic term $z(1 - qz) + p$ on the left–hand side of (5.2) has zeros

$$\begin{aligned} z_0 &= z_0(p) = \frac{1}{2q}\left[1 - \sqrt{1 + 4pq}\right] \in (-1, 0), \\ z_3 &= z_3(p) = \frac{1}{2q}\left[1 + \sqrt{1 + 4pq}\right] \in (1, \infty). \end{aligned} \tag{5.3}$$

We must have $N(z_0) = 0$ for a nontrivial solution to exist, so that

$$[q^2 z_0^2 (3 - 2qz_0) - p] b_1 = p z_0 (1 - qz_0)^2 b_2.$$

Since

$$1 - qz_0 = qz_3 \quad \text{and} \quad p = -qz_0 z_3, \tag{5.4}$$

this simplifies to

$$[qz_0(1 + 2qz_3) + z_3] b_1 = pqz_3^2 b_2$$

or

$$(5.5)\qquad (1 + pz_3 - 2pq)b_1 = pqz_3^2 b_2,$$

which shows that if $b_2 \neq 0$ then b_1/b_2 is positive.

A common factor $z = z_0$ may be removed from both sides of (5.2) and division by pz_3 yields

$$\left(1 - \frac{z}{z_3}\right)\left(1 - \frac{qz}{p}\right)(1 - z)B(z) = pzN_1(z),$$

where $N_1(z)$ is a quadratic in z. The coefficient of $B(z)$ is nonvanishing on the interior of the unit disk, so that $B(z)$ may be written

$$(5.6)\qquad B(z) = \frac{pzN_1(z)}{(1 - z/z_3)(1 - qz/p)(1 - z)} \quad \text{for} \quad |z| < 1.$$

It remains to show that if b_1 and b_2 are positive and satisfy (5.5), then the constants b_n defined through (5.6) are all positive.

The power series $B(z)$ has radius of convergence unity provided that $N_1(1) \neq 0$. To establish this inequality, it suffices to show that $N(1) \neq 0$. We have

$$N(1) = [q^2(3 - 2q) - 1 + q]b_1 - p^3 b_2.$$

For $q \in [0, 1/2]$, the expression in brackets is strictly increasing in q and achieves value zero for $q = 1/2$, providing the required result that $N_1(1) \neq 0$.

We consider

$$h(z) = \frac{N_1(z)}{(1-z)(1-qz/p)(1-z/z_3)}$$
$$= \frac{A_1}{1-z} + \frac{A_2}{1-qz/p} + \frac{A_3}{1-z/z_3}.$$

By applying the cover–up rule to

$$h(z) = \frac{N(z)}{-p^2(1-z)(1-z/z_0)(1-z/z_3)(1-qz/p)},$$

we derive that

$$A_1 = \frac{N(1)}{-p^2(1-1/z_0)(1-1/z_3)(1-q/p)},$$

$$A_3 = \frac{[q^2z_3^2(3-2qz_3)-p]b_1 - pz_3(1-qz_3)^2b_2}{-p^2(1-z_3/z_0)(1-z_3)(1-qz_3/p)}. \tag{5.7}$$

Note from (5.3) that

$$z_3 > \frac{1}{q} > \frac{p}{q} = z_2 > 1, \tag{5.8}$$

so that the notation z_2, z_3 adopted in this section is consistent with the usage of the lemma.

Since $N(1) < 0$ for $q \in [0, 1/2]$, we have that $A_1 > 0$. Also

$$A_1 + A_2 + A_3 = g(0) = \frac{b_1}{p} > 0.$$

We shall prove that $A_3 < 0$, from which it follows that $A_1 + A_2 > 0$ and thus that the conditions of the lemma are satisfied.

By (5.8) and $z_0 < 0$, the denominator of the fraction in (5.7) is negative, so that we need to establish that the numerator is positive. By exploiting (5.4), the numerator may be expressed as

$$qz_3 \left[\{ qz_3(1 + 2qz_0) + z_0 \} \, b_1 - pqz_0^2 b_2 \right].$$

By (5.4), the expression in brackets reduces further to

$$\{ pz_0 + 1 - 2pq \} \, b_1 - pqz_0^2 b_2.$$

We wish to show that this must be positive. By (5.5),

$$\{ pz_3 + 1 - 2pq \} \, b_1 - pqz_3^2 b_2 = 0,$$

so our task is equivalent to deriving that

$$p(z_0 - z_3)b_1 - pq(z_0^2 - z_3^2)b_2 > 0$$

or equivalently that

$$b_1 - q(z_0 + z_3)b_2 < 0,$$

which by (5.4) reduces further to

$$b_1 - b_2 < 0.$$

Substitution for b_1/b_2 from (5.5) converts this condition to

$$pqz_3^2 - pz_3 + 2pq - 1 < 0.$$

Since $qz_3^2 = z_3 + p$, the left–hand side may be cast as

$$p^2 + 2pq - 1 = -p^2 + 2p - 1 = -q^2,$$

so the condition is satisfied. Thus the conditions of the lemma apply so that a positive, bounded solution (h_n) exists in the case $0 < q < 1/2$. The relation

$$B(z) = pzh(z) \tag{5.9}$$

provides $b_n = ph_{n-1}$, so the maximal solution (b_n) to (4.2) subject to (4.3) is positive. This completes the proof. □

Remark 1. *By Abel's theorem, $b_n \to b$ as $n \to \infty$ where*

$$b = \lim_{z\to 1}(1-z)B(z) = A_1.$$

Taking $b = 1$ gives $A_1 = 1$ or

$$[q^2(3-2q) - 1 + q]b_1 - p^3 b_2 = -p^2\left(1 - \frac{1}{z_0}\right)\left(1 - \frac{1}{z_3}\right)\left(1 - \frac{q}{p}\right).$$

The values of b_1, b_2 may be found by solving this equation with (5.5), whence the values of b_n for all $n > 0$ follow from (5.9).

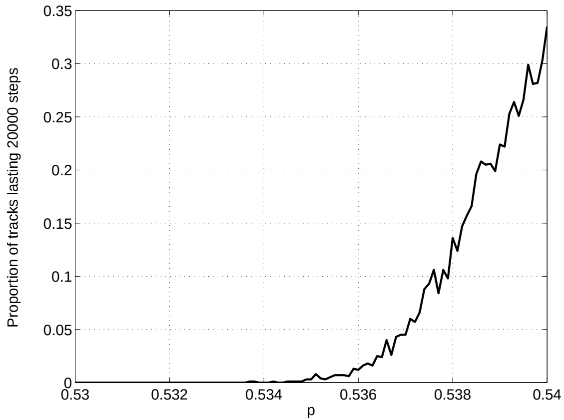

Figure 5: *Monte Carlo simulation results for the oriented site percolation process on* $\vec{\mathbb{L}}^2_{alt}$.

6. Simulations

A Monte Carlo simulation has been performed of the site percolation process on $\vec{\mathbb{L}}^2_{alt}$. Tracks were able to run for 20,000 time steps and those still alive at this time were deemed to have lasted infinitely long. After some initial testing over shorter periods of time, values of p were varied from 0.53 to 0.54 in steps of size 0.0001. One thousand Monte Carlo runs were performed for each of these probabilities. The results of this simulation are illustrated in Figure 5.

There are tracks lasting 20,000 steps for probabilities greater than approximately $p = 0.535$, suggesting that $p_{cs} \approx 0.535$.

References

[1] P. BALISTER, B. BOLLOBÁS AND A. STACEY, Upper bounds for the critical probability of oriented percolation in two dimensions, *Proc. Roy. Soc. London Ser. A*, **440** (1993), 201–220.

[2] P. BALISTER, B. BOLLOBÁS AND A. STACEY, Improved upper bounds for the critical probability of oriented percolation in two dimensions, *Random Structures Algorithms*, **5** (1994), 573–589.

[3] J. BISHIR, A lower bound for the critical probability in the one–quadrant oriented–atom percolation process, *J. Roy. Statist. Soc. Ser. B*, **25** (1963), 401–404.

[4] B. BOLLABÁS AND A. STACEY, Approximate upper bounds for the critical probability of oriented percolation in two dimensions based on rapidly mixing Markov chains, *J. Appl. Probab.*, **34** (1997), 859–867.

[5] S.R. BROADBENT, in discussion on Symposium on Monte Carlo Methods, *J. Roy. Statist. Soc. Ser. B*, **16** (1954), 68.

[6] S.R. BROADBENT AND J.M. HAMMERSLEY, Percolation processes. I. Crystals and mazes, *Proc. Camb. Phil. Soc.*, **53** (1957), 629–641.

[7] R. DURRETT, Oriented percolation in two dimensions, *Ann. Probab.*, **12** (1984), 999–1040.

[8] R. DURRETT, Stochastic growth models: bounds on critical values, *J. Appl. Probab.*, **29** (1992), 11–20.

[9] W. FELLER, *An Introduction to Probability Theory and its Applications*, 2nd Edition, John Wiley and Sons, New York (1957).

[10] H.L. FRISCH, E. SONNENBLICK, V.A. VYSSOTSKY AND J.M. HAMMERSLEY, Critical percolation probabilities (site problem), *Phys. Rev.*, **124** (1961), 1021–1022.

[11] H.L. FRISCH, J.M. HAMMERSLEY AND D.J.A. WELSH, Monte Carlo estimates of percolation probabilities for various lattices, *Phys. Rev.*, **126** (1962), 949–951.

[12] H.L. FRISCH AND J.M. HAMMERSLEY, Percolation processes and related topics, *SIAM J.*, **11** (1963), 894–918.

[13] G. GRIMMETT, Percolation and disordered systems, in Lectures on Probability Theory and Statistics (Saint-Flour, 1996), *Lecture Notes in Math.*, **1665** (1997), 153–300.

[14] G. GRIMMETT, *Percolation*, 2nd Edition, Springer–Verlag, Berlin (1999).

[15] G. GRIMMETT AND P. HIEMER, Directed percolation and random walk, *In and Out of Equilibrium*, Ed. V. Sidoravicius, Birkhauser, Berlin (2002), 273–297.

[16] J.M. HAMMERSLEY, Comparison of atom and bond percolation processes, *J. Math. Phys.*, **2** (1961), 728–733.

[17] B.D. HUGHES, *Random Walks and Random Environments, Volume 2: Random Environments*, Oxford University Press, Oxford (1996).

[18] H. KESTEN, The critical probability of bond percolation on the square lattice equals 1/2, *Comm. Math. Phys.*, **74** (1980), 41–59.

[19] H. KESTEN, *Percolation Theory for Mathematicians*, Progress in Probability and Statistics, vol. 2, Birkhäuser Boston, Mass. (1982).

[20] H. KESTEN, Percolation theory and first–passage percolation, *Ann. Probab.*, **15** (1987), 1231–1271.

[21] T.M. LIGGETT, Survival of discrete time growth models, with applications to oriented percolation, *Ann. Appl. Prob.*, **5** (1995), 613–636.

[22] T. ŁUCZAK AND J.C. WIERMAN, Critical probability bounds for two–dimensional site percolation models, *J. Physics A*, **21** (1988), 3131–3138.

[23] R.N. ONODY AND U.P.C. NEVES, Series expansion of the directed percolation probability, *J. Phys. A*, **25** (1992), 6609–6615.

[24] J.G. OXLEY AND D.J.A. WELSH, On some percolation results of J.M. Hammersley, *J. Appl. Probab.*, **16** (1979), 526–540.

[25] J.C. WIERMAN, Bond percolation on honeycomb and triangular lattices, *Adv. Appl. Probab.*, **13** (1981), 298–313.

[26] J.C. WIERMAN, A bond percolation critical probability determination based on the star–triangle transformation, *J. Physics A.*, **17** (1984), 1525–1530.

[27] J.C. WIERMAN, Duality for directed site percolation, in *Particle Systems, Random Media and Large Deviations*, Ed. R. Durrett, *Contemp. Math.*, **41** (1985), 363–380.

[28] J.C. WIERMAN, Bond percolation critical probability bounds derived by edge contraction, *J. Physics A*, **21** (1988), 1487–1492.

[29] J.C. WIERMAN, Substitution method critical probability bounds for the square lattice site percolation model, *Combinatorics, Probability and Computing*, **4** (1995), 181–188.

[30] J.C. WIERMAN, An improved upper bound for the hexagonal lattice site percolation critical probability, *Combinatorics, Probability and Computing*, **11** (2002), 629–643.