journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 4, Article 98, 4 pp.

ON A DISCRETE OPIAL-TYPE INEQUALITY

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Received 06 August, 2007; accepted 20 August, 2007
Communicated by R.P. Agarwal

ABSTRACT. The main purpose of the present paper is to establish a new discrete Opial-type inequality. Our result provide a new estimates on such type of inequality.

Key words and phrases: Opial's inequality, discrete Opial's inequality, Hölder inequality.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

In 1960, Z. Opial [14] established the following integral inequality:

Theorem A. *Suppose* $f \in C^1[0,h]$ *satisfies* $f(0) = f(h) = 0$ *and* $f(x) > 0$ *for all* $x \in (0,h)$. *Then the following integral inequality holds*

$$\int_0^h |f(x)f'(x)|\, dx \le \frac{h}{4}\int_0^h (f')^2\, dx, \tag{1.1}$$

where the constant $\frac{h}{4}$ *is best possible.*

Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [1, 2, 3, 10, 12]. In recent years, inequality (1.1) has received further attention and a large number of papers dealing with new proofs, extensions, generalizations and variants of Opial's inequality have appeared in

Research is partially supported by the Research Grants Council of the Hong Kong SAR, China (Project No.: HKU7016/07P)..

Research is supported Zhejiang Provincial Natural Science Foundation of China (Y605065), Foundation of the Education Department of Zhejiang Province of China (20050392), the Academic Mainstay of Middle-age and Youth Foundation of Shandong Province of China (200203).

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the literature [4] – [9], [13], [15], [16], [18] – [20]. For an extensive survey on these inequalities, see [1, 12].

For discrete analogues of Opial-type inequalities, good accounts of the recent works in this aspect are given in [1, 12], etc. In particular, an inequality involving two sequences was established by Pachpatte in [17] as follows:

Theorem B. *Let x_i and y_i $(i = 0, 1, \dots, \tau)$ be non-decreasing sequences of non-negative numbers, and $x_0 = y_0 = 0$. Then, the following inequality holds*

$$\sum_{i=0}^{\tau-1} [x_i \Delta y_i + y_{i+1} \Delta x_i] \le \frac{\tau}{2} \sum_{i=0}^{\tau-1} \left[(\Delta x_i)^2 + (\Delta y_i)^2\right]. \tag{1.2}$$

The main purpose of the present paper is to establish a new discrete Opial-type inequality involving two sequences as follows.

Theorem 1.1. *Let $\{x_{i,j}\}$ and $\{y_{i,j}\}$ be non-decreasing sequences of non-negative numbers defined for $i = 0, 1, \dots, \tau$, $j = 0, 1, \dots, \sigma$, where τ, σ are natural numbers, and $x_{0,j} = x_{i,0} = 0$, $y_{0,j} = y_{i,0} = 0$ $(i = 0, 1, \dots, \tau;\ j = 0, 1, \dots, \sigma)$. Let*

$$\Delta_1 x_{i,j} = x_{i+1,j} - x_{i,j}, \quad \Delta_2 x_{i,j} = x_{i,j+1} - x_{i,j},$$

then

$$\begin{aligned} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \Big[x_{i,j} \cdot \Delta_2 \Delta_1 y_{i,j} &+ \Delta_1 y_{i,j+1} \cdot \Delta_2 x_{i+1,j} \\ &+ y_{i,j} \cdot \Delta_2 \Delta_1 x_{i,j} + \Delta_1 x_{i,j+1} \cdot \Delta_2 y_{i+1,j+1} \Big] \\ &\le \frac{\sigma\tau}{2} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \Big[(\Delta_2 \Delta_1 x_{i,j})^2 + (\Delta_2 \Delta_1 y_{i,j})^2 \Big]. \end{aligned} \tag{1.3}$$

Our result in special cases yields some of the recent results on Opial's inequality and provides a new estimate on such types of inequalities.

2. Main Results

Theorem 2.1. *Let $\{x_{i,j}\}$ and $\{y_{i,j}\}$ be non-decreasing sequences of non-negative numbers defined for $i = 0, 1, \dots, \tau$, $j = 0, 1, \dots, \sigma$, where τ, σ are natural numbers, with $x_{0,j} = x_{i,0} = 0$, $y_{0,j} = y_{i,0} = 0$ $(i = 0, 1, \dots, \tau; j = 0, 1, \dots, \sigma)$. Let $\frac{1}{p} + \frac{1}{q} = 1, p > 1$, and*

$$\Delta_1 x_{i,j} = x_{i+1,j} - x_{i,j}, \quad \Delta_2 x_{i,j} = x_{i,j+1} - x_{i,j},$$

then

$$\begin{aligned} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \Big[x_{i,j} \cdot \Delta_2 \Delta_1 y_{i,j} &+ \Delta_1 y_{i,j+1} \cdot \Delta_2 x_{i+1,j} \\ &+ y_{i,j} \cdot \Delta_2 \Delta_1 x_{i,j} + \Delta_1 x_{i,j+1} \cdot \Delta_2 y_{i+1,j+1} \Big] \\ &\le \frac{1}{p} (\sigma\tau)^{p/q} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} (\Delta_2 \Delta_1 x_{i,j})^p + \frac{1}{q} (\sigma\tau)^{q/p} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} (\Delta_2 \Delta_1 y_{i,j})^q. \end{aligned} \tag{2.1}$$

Proof. We have

$$\begin{aligned}\Delta_2\Delta_1(x_{ij}y_{ij}) &= \Delta_2(x_{i,j}\Delta_1 y_{i,j} + y_{i+1,j}\Delta_1 x_{i,j}) \\ &= \Delta_2(x_{i,j}\Delta_1 y_{i,j}) + \Delta_2(y_{i+1,j}\Delta_1 x_{i,j}) \\ &= x_{i,j}\cdot\Delta_2\Delta_1 y_{i,j} + \Delta_1 y_{i,j+1}\Delta_2 x_{i,j} + y_{i+1,j}\cdot\Delta_2\Delta_1 x_{i,j} + \Delta_1 x_{i,j+1}\Delta_2 y_{i+1,j+1}.\end{aligned}$$

On the other hand, in view of $x_{0,j} = x_{i,0} = 0$, $y_{0,j} = y_{i,0} = 0$ $(i = 0, 1, \ldots, \tau;\ j = 0, 1, \ldots, \sigma)$, it follows that

$$\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Big[x_{i,j}\cdot\Delta_2\Delta_1 y_{i,j} + \Delta_1 y_{i,j+1}\cdot\Delta_2 x_{i+1,j} + y_{i,j}\cdot\Delta_2\Delta_1 x_{i,j} + \Delta_1 x_{i,j+1}\cdot\Delta_2 y_{i+1,j+1}\Big] \\ = x_{\tau,\sigma}\cdot y_{\tau,\sigma}.$$

Now, using the elementary inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1,$$

the facts that

$$x_{\tau,\sigma} = \sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Delta_2\Delta_1 x_{i,j},$$

$$y_{\tau,\sigma} = \sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Delta_2\Delta_1 y_{i,j},$$

and Hölder's inequality, we obtain

$$\begin{aligned}&\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Big[x_{i,j}\cdot\Delta_2\Delta_1 y_{i,j} + \Delta_1 y_{i,j+1}\cdot\Delta_2 x_{i+1,j} + y_{i,j}\cdot\Delta_2\Delta_1 x_{i,j} + \Delta_1 x_{i,j+1}\cdot\Delta_2 y_{i+1,j+1}\Big] \\ &\le \frac{x^p_{\tau,\sigma}}{p} + \frac{y^q_{\tau,\sigma}}{q} \\ &= \frac{1}{p}\left(\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Delta_2\Delta_1 x_{i,j}\right)^p + \frac{1}{q}\left(\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}\Delta_2\Delta_1 y_{i,j}\right)^q \\ &\le \frac{1}{p}(\sigma\tau)^{p/q}\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}(\Delta_2\Delta_1 x_{i,j})^p + \frac{1}{q}(\sigma\tau)^{q/p}\sum_{i=0}^{\tau-1}\sum_{j=0}^{\sigma-1}(\Delta_2\Delta_1 y_{i,j})^q.\end{aligned}$$

□

Remark 2.2. Taking $p = q = 2$, Theorem 2.1 reduces to Theorem 1.1.

Furthermore, by reducing $\{x_{i,j}\}$ and $\{y_{i,j}\}$ to $\{x_i\}$ and $\{y_i\}$ $(i = 0, 1, \ldots, \tau)$, respectively, and with suitable changes, we have

$$\sum_{i=0}^{\tau-1}\Big[x_i\Delta y_i + y_{i+1}\Delta x_i\Big] \le \frac{\tau}{2}\sum_{i=0}^{\tau-1}\Big[(\Delta x_i)^2 + (\Delta y_i)^2\Big].$$

This result was given by Pachpatte in [17].

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