

ON A DISCRETE OPIAL-TYPE INEQUALITY

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Abstract: The main purpose of the present paper is to establish a new discrete Opial-type inequality. Our result provide a new estimates on such type of inequality.

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1. Introduction

In 1960, Z. Opial [14] established the following integral inequality:

Theorem A. *Suppose $f \in C^1[0, h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0, h)$. Then the following integral inequality holds*

$$(1.1) \quad \int_0^h |f(x)f'(x)| dx \leq \frac{h}{4} \int_0^h (f')^2 dx,$$

where the constant $\frac{h}{4}$ is best possible.

Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [1, 2, 3, 10, 12]. In recent years, inequality (1.1) has received further attention and a large number of papers dealing with new proofs, extensions, generalizations and variants of Opial's inequality have appeared in the literature [4] – [9], [13], [15], [16], [18] – [20]. For an extensive survey on these inequalities, see [1, 12].

For discrete analogues of Opial-type inequalities, good accounts of the recent works in this aspect are given in [1, 12], etc. In particular, an inequality involving two sequences was established by Pachpatte in [17] as follows:

Theorem B. *Let x_i and y_i ($i = 0, 1, \dots, \tau$) be non-decreasing sequences of non-negative numbers, and $x_0 = y_0 = 0$. Then, the following inequality holds*

$$(1.2) \quad \sum_{i=0}^{\tau-1} [x_i \Delta y_i + y_{i+1} \Delta x_i] \leq \frac{\tau}{2} \sum_{i=0}^{\tau-1} [(\Delta x_i)^2 + (\Delta y_i)^2].$$

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The main purpose of the present paper is to establish a new discrete Opial-type inequality involving two sequences as follows.

Theorem 1.1. *Let $\{x_{i,j}\}$ and $\{y_{i,j}\}$ be non-decreasing sequences of non-negative numbers defined for $i = 0, 1, \dots, \tau$, $j = 0, 1, \dots, \sigma$, where τ, σ are natural numbers, and $x_{0,j} = x_{i,0} = 0$, $y_{0,j} = y_{i,0} = 0$ ($i = 0, 1, \dots, \tau$; $j = 0, 1, \dots, \sigma$). Let*

$$\Delta_1 x_{i,j} = x_{i+1,j} - x_{i,j}, \quad \Delta_2 x_{i,j} = x_{i,j+1} - x_{i,j},$$

then

$$(1.3) \quad \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \left[x_{i,j} \cdot \Delta_2 \Delta_1 y_{i,j} + \Delta_1 y_{i,j+1} \cdot \Delta_2 x_{i+1,j} \right. \\ \left. + y_{i,j} \cdot \Delta_2 \Delta_1 x_{i,j} + \Delta_1 x_{i,j+1} \cdot \Delta_2 y_{i+1,j+1} \right] \\ \leq \frac{\sigma\tau}{2} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \left[(\Delta_2 \Delta_1 x_{i,j})^2 + (\Delta_2 \Delta_1 y_{i,j})^2 \right].$$

Our result in special cases yields some of the recent results on Opial's inequality and provides a new estimate on such types of inequalities.



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2. Main Results

Theorem 2.1. Let $\{x_{i,j}\}$ and $\{y_{i,j}\}$ be non-decreasing sequences of non-negative numbers defined for $i = 0, 1, \dots, \tau$, $j = 0, 1, \dots, \sigma$, where τ, σ are natural numbers, with $x_{0,j} = x_{i,0} = 0$, $y_{0,j} = y_{i,0} = 0$ ($i = 0, 1, \dots, \tau$; $j = 0, 1, \dots, \sigma$). Let $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, and

$$\Delta_1 x_{i,j} = x_{i+1,j} - x_{i,j}, \quad \Delta_2 x_{i,j} = x_{i,j+1} - x_{i,j},$$

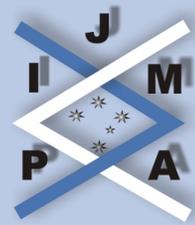
then

$$(2.1) \quad \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \left[x_{i,j} \cdot \Delta_2 \Delta_1 y_{i,j} + \Delta_1 y_{i,j+1} \cdot \Delta_2 x_{i+1,j} \right. \\ \left. + y_{i,j} \cdot \Delta_2 \Delta_1 x_{i,j} + \Delta_1 x_{i,j+1} \cdot \Delta_2 y_{i+1,j+1} \right] \\ \leq \frac{1}{p} (\sigma\tau)^{p/q} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} (\Delta_2 \Delta_1 x_{i,j})^p + \frac{1}{q} (\sigma\tau)^{q/p} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} (\Delta_2 \Delta_1 y_{i,j})^q.$$

Proof. We have

$$\begin{aligned} \Delta_2 \Delta_1 (x_{ij} y_{ij}) &= \Delta_2 (x_{i,j} \Delta_1 y_{i,j} + y_{i+1,j} \Delta_1 x_{i,j}) \\ &= \Delta_2 (x_{i,j} \Delta_1 y_{i,j}) + \Delta_2 (y_{i+1,j} \Delta_1 x_{i,j}) \\ &= x_{i,j} \cdot \Delta_2 \Delta_1 y_{i,j} + \Delta_1 y_{i,j+1} \Delta_2 x_{i,j} \\ &\quad + y_{i+1,j} \cdot \Delta_2 \Delta_1 x_{i,j} + \Delta_1 x_{i,j+1} \Delta_2 y_{i+1,j+1}. \end{aligned}$$

On the other hand, in view of $x_{0,j} = x_{i,0} = 0$, $y_{0,j} = y_{i,0} = 0$ ($i = 0, 1, \dots, \tau$; $j =$



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$0, 1, \dots, \sigma)$, it follows that

$$\sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \left[x_{i,j} \cdot \Delta_2 \Delta_1 y_{i,j} + \Delta_1 y_{i,j+1} \cdot \Delta_2 x_{i+1,j} + y_{i,j} \cdot \Delta_2 \Delta_1 x_{i,j} + \Delta_1 x_{i,j+1} \cdot \Delta_2 y_{i+1,j+1} \right] = x_{\tau,\sigma} \cdot y_{\tau,\sigma}.$$

Now, using the elementary inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad p > 1,$$

the facts that

$$x_{\tau,\sigma} = \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \Delta_2 \Delta_1 x_{i,j},$$

$$y_{\tau,\sigma} = \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \Delta_2 \Delta_1 y_{i,j},$$

and Hölder's inequality, we obtain

$$\begin{aligned} & \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \left[x_{i,j} \cdot \Delta_2 \Delta_1 y_{i,j} + \Delta_1 y_{i,j+1} \cdot \Delta_2 x_{i+1,j} + y_{i,j} \cdot \Delta_2 \Delta_1 x_{i,j} + \Delta_1 x_{i,j+1} \cdot \Delta_2 y_{i+1,j+1} \right] \\ & \leq \frac{x_{\tau,\sigma}^p}{p} + \frac{y_{\tau,\sigma}^q}{q} \\ & = \frac{1}{p} \left(\sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \Delta_2 \Delta_1 x_{i,j} \right)^p + \frac{1}{q} \left(\sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} \Delta_2 \Delta_1 y_{i,j} \right)^q \end{aligned}$$

$$\leq \frac{1}{p}(\sigma\tau)^{p/q} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} (\Delta_2 \Delta_1 x_{i,j})^p + \frac{1}{q}(\sigma\tau)^{q/p} \sum_{i=0}^{\tau-1} \sum_{j=0}^{\sigma-1} (\Delta_2 \Delta_1 y_{i,j})^q.$$

□

Remark 1. Taking $p = q = 2$, Theorem 2.1 reduces to Theorem 1.1.

Furthermore, by reducing $\{x_{i,j}\}$ and $\{y_{i,j}\}$ to $\{x_i\}$ and $\{y_i\}$ ($i = 0, 1, \dots, \tau$), respectively, and with suitable changes, we have

$$\sum_{i=0}^{\tau-1} [x_i \Delta y_i + y_{i+1} \Delta x_i] \leq \frac{\tau}{2} \sum_{i=0}^{\tau-1} [(\Delta x_i)^2 + (\Delta y_i)^2].$$

This result was given by Pachpatte in [17].



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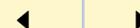


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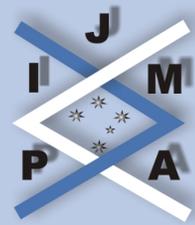
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