# OPERATOR NORM INEQUALITIES OF MINKOWSKI TYPE

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*Abstract:* Operator norm inequalities of Minkowski type are presented for unitarily invariant norm. Some of these inequalities generalize an earlier work of Hiai and Zhan.

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# Contents





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## 1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a separable complex Hilbert space $H$. A unitarily invariant norm $|||\cdot|||$ is a norm on the space of operators satisfying $|||A||| = |||UAV|||$ for all $A$ and all unitary operators $U$ and $V$ in $B(H)$. Except for the operator norm, which is defined on all of $B(H)$, each unitarily invariant norm $|||\cdot|||$ is defined on a norm ideal $C_{|||\cdot|||}$ contained in the ideal of compact operators. When we talk of $|||A|||$ we are implicitly assuming that $A$ belongs to $C_{|||\cdot|||}$.

The absolute value of an operator $A \in B(H)$, denoted by $|A|$, is defined by $|A| = (A^*A)^{1/2}$. Let $s_1(A), s_2(A), \ldots$ be the singular values of the compact operator $A$, i.e., the eigenvalues of $|A|$, rearranged such that $s_1(A) \geq s_2(A) \geq \cdots$.

For $p > 0$ and for every unitarily invariant norm $|||\cdot|||$ on $B(H)$, define

$$|||A|||^{(p)} = |||\ |A|^p|||^{1/p}.$$

It is known that

$$|||\ |A+B|^p|||^{1/p} \leq |||\ |A|^p|||^{1/p} + |||\ |B|^p|||^{1/p} \tag{1.1}$$

for $p \geq 1$ and

$$|||\ |A+B|^p|||^{1/p} \leq 2^{1/p-1}\left(|||\ |A|^p|||^{1/p} + |||\ |B|^p|||^{1/p}\right) \tag{1.2}$$

for $0 < p < 1$ (see e.g., [1, p.p. 95,108]). Based on the definition of $|||\cdot|||^{(p)}$ and inequality (1.1), it can be easily seen that $|||\cdot|||^{(p)}$ is a unitarily invariant norm for $p \geq 1$.

For $0 < p < \infty$, let

$$\|A\|_p = \left(\sum_{i=1}^{\infty} s_i^p(A)\right)^{\frac{1}{p}}.$$

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If $p \geq 1$, then $\|\cdot\|_p$ is a norm, called the Schatten $p$-norm. So,

$$\|A\|_p = \left(\operatorname{tr} |A|^p\right)^{1/p},$$

where $\operatorname{tr}$ is the usual trace functional. When $p = 1$, $\|A\|_1$ is called the trace norm of $A$. Note that for all positive real numbers $r$ and $p$, we have

$$\| \, |A|^r \|_p = \|A\|_{rp}^r \,. \tag{1.3}$$

For the theory of unitarily invariant norms, the reader is referred to [1], [3], [8], [9], [10], and the references therein.

The Minkowski's inequality for scalars asserts that if $a_i$, $b_i$ $(i = 1, 2, \dots, n)$ are complex numbers and $p \geq 1$, then

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}}.$$

Hiai and Zhan [4], proved that if $A_1$, $A_2$, $B_1$, $B_2$ are matrices of order $n$ and $1 \leq p, r < \infty$, then

$$\begin{aligned} &||| \; |A_1 + A_2|^p + |B_1 + B_2|^p |||^{1/p} \\ &\qquad \leq 2^{|1/p - 1/2|} \left( ||| \; |A_1|^p + |B_1|^p |||^{1/p} + ||| \; |A_2|^p + |B_2|^p |||^{1/p} \right), \end{aligned} \tag{1.4}$$

$$\begin{aligned} &\| |A_1 + A_2|^p + |B_1 + B_2|^p \|_r^{1/p} \\ &\qquad \leq 2^{(1-1/r)/p} \left( \| |A_1|^p + |B_1|^p \|_r^{1/p} + \| |A_2|^p + |B_2|^p \|_r^{1/p} \right), \end{aligned} \tag{1.5}$$


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and

$$\left\| \left( |A_1 + A_2|^p + |B_1 + B_2|^p \right)^{1/p} \right\|_r \leq 2^{|1/p - 1/r|} \left( \left\| \left( |A_1|^p + |B_1|^p \right)^{1/p} \right\|_r + \left\| \left( |A_2|^p + |B_2|^p \right)^{1/p} \right\|_r \right). \tag{1.6}$$

These inequalities are norm inequalities of Minkowski type.

The purpose of this paper is to establish new operator norm inequalities. Our inequalities generalize the inequalities (1.4), (1.5), and (1.6) for $n-$tuple of operators. Our analysis is based on some recent results on convexity and concavity of functions and on some operator inequalities.


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## 2. Norm Inequalities of Minkowski Type

In this section, we generalize inequality (1.4) for operators $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$, and other norm inequalities of Minkowski type. To achieve our goal we need the following two lemmas. The first lemma can be found in [2] and a stronger version of the second lemma can be found in [5].

**Lemma 2.1.** *Let* $A_1, \ldots, A_n \in B(H)$ *be positive operators. Then, for every unitarily invariant norm,*

$$\left|\left|\left|\sum_{i=1}^{n} A_i^r\right|\right|\right| \leq \left|\left|\left|\left(\sum_{i=1}^{n} A_i\right)^r\right|\right|\right| \tag{2.1}$$

*for* $r \geq 1$ *and*

$$\left|\left|\left|\left(\sum_{i=1}^{n} A_i\right)^r\right|\right|\right| \leq \left|\left|\left|\sum_{i=1}^{n} A_i^r\right|\right|\right| \tag{2.2}$$

*for* $0 < r \leq 1$.

**Lemma 2.2.** *Let* $A_1, \ldots, A_n \in B(H)$ *be positive operators. Then, for every unitarily invariant norm,*

$$\left|\left|\left|\left(\sum_{i=1}^{n} A_i\right)^r\right|\right|\right| \leq n^{r-1} \left|\left|\left|\sum_{i=1}^{n} A_i^r\right|\right|\right| \tag{2.3}$$

*for* $r \geq 1$ *and*

$$\left|\left|\left|\sum_{i=1}^{n} A_i^r\right|\right|\right| \leq n^{1-r} \left|\left|\left|\left(\sum_{i=1}^{n} A_i\right)^r\right|\right|\right| \tag{2.4}$$

*for* $0 < r \leq 1$.



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Now, we are in a position to generalize (1.4).

**Theorem 2.3.** *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \dots, n)$ *and* $p \geq 1$. *Then, for every unitarily invariant norm,*

$$n^{-|1/p-1/2|} \left|\left|\left| \sum_{i=1}^{n} |A_i + B_i|^p \right|\right|\right|^{\frac{1}{p}} \leq \left|\left|\left| \sum_{i=1}^{n} |A_i|^p \right|\right|\right|^{\frac{1}{p}} + \left|\left|\left| \sum_{i=1}^{n} |B_i|^p \right|\right|\right|^{\frac{1}{p}} \tag{2.5}$$

*and*

$$\begin{aligned} \left|\left|\left| \sum_{i=1}^{n} |A_i|^p \right|\right|\right|^{\frac{1}{p}} &+ \left|\left|\left| \sum_{i=1}^{n} |B_i|^p \right|\right|\right|^{\frac{1}{p}} \\ &\leq n^{|1/p-1/2|} \left( \left|\left|\left| \sum_{i=1}^{n} |A_i + B_i|^p \right|\right|\right|^{\frac{1}{p}} + \left|\left|\left| \sum_{i=1}^{n} |A_i - B_i|^p \right|\right|\right|^{\frac{1}{p}} \right). \end{aligned} \tag{2.6}$$

*Proof.* Let

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_n & 0 & \cdots & 0 \end{bmatrix}$$

be operators in $B\left(\bigoplus_{i=1}^{n} H\right)$. Then

$$|A|^2 = \begin{bmatrix} \sum_{i=1}^{n} |A_i|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad |B|^2 = \begin{bmatrix} \sum_{i=1}^{n} |B_i|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$



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and

$$|A+B|^2 = \begin{bmatrix} \sum_{i=1}^n |A_i+B_i|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By applying (1.1) to the operators $A$ and $B$, we get

$$\left|\left|\left| \left( \sum_{i=1}^n |A_i+B_i|^2 \right)^{\frac{p}{2}} \right|\right|\right|^{\frac{1}{p}} \leq \left|\left|\left| \left( \sum_{i=1}^n |A_i|^2 \right)^{\frac{p}{2}} \right|\right|\right|^{\frac{1}{p}} + \left|\left|\left| \left( \sum_{i=1}^n |B_i|^2 \right)^{\frac{p}{2}} \right|\right|\right|^{\frac{1}{p}}. \tag{2.7}$$

For $1 \leq p \leq 2$, it follows, from (2.2) and (2.4), that

$$\left|\left|\left| \left( \sum_{i=1}^n |A_i|^2 \right)^{\frac{p}{2}} \right|\right|\right| \leq \left|\left|\left| \sum_{i=1}^n |A_i|^p \right|\right|\right|, \tag{2.8}$$

$$\left|\left|\left| \left( \sum_{i=1}^n |B_i|^2 \right)^{\frac{p}{2}} \right|\right|\right| \leq \left|\left|\left| \sum_{i=1}^n |B_i|^p \right|\right|\right|, \tag{2.9}$$

and

$$\left|\left|\left| \sum_{i=1}^n |A_i+B_i|^p \right|\right|\right| \leq n^{1-p/2} \left|\left|\left| \left( \sum_{i=1}^n |A_i+B_i|^2 \right)^{\frac{p}{2}} \right|\right|\right|. \tag{2.10}$$

Now, inequality (2.5) follows by combining (2.8), (2.9), and (2.10) by (2.7).



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For $p > 2$, it follows, from (2.1) and (2.3), that

$$\left|\left|\left| \sum_{i=1}^{n} |A_i + B_i|^p \right|\right|\right| \le \left|\left|\left| \left( \sum_{i=1}^{n} |A_i + B_i|^2 \right)^{\frac{p}{2}} \right|\right|\right|, \tag{2.11}$$

$$\left|\left|\left| \left( \sum_{i=1}^{n} |A_i|^2 \right)^{\frac{p}{2}} \right|\right|\right| \le n^{p/2-1} \left|\left|\left| \sum_{i=1}^{n} |A_i|^p \right|\right|\right|, \tag{2.12}$$

and

$$\left|\left|\left| \left( \sum_{i=1}^{n} |B_i|^2 \right)^{\frac{p}{2}} \right|\right|\right| \le n^{p/2-1} \left|\left|\left| \sum_{i=1}^{n} |B_i|^p \right|\right|\right|. \tag{2.13}$$

Consequently, inequality (2.5) follows, by combining (2.11), (2.12), and (2.13) by (2.7). This completes the proof of inequality (2.5).

For inequality (2.6), replacing $A_i$ and $B_i$ in (2.5) by $A_i + B_i$ and $A_i - B_i$, respectively, we have

$$\begin{aligned} 2 \left|\left|\left| \sum_{i=1}^{n} |A_i|^p \right|\right|\right|^{\frac{1}{p}} & \\ \le n^{|1/p-1/2|} & \left( \left|\left|\left| \sum_{i=1}^{n} |A_i + B_i|^p \right|\right|\right|^{\frac{1}{p}} + \left|\left|\left| \sum_{i=1}^{n} |A_i - B_i|^p \right|\right|\right|^{\frac{1}{p}} \right). \end{aligned} \tag{2.14}$$


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Again, replacing $A_i$ and $B_i$ in (2.5) by $A_i + B_i$ and $B_i - A_i$, respectively, we have

$$(2.15)\qquad 2\left|\left|\left|\sum_{i=1}^{n}|B_i|^p\right|\right|\right|^{\frac{1}{p}} \leq n^{|1/p-1/2|}\left(\left|\left|\left|\sum_{i=1}^{n}|A_i+B_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n}|A_i-B_i|^p\right|\right|\right|^{\frac{1}{p}}\right).$$

Now, inequality (2.6) follows, by adding inequalities (2.14) and (2.15). This completes the proof of the theorem. □

Based on inequality (1.2) and using a procedure similar to that given in the proof of Theorem 2.3, we have the following result.

**Theorem 2.4.** *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ *and* $0 < p \leq 1$. *Then, for every unitarily invariant norm,*

$$(2.16)\qquad 2^{1-1/p}n^{-|1/p-1/2|}\left|\left|\left|\sum_{i=1}^{n}|A_i+B_i|^p\right|\right|\right|^{\frac{1}{p}} \leq \left(\left|\left|\left|\sum_{i=1}^{n}|A_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n}|B_i|^p\right|\right|\right|^{\frac{1}{p}}\right)$$

*and*

$$(2.17)\qquad \left|\left|\left|\sum_{i=1}^{n}|A_i|^p\right|\right|\right|^{\frac{1}{p}} + \left|\left|\left|\sum_{i=1}^{n}|B_i|^p\right|\right|\right|^{\frac{1}{p}}$$

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$$\leq 2^{1/p-1} n^{|1/p-1/2|} \left( \left|\left|\left| \sum_{i=1}^{n} |A_i + B_i|^p \right|\right|\right|^{\frac{1}{p}} + \left|\left|\left| \sum_{i=1}^{n} |A_i - B_i|^p \right|\right|\right|^{\frac{1}{p}} \right).$$

For $p \geq 1$ inequalities (2.5) and (2.6) can be written in equivalent forms as follow:

$$n^{-|1/p-1/2|} \left|\left|\left| \left( \sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right|\right|\right|^{(p)} \leq \left|\left|\left| \left( \sum_{i=1}^{n} |A_i|^p \right)^{\frac{1}{p}} \right|\right|\right|^{(p)} + \left|\left|\left| \left( \sum_{i=1}^{n} |B_i|^p \right)^{\frac{1}{p}} \right|\right|\right|^{(p)} \tag{2.18}$$

and

$$\left|\left|\left| \left( \sum_{i=1}^{n} |A_i|^p \right)^{\frac{1}{p}} \right|\right|\right|^{(p)} + \left|\left|\left| \left( \sum_{i=1}^{n} |B_i|^p \right)^{\frac{1}{p}} \right|\right|\right|^{(p)} \leq n^{|1/p-1/2|} \left( \left|\left|\left| \left( \sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right|\right|\right|^{(p)} + \left|\left|\left| \left( \sum_{i=1}^{n} |A_i - B_i|^p \right)^{\frac{1}{p}} \right|\right|\right|^{(p)} \right). \tag{2.19}$$

In the following theorem we give inequalities related to inequalities (2.18) and (2.19). In order to do that we need the following lemma, which is a particular case of Theorem 2 in [7].


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**Lemma 2.5.** *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ *and* $p \geq 2$. *Then*

$$\left|\left|\left| \left( \sum_{i=1}^{n} |A_i|^2 \right)^{\frac{1}{2}} \right|\right|\right| \leq n^{1/2-1/p} \left( \left|\left|\left| \left( \sum_{i=1}^{n} |A_i|^p \right)^{\frac{1}{p}} \right|\right|\right| \right) \tag{2.20}$$

*for every unitarily invariant norm.*

**Theorem 2.6.** *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ *and* $p \geq 2$. *Then, for every unitarily invariant norm,*

$$\begin{aligned} n^{-(1-1/p)} \left|\left|\left| \left( \sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| \\ \leq \left|\left|\left| \left( \sum_{i=1}^{n} |A_i|^p \right)^{\frac{1}{p}} \right|\right|\right| + \left|\left|\left| \left( \sum_{i=1}^{n} |B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| \end{aligned} \tag{2.21}$$

*and*

$$\begin{aligned} \left|\left|\left| \left( \sum_{i=1}^{n} |A_i|^p \right)^{\frac{1}{p}} \right|\right|\right| + \left|\left|\left| \left( \sum_{i=1}^{n} |B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| \\ \leq n^{1-1/p} \left( \left|\left|\left| \left( \sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| + \left|\left|\left| \left( \sum_{i=1}^{n} |A_i - B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| \right). \end{aligned} \tag{2.22}$$

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*Proof.* By using (2.2), (2.4), (2.7), and (2.20), respectively, we have

$$\begin{aligned}\left|\left|\left|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right|\right|\right| &\le \left|\left|\left|\sum_{i=1}^{n}|A_i+B_i|\right|\right|\right| \\ &\le n^{1/2}\left|\left|\left|\left(\sum_{i=1}^{n}|A_i+B_i|^2\right)^{\frac{1}{2}}\right|\right|\right| \\ &\le n^{1/2}\left(\left|\left|\left|\left(\sum_{i=1}^{n}|A_i|^2\right)^{\frac{1}{2}}\right|\right|\right| + \left|\left|\left|\left(\sum_{i=1}^{n}|B_i|^2\right)^{\frac{1}{2}}\right|\right|\right|\right) \\ &\le n^{1-1/p}\left(\left|\left|\left|\left(\sum_{i=1}^{n}|A_i|^p\right)^{\frac{1}{p}}\right|\right|\right| + \left|\left|\left|\left(\sum_{i=1}^{n}|B_i|^p\right)^{\frac{1}{p}}\right|\right|\right|\right).\end{aligned}$$

This proves inequality (2.21). Inequality (2.22) follows from inequality (2.21) by a proof similar to that given for inequality (2.6) in Theorem 2.3. The proof is complete. □

It is known that for a positive operator $A$ and for $0 < r \le 1$, we have

$$|||A|||^r \le |||A^r||| \tag{2.23}$$

for every unitarily invariant norm; and the reverse inequality holds for $r \ge 1$.

Using inequality (2.23) we have the following application of Theorem 2.6.

**Corollary 2.7.** *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \dots, n)$ *and* $p \ge 2$. *Then, for every*


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*unitarily invariant norm,*

$$n^{-(1-1/p)} \left|\left|\left| \sum_{i=1}^{n} |A_i + B_i|^p \right|\right|\right|^{\frac{1}{p}} \le \left|\left|\left| \left( \sum_{i=1}^{n} |A_i|^p \right)^{\frac{1}{p}} \right|\right|\right| + \left|\left|\left| \left( \sum_{i=1}^{n} |B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| \tag{2.24}$$

*and*

$$\begin{aligned} \left|\left|\left| \sum_{i=1}^{n} |A_i|^p \right|\right|\right|^{\frac{1}{p}} + \left|\left|\left| \sum_{i=1}^{n} |B_i|^p \right|\right|\right|^{\frac{1}{p}} \\ \le n^{1-1/p} \left( \left|\left|\left| \left( \sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| + \left|\left|\left| \left( \sum_{i=1}^{n} |A_i - B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| \right). \end{aligned} \tag{2.25}$$

*Remark* 1. In view of (2.5), (2.21), and (2.23), one might conjecture that if $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$, then, for every unitarily invariant norm,

$$\begin{aligned} \left|\left|\left| \left( \sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| \\ \le n^{|1/p-1/2|} \left( \left|\left|\left| \sum_{i=1}^{n} |A_i|^p \right|\right|\right|^{\frac{1}{p}} + \left|\left|\left| \sum_{i=1}^{n} |B_i|^p \right|\right|\right|^{\frac{1}{p}} \right) \end{aligned} \tag{2.26}$$

for $p \ge 1$ and

$$\left|\left|\left| \left( \sum_{i=1}^{n} |A_i + B_i|^p \right)^{\frac{1}{p}} \right|\right|\right| \le n^{1-1/p} \left( \left|\left|\left| \sum_{i=1}^{n} |A_i|^p \right|\right|\right|^{\frac{1}{p}} + \left|\left|\left| \sum_{i=1}^{n} |B_i|^p \right|\right|\right|^{\frac{1}{p}} \right) \tag{2.27}$$



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for $p \geq 2$.

*Remark* 2. Using the same procedure used in the proof of inequality (2.6) in Theorem 2.3, inequalities (1.1) and (1.2) imply that

$$|||\ |A|^p|||^{1/p} + |||\ |B|^p|||^{1/p} \leq |||\ |A+B|^p|||^{1/p} + |||\ |A-B|^p|||^{1/p} \tag{2.28}$$

for $p \geq 1$ and

$$|||\ |A|^p|||^{1/p} + |||\ |B|^p|||^{1/p} \leq 2^{1/p-1}(|||\ |A+B|^p|||^{1/p} + |||\ |A-B|^p|||^{1/p}) \tag{2.29}$$

for $0 < p \leq 1$. For $p \geq 1$, it follows, from the triangle inequality for norms and a scalar inequality, that

$$|||\ |A+B|^p + |A-B|^p|||^{1/p} \leq |||\ |A+B|^p|||^{1/p} + |||\ |A-B|^p|||^{1/p}. \tag{2.30}$$

For $p \geq 2$, the left hand side of (2.30) is the right hand side of the famous Clarkson inequality

$$2\,|||\ |A|^p + |B|^p||| \leq |||\ |A+B|^p + |A+B|^p|||, \tag{2.31}$$

see e.g., [6]. In view of the inequalities (2.29) and (2.30) we may introduce the following question: For $p \geq 2$ are the following inequalities:

$$|||\ |A|^p|||^{1/p} + |||\ |B|^p|||^{1/p} \leq |||\ |A+B|^p + |A-B|^p|||^{1/p} \tag{2.32}$$

and

$$2\,|||\ |A|^p + |B|^p||| \leq \left(|||\ |A|^p|||^{1/p} + |||\ |B|^p|||^{1/p}\right)^p \tag{2.33}$$

true?

Inequalities (2.32) and (2.33), if true, form a refinement of the Clarkson inequality (2.31).


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## 3. Norm Inequalities of Minkowski Type for the Schatten $P-$Norm

In this section, we present some norm inequalities of Minkowski type for the Schatten $p-$norm. These inequalities generalize the inequalities (1.5) and (1.6) for an $n-$tuple of operators.

**Theorem 3.1.** *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \dots, n)$ *and* $1 \le p, r < \infty$. *Then*

$$n^{-(1-1/r)/p} \left\| \sum_{i=1}^{n} |A_i + B_i|^p \right\|_r^{\frac{1}{p}} \le \left\| \sum_{i=1}^{n} |A_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^{n} |B_i|^p \right\|_r^{\frac{1}{p}} \tag{3.1}$$

*and*

$$\begin{aligned} &\left\| \sum_{i=1}^{n} |A_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^{n} |B_i|^p \right\|_r^{\frac{1}{p}} \\ &\qquad \le n^{(1-1/r)/p} \left( \left\| \sum_{i=1}^{n} |A_i + B_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^{n} |A_i - B_i|^p \right\|_r^{\frac{1}{p}} \right). \end{aligned} \tag{3.2}$$

*Proof.* It follows, from (1.3) and the triangle inequality, that

$$\left\| \sum_{i=1}^{n} |A_i + B_i|^{pr} \right\|_1^{\frac{1}{pr}} = \left\| \begin{bmatrix} A_1 + B_1 & 0 & \cdots & 0 \\ 0 & A_2 + B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n + B_n \end{bmatrix} \right\|_{pr}$$


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$$= \left\| \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} + \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{bmatrix} \right\|_{pr}$$

$$\leq \left\| \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} \right\|_{pr} + \left\| \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{bmatrix} \right\|_{pr}$$

$$(3.3)\qquad = \left\| \sum_{i=1}^{n} |A_i|^{pr} \right\|_1^{\frac{1}{pr}} + \left\| \sum_{i=1}^{n} |B_i|^{pr} \right\|_1^{\frac{1}{pr}} .$$

Now, by using (1.3), (2.3), (3.3), and (2.2), respectively, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n} |A_i + B_i|^p \right\|_r^{\frac{1}{p}} &= \left\| \left( \sum_{i=1}^{n} |A_i + B_i|^p \right)^r \right\|_1^{\frac{1}{pr}} \\ &\leq n^{(r-1)/pr} \left\| \sum_{i=1}^{n} |A_i + B_i|^{pr} \right\|_1^{\frac{1}{pr}} \\ &\leq n^{(1-1/r)/p} \left( \left\| \sum_{i=1}^{n} |A_i|^{pr} \right\|_1^{\frac{1}{pr}} + \left\| \sum_{i=1}^{n} |B_i|^{pr} \right\|_1^{\frac{1}{pr}} \right) \\ &= n^{(1-1/r)/p} \left( \left\| \left( \sum_{i=1}^{n} |A_i|^{pr} \right)^{\frac{1}{r}} \right\|_r^{\frac{1}{p}} + \left\| \left( \sum_{i=1}^{n} |B_i|^{pr} \right)^{\frac{1}{r}} \right\|_r^{\frac{1}{p}} \right) \end{aligned}$$


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$$\leq n^{(1-1/r)/p}\left(\left\|\sum_{i=1}^{n}|A_i|^p\right\|_r^{\frac{1}{p}}+\left\|\sum_{i=1}^{n}|B_i|^p\right\|_r^{\frac{1}{p}}\right).$$

This proves inequality (3.1). The proof of inequality (3.2) follows from (3.1) by a proof similar to that given for inequality (2.6) in Theorem 2.3. The proof is complete. □

The following is our final result.

**Theorem 3.2.** *Let* $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ *and* $1 \leq p, r < \infty$. *Then*

$$n^{-|1/p-1/r|}\left\|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right\|_r \leq \left\|\left(\sum_{i=1}^{n}|A_i|^p\right)^{\frac{1}{p}}\right\|_r+\left\|\left(\sum_{i=1}^{n}|B_i|^p\right)^{\frac{1}{p}}\right\|_r \tag{3.4}$$

*and*

$$\left\|\left(\sum_{i=1}^{n}|A_i|^p\right)^{\frac{1}{p}}\right\|_r+\left\|\left(\sum_{i=1}^{n}|B_i|^p\right)^{\frac{1}{p}}\right\|_r \leq n^{|1/p-1/r|}\left(\left\|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right\|_r+\left\|\left(\sum_{i=1}^{n}|A_i-B_i|^p\right)^{\frac{1}{p}}\right\|_r\right). \tag{3.5}$$

*Proof.* First suppose that $r \leq p$. By using (1.3), (2.2), (3.3), and (2.4), respectively,



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we have

$$
\begin{aligned}
\left\|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right\|_r &= \left\|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{r}{p}}\right\|_1^{\frac{1}{r}} \\
&\leq \left\|\sum_{i=1}^{n}|A_i+B_i|^r\right\|_1^{\frac{1}{r}} \\
&\leq \left\|\sum_{i=1}^{n}|A_i|^r\right\|_1^{\frac{1}{r}} + \left\|\sum_{i=1}^{n}|B_i|^r\right\|_1^{\frac{1}{r}} \\
&\leq n^{1/r-1/p}\left(\left\|\left(\sum_{i=1}^{n}|A_i|^p\right)^{\frac{r}{p}}\right\|_1^{\frac{1}{r}} + \left\|\left(\sum_{i=1}^{n}|B_i|^p\right)^{\frac{r}{p}}\right\|_1^{\frac{1}{r}}\right) \\
&= n^{1/r-1/p}\left(\left\|\left(\sum_{i=1}^{n}|A_i|^p\right)^{\frac{1}{p}}\right\|_r + \left\|\left(\sum_{i=1}^{n}|B_i|^p\right)^{\frac{1}{p}}\right\|_r\right).
\end{aligned}
$$

Next, for $p < r$, by using (1.3) and (3.1), we have

$$
\begin{aligned}
\left\|\left(\sum_{i=1}^{n}|A_i+B_i|^p\right)^{\frac{1}{p}}\right\|_r &= \left\|\sum_{i=1}^{n}|A_i+B_i|^p\right\|_{\frac{r}{p}}^{\frac{1}{p}} \\
&\leq n^{1/p(1-p/r)}\left(\left\|\sum_{i=1}^{n}|A_i|^p\right\|_{\frac{r}{p}}^{\frac{1}{p}} + \left\|\sum_{i=1}^{n}|B_i|^p\right\|_{\frac{r}{p}}^{\frac{1}{p}}\right)
\end{aligned}
$$

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$$= n^{1/p-1/r}\left(\left\|\left(\sum_{i=1}^{n}|A_i|^p\right)^{\frac{1}{p}}\right\|_r + \left\|\left(\sum_{i=1}^{n}|B_i|^p\right)^{\frac{1}{p}}\right\|_r\right).$$

This proves inequality (3.4). The proof of inequality (3.5) follows from (3.4) by a proof similar to that given for inequality (2.6) in Theorem 2.3. The proof is complete. □

*Remark* 3. For the Schatten $p-$norm, (3.4) is better than (2.21), and if $rp \leq 2$ or $r(4-p) \leq 2$, then (3.1) is better than (2.5).

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