



# OPERATOR NORM INEQUALITIES OF MINKOWSKI TYPE

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*Abstract:* Operator norm inequalities of Minkowski type are presented for unitarily invariant norm. Some of these inequalities generalize an earlier work of Hiai and Zhan.

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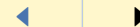
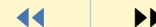
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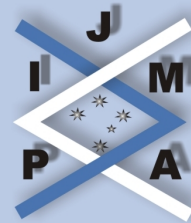
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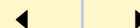
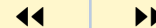
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# 1. Introduction

Let  $B(H)$  be the space of all bounded linear operators on a separable complex Hilbert space  $H$ . A unitarily invariant norm  $|||\cdot|||$  is a norm on the space of operators satisfying  $|||A||| = |||UAV|||$  for all  $A$  and all unitary operators  $U$  and  $V$  in  $B(H)$ . Except for the operator norm, which is defined on all of  $B(H)$ , each unitarily invariant norm  $|||\cdot|||$  is defined on a norm ideal  $C_{|||\cdot|||}$  contained in the ideal of compact operators. When we talk of  $|||A|||$  we are implicitly assuming that  $A$  belongs to  $C_{|||\cdot|||}$ .

The absolute value of an operator  $A \in B(H)$ , denoted by  $|A|$ , is defined by  $|A| = (A^*A)^{1/2}$ . Let  $s_1(A), s_2(A), \dots$  be the singular values of the compact operator  $A$ , i.e., the eigenvalues of  $|A|$ , rearranged such that  $s_1(A) \geq s_2(A) \geq \dots$ .

For  $p > 0$  and for every unitarily invariant norm  $|||\cdot|||$  on  $B(H)$ , define

$$|||A|||^{(p)} = ||| |A|^p |||^{1/p}.$$

It is known that

$$(1.1) \quad ||| |A + B|^p |||^{1/p} \leq ||| |A|^p |||^{1/p} + ||| |B|^p |||^{1/p}$$

for  $p \geq 1$  and

$$(1.2) \quad ||| |A + B|^p |||^{1/p} \leq 2^{1/p-1} \left( ||| |A|^p |||^{1/p} + ||| |B|^p |||^{1/p} \right)$$

for  $0 < p < 1$  (see e.g., [1, p.p. 95,108]). Based on the definition of  $|||\cdot|||^{(p)}$  and inequality (1.1), it can be easily seen that  $|||\cdot|||^{(p)}$  is a unitarily invariant norm for  $p \geq 1$ .

For  $0 < p < \infty$ , let

$$\|A\|_p = \left( \sum_{i=1}^{\infty} s_i^p(A) \right)^{\frac{1}{p}}.$$



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If  $p \geq 1$ , then  $\|\cdot\|_p$  is a norm, called the Schatten  $p$ -norm. So,

$$\|A\|_p = (\operatorname{tr} |A|^p)^{1/p},$$

where  $\operatorname{tr}$  is the usual trace functional. When  $p = 1$ ,  $\|A\|_1$  is called the trace norm of  $A$ . Note that for all positive real numbers  $r$  and  $p$ , we have

$$(1.3) \quad \| |A|^r \|_p = \|A\|_{rp}^r.$$

For the theory of unitarily invariant norms, the reader is referred to [1], [3], [8], [9], [10], and the references therein.

The Minkowski's inequality for scalars asserts that if  $a_i, b_i$  ( $i = 1, 2, \dots, n$ ) are complex numbers and  $p \geq 1$ , then

$$\left( \sum_{i=1}^n |a_i + b_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n |a_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |b_i|^p \right)^{\frac{1}{p}}.$$

Hiai and Zhan [4], proved that if  $A_1, A_2, B_1, B_2$  are matrices of order  $n$  and  $1 \leq p, r < \infty$ , then

$$(1.4) \quad \left\| \| |A_1 + A_2|^p + |B_1 + B_2|^p \|^{1/p} \right\| \leq 2^{|1/p-1/2|} \left( \left\| \| |A_1|^p + |B_1|^p \|^{1/p} \right\| + \left\| \| |A_2|^p + |B_2|^p \|^{1/p} \right\| \right),$$

$$(1.5) \quad \left\| \| |A_1 + A_2|^p + |B_1 + B_2|^p \|_r^{1/p} \right\| \leq 2^{(1-1/r)/p} \left( \left\| \| |A_1|^p + |B_1|^p \|_r^{1/p} \right\| + \left\| \| |A_2|^p + |B_2|^p \|_r^{1/p} \right\| \right),$$

and

$$(1.6) \quad \left\| (|A_1 + A_2|^p + |B_1 + B_2|^p)^{1/p} \right\|_r \\ \leq 2^{|1/p-1/r|} \left( \left\| (|A_1|^p + |B_1|^p)^{1/p} \right\|_r + \left\| (|A_2|^p + |B_2|^p)^{1/p} \right\|_r \right).$$

These inequalities are norm inequalities of Minkowski type.

The purpose of this paper is to establish new operator norm inequalities. Our inequalities generalize the inequalities (1.4), (1.5), and (1.6) for  $n$ -tuple of operators. Our analysis is based on some recent results on convexity and concavity of functions and on some operator inequalities.



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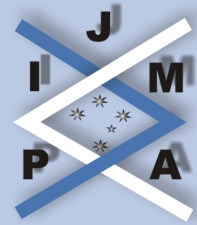
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## 2. Norm Inequalities of Minkowski Type

In this section, we generalize inequality (1.4) for operators  $A_i, B_i \in B(H)$  ( $i = 1, 2, \dots, n$ ), and other norm inequalities of Minkowski type. To achieve our goal we need the following two lemmas. The first lemma can be found in [2] and a stronger version of the second lemma can be found in [5].

**Lemma 2.1.** *Let  $A_1, \dots, A_n \in B(H)$  be positive operators. Then, for every unitarily invariant norm,*

$$(2.1) \quad \left\| \left\| \sum_{i=1}^n A_i^r \right\| \right\| \leq \left\| \left\| \left( \sum_{i=1}^n A_i \right)^r \right\| \right\|$$

for  $r \geq 1$  and

$$(2.2) \quad \left\| \left\| \left( \sum_{i=1}^n A_i \right)^r \right\| \right\| \leq \left\| \left\| \sum_{i=1}^n A_i^r \right\| \right\|$$

for  $0 < r \leq 1$ .

**Lemma 2.2.** *Let  $A_1, \dots, A_n \in B(H)$  be positive operators. Then, for every unitarily invariant norm,*

$$(2.3) \quad \left\| \left\| \left( \sum_{i=1}^n A_i \right)^r \right\| \right\| \leq n^{r-1} \left\| \left\| \sum_{i=1}^n A_i^r \right\| \right\|$$

for  $r \geq 1$  and

$$(2.4) \quad \left\| \left\| \sum_{i=1}^n A_i^r \right\| \right\| \leq n^{1-r} \left\| \left\| \left( \sum_{i=1}^n A_i \right)^r \right\| \right\|$$

for  $0 < r \leq 1$ .

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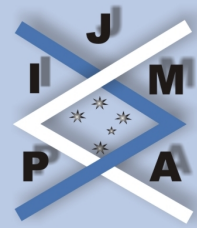
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Now, we are in a position to generalize (1.4).

**Theorem 2.3.** Let  $A_i, B_i \in B(H)$  ( $i = 1, 2, \dots, n$ ) and  $p \geq 1$ . Then, for every unitarily invariant norm,

$$(2.5) \quad n^{-|1/p-1/2|} \left\| \left\| \sum_{i=1}^n |A_i + B_i|^p \right\| \right\|^{1/p} \leq \left\| \left\| \sum_{i=1}^n |A_i|^p \right\| \right\|^{1/p} + \left\| \left\| \sum_{i=1}^n |B_i|^p \right\| \right\|^{1/p}$$

and

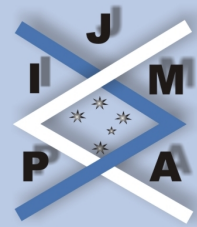
$$(2.6) \quad \left\| \left\| \sum_{i=1}^n |A_i|^p \right\| \right\|^{1/p} + \left\| \left\| \sum_{i=1}^n |B_i|^p \right\| \right\|^{1/p} \\ \leq n^{|1/p-1/2|} \left( \left\| \left\| \sum_{i=1}^n |A_i + B_i|^p \right\| \right\|^{1/p} + \left\| \left\| \sum_{i=1}^n |A_i - B_i|^p \right\| \right\|^{1/p} \right).$$

*Proof.* Let

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_n & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ B_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_n & 0 & \cdots & 0 \end{bmatrix}$$

be operators in  $B(\bigoplus_{i=1}^n H)$ . Then

$$|A|^2 = \begin{bmatrix} \sum_{i=1}^n |A_i|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad |B|^2 = \begin{bmatrix} \sum_{i=1}^n |B_i|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$



and

$$|A + B|^2 = \begin{bmatrix} \sum_{i=1}^n |A_i + B_i|^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By applying (1.1) to the operators  $A$  and  $B$ , we get

$$(2.7) \quad \left\| \left( \sum_{i=1}^n |A_i + B_i|^2 \right)^{\frac{p}{2}} \right\|^{\frac{1}{p}} \leq \left\| \left( \sum_{i=1}^n |A_i|^2 \right)^{\frac{p}{2}} \right\|^{\frac{1}{p}} + \left\| \left( \sum_{i=1}^n |B_i|^2 \right)^{\frac{p}{2}} \right\|^{\frac{1}{p}}.$$

For  $1 \leq p \leq 2$ , it follows, from (2.2) and (2.4), that

$$(2.8) \quad \left\| \left( \sum_{i=1}^n |A_i|^2 \right)^{\frac{p}{2}} \right\| \leq \left\| \sum_{i=1}^n |A_i|^p \right\|,$$

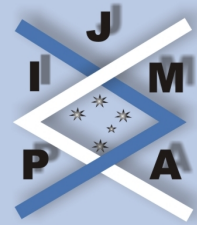
$$(2.9) \quad \left\| \left( \sum_{i=1}^n |B_i|^2 \right)^{\frac{p}{2}} \right\| \leq \left\| \sum_{i=1}^n |B_i|^p \right\|,$$

and

$$(2.10) \quad \left\| \sum_{i=1}^n |A_i + B_i|^p \right\| \leq n^{1-p/2} \left\| \left( \sum_{i=1}^n |A_i + B_i|^2 \right)^{\frac{p}{2}} \right\|.$$

Now, inequality (2.5) follows by combining (2.8), (2.9), and (2.10) by (2.7).





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For  $p > 2$ , it follows, from (2.1) and (2.3), that

$$(2.11) \quad \left\| \left\| \sum_{i=1}^n |A_i + B_i|^p \right\| \right\| \leq \left\| \left\| \left( \sum_{i=1}^n |A_i + B_i|^2 \right)^{\frac{p}{2}} \right\| \right\|,$$

$$(2.12) \quad \left\| \left\| \left( \sum_{i=1}^n |A_i|^2 \right)^{\frac{p}{2}} \right\| \right\| \leq n^{p/2-1} \left\| \left\| \sum_{i=1}^n |A_i|^p \right\| \right\|,$$

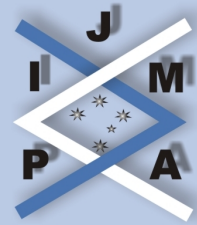
and

$$(2.13) \quad \left\| \left\| \left( \sum_{i=1}^n |B_i|^2 \right)^{\frac{p}{2}} \right\| \right\| \leq n^{p/2-1} \left\| \left\| \sum_{i=1}^n |B_i|^p \right\| \right\|.$$

Consequently, inequality (2.5) follows, by combining (2.11), (2.12), and (2.13) by (2.7). This completes the proof of inequality (2.5).

For inequality (2.6), replacing  $A_i$  and  $B_i$  in (2.5) by  $A_i + B_i$  and  $A_i - B_i$ , respectively, we have

$$(2.14) \quad 2 \left\| \left\| \sum_{i=1}^n |A_i|^p \right\| \right\|^{\frac{1}{p}} \leq n^{|1/p-1/2|} \left( \left\| \left\| \sum_{i=1}^n |A_i + B_i|^p \right\| \right\|^{\frac{1}{p}} + \left\| \left\| \sum_{i=1}^n |A_i - B_i|^p \right\| \right\|^{\frac{1}{p}} \right).$$



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Again, replacing  $A_i$  and  $B_i$  in (2.5) by  $A_i + B_i$  and  $B_i - A_i$ , respectively, we have

$$(2.15) \quad 2 \left\| \left\| \sum_{i=1}^n |B_i|^p \right\| \right\|^{\frac{1}{p}} \\ \leq n^{|1/p-1/2|} \left( \left\| \left\| \sum_{i=1}^n |A_i + B_i|^p \right\| \right\|^{\frac{1}{p}} + \left\| \left\| \sum_{i=1}^n |A_i - B_i|^p \right\| \right\|^{\frac{1}{p}} \right).$$

Now, inequality (2.6) follows, by adding inequalities (2.14) and (2.15). This completes the proof of the theorem.  $\square$

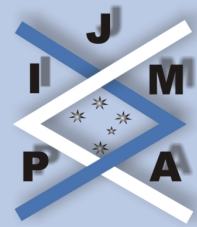
Based on inequality (1.2) and using a procedure similar to that given in the proof of Theorem 2.3, we have the following result.

**Theorem 2.4.** *Let  $A_i, B_i \in B(H)$  ( $i = 1, 2, \dots, n$ ) and  $0 < p \leq 1$ . Then, for every unitarily invariant norm,*

$$(2.16) \quad 2^{1-1/p} n^{-|1/p-1/2|} \left\| \left\| \sum_{i=1}^n |A_i + B_i|^p \right\| \right\|^{\frac{1}{p}} \\ \leq \left( \left\| \left\| \sum_{i=1}^n |A_i|^p \right\| \right\|^{\frac{1}{p}} + \left\| \left\| \sum_{i=1}^n |B_i|^p \right\| \right\|^{\frac{1}{p}} \right)$$

and

$$(2.17) \quad \left\| \left\| \sum_{i=1}^n |A_i|^p \right\| \right\|^{\frac{1}{p}} + \left\| \left\| \sum_{i=1}^n |B_i|^p \right\| \right\|^{\frac{1}{p}}$$



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$$\leq 2^{1/p-1} n^{|1/p-1/2|} \left( \left\| \left\| \sum_{i=1}^n |A_i + B_i|^p \right\| \right\|^{1/p} + \left\| \left\| \sum_{i=1}^n |A_i - B_i|^p \right\| \right\|^{1/p} \right).$$

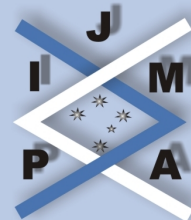
For  $p \geq 1$  inequalities (2.5) and (2.6) can be written in equivalent forms as follow:

$$(2.18) \quad n^{-|1/p-1/2|} \left\| \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{1/p} \right\| \right\|^{(p)} \\ \leq \left\| \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{1/p} \right\| \right\|^{(p)} + \left\| \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{1/p} \right\| \right\|^{(p)}$$

and

$$(2.19) \quad \left\| \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{1/p} \right\| \right\|^{(p)} + \left\| \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{1/p} \right\| \right\|^{(p)} \\ \leq n^{|1/p-1/2|} \left( \left\| \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{1/p} \right\| \right\|^{(p)} + \left\| \left\| \left( \sum_{i=1}^n |A_i - B_i|^p \right)^{1/p} \right\| \right\|^{(p)} \right).$$

In the following theorem we give inequalities related to inequalities (2.18) and (2.19). In order to do that we need the following lemma, which is a particular case of Theorem 2 in [7].



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**Lemma 2.5.** Let  $A_i, B_i \in B(H)$  ( $i = 1, 2, \dots, n$ ) and  $p \geq 2$ . Then

$$(2.20) \quad \left\| \left\| \left( \sum_{i=1}^n |A_i|^2 \right)^{\frac{1}{2}} \right\| \right\| \leq n^{1/2-1/p} \left( \left\| \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{\frac{1}{p}} \right\| \right\| \right)$$

for every unitarily invariant norm.

**Theorem 2.6.** Let  $A_i, B_i \in B(H)$  ( $i = 1, 2, \dots, n$ ) and  $p \geq 2$ . Then, for every unitarily invariant norm,

$$(2.21) \quad n^{-(1-1/p)} \left\| \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{\frac{1}{p}} \right\| \right\| \leq \left\| \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{\frac{1}{p}} \right\| \right\| + \left\| \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{\frac{1}{p}} \right\| \right\|$$

and

$$(2.22) \quad \left\| \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{\frac{1}{p}} \right\| \right\| + \left\| \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{\frac{1}{p}} \right\| \right\| \leq n^{1-1/p} \left( \left\| \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{\frac{1}{p}} \right\| \right\| + \left\| \left\| \left( \sum_{i=1}^n |A_i - B_i|^p \right)^{\frac{1}{p}} \right\| \right\| \right).$$



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*Proof.* By using (2.2), (2.4), (2.7), and (2.20), respectively, we have

$$\begin{aligned}
 \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{\frac{1}{p}} \right\| &\leq \left\| \sum_{i=1}^n |A_i + B_i| \right\| \\
 &\leq n^{1/2} \left\| \left( \sum_{i=1}^n |A_i + B_i|^2 \right)^{\frac{1}{2}} \right\| \\
 &\leq n^{1/2} \left( \left\| \left( \sum_{i=1}^n |A_i|^2 \right)^{\frac{1}{2}} \right\| + \left\| \left( \sum_{i=1}^n |B_i|^2 \right)^{\frac{1}{2}} \right\| \right) \\
 &\leq n^{1-1/p} \left( \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{\frac{1}{p}} \right\| + \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{\frac{1}{p}} \right\| \right).
 \end{aligned}$$

This proves inequality (2.21). Inequality (2.22) follows from inequality (2.21) by a proof similar to that given for inequality (2.6) in Theorem 2.3. The proof is complete.  $\square$

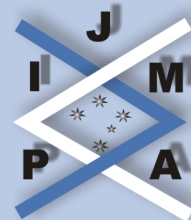
It is known that for a positive operator  $A$  and for  $0 < r \leq 1$ , we have

$$(2.23) \quad \left\| |A| \right\|^r \leq \left\| A^r \right\|$$

for every unitarily invariant norm; and the reverse inequality holds for  $r \geq 1$ .

Using inequality (2.23) we have the following application of Theorem 2.6.

**Corollary 2.7.** *Let  $A_i, B_i \in B(H)$  ( $i = 1, 2, \dots, n$ ) and  $p \geq 2$ . Then, for every*



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unitarily invariant norm,

$$(2.24) \quad n^{-(1-1/p)} \left\| \left\| \sum_{i=1}^n |A_i + B_i|^p \right\| \right\|^{1/p} \leq \left\| \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{1/p} \right\| \right\| + \left\| \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{1/p} \right\| \right\|$$

and

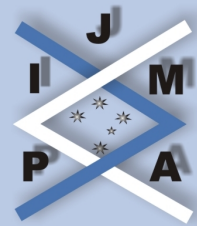
$$(2.25) \quad \left\| \left\| \sum_{i=1}^n |A_i|^p \right\| \right\|^{1/p} + \left\| \left\| \sum_{i=1}^n |B_i|^p \right\| \right\|^{1/p} \\ \leq n^{1-1/p} \left( \left\| \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{1/p} \right\| \right\| + \left\| \left\| \left( \sum_{i=1}^n |A_i - B_i|^p \right)^{1/p} \right\| \right\| \right).$$

*Remark 1.* In view of (2.5), (2.21), and (2.23), one might conjecture that if  $A_i, B_i \in B(H)$  ( $i = 1, 2, \dots, n$ ), then, for every unitarily invariant norm,

$$(2.26) \quad \left\| \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{1/p} \right\| \right\| \\ \leq n^{|1/p-1/2|} \left( \left\| \left\| \sum_{i=1}^n |A_i|^p \right\| \right\|^{1/p} + \left\| \left\| \sum_{i=1}^n |B_i|^p \right\| \right\|^{1/p} \right)$$

for  $p \geq 1$  and

$$(2.27) \quad \left\| \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{1/p} \right\| \right\| \leq n^{1-1/p} \left( \left\| \left\| \sum_{i=1}^n |A_i|^p \right\| \right\|^{1/p} + \left\| \left\| \sum_{i=1}^n |B_i|^p \right\| \right\|^{1/p} \right)$$



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for  $p \geq 2$ .

*Remark 2.* Using the same procedure used in the proof of inequality (2.6) in Theorem 2.3, inequalities (1.1) and (1.2) imply that

$$(2.28) \quad ||| |A|^p |||^{1/p} + ||| |B|^p |||^{1/p} \leq ||| |A + B|^p |||^{1/p} + ||| |A - B|^p |||^{1/p}$$

for  $p \geq 1$  and

$$(2.29) \quad ||| |A|^p |||^{1/p} + ||| |B|^p |||^{1/p} \leq 2^{1/p-1} (||| |A + B|^p |||^{1/p} + ||| |A - B|^p |||^{1/p})$$

for  $0 < p \leq 1$ . For  $p \geq 1$ , it follows, from the triangle inequality for norms and a scalar inequality, that

$$(2.30) \quad ||| |A + B|^p + |A - B|^p |||^{1/p} \leq ||| |A + B|^p |||^{1/p} + ||| |A - B|^p |||^{1/p}.$$

For  $p \geq 2$ , the left hand side of (2.30) is the right hand side of the famous Clarkson inequality

$$(2.31) \quad 2 ||| |A|^p + |B|^p ||| \leq ||| |A + B|^p + |A - B|^p |||,$$

see e.g., [6]. In view of the inequalities (2.29) and (2.30) we may introduce the following question: For  $p \geq 2$  are the following inequalities:

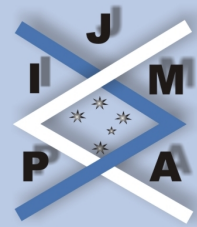
$$(2.32) \quad ||| |A|^p |||^{1/p} + ||| |B|^p |||^{1/p} \leq ||| |A + B|^p + |A - B|^p |||^{1/p}$$

and

$$(2.33) \quad 2 ||| |A|^p + |B|^p ||| \leq \left( ||| |A|^p |||^{1/p} + ||| |B|^p |||^{1/p} \right)^p$$

true?

Inequalities (2.32) and (2.33), if true, form a refinement of the Clarkson inequality (2.31).



### 3. Norm Inequalities of Minkowski Type for the Schatten $P$ -Norm

In this section, we present some norm inequalities of Minkowski type for the Schatten  $p$ -norm. These inequalities generalize the inequalities (1.5) and (1.6) for an  $n$ -tuple of operators.

**Theorem 3.1.** *Let  $A_i, B_i \in B(H)$  ( $i = 1, 2, \dots, n$ ) and  $1 \leq p, r < \infty$ . Then*

$$(3.1) \quad n^{-(1-1/r)/p} \left\| \sum_{i=1}^n |A_i + B_i|^p \right\|_r^{\frac{1}{p}} \leq \left\| \sum_{i=1}^n |A_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^n |B_i|^p \right\|_r^{\frac{1}{p}}$$

and

$$(3.2) \quad \left\| \sum_{i=1}^n |A_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^n |B_i|^p \right\|_r^{\frac{1}{p}} \leq n^{(1-1/r)/p} \left( \left\| \sum_{i=1}^n |A_i + B_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^n |A_i - B_i|^p \right\|_r^{\frac{1}{p}} \right).$$

*Proof.* It follows, from (1.3) and the triangle inequality, that

$$\left\| \sum_{i=1}^n |A_i + B_i|^{pr} \right\|_1^{\frac{1}{pr}} = \left\| \begin{bmatrix} A_1 + B_1 & 0 & \cdots & 0 \\ 0 & A_2 + B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n + B_n \end{bmatrix} \right\|_{pr}$$

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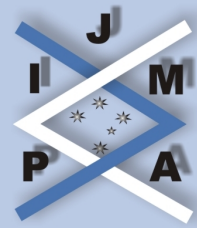
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$$\begin{aligned}
 &= \left\| \left[ \begin{array}{cccc} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{array} \right] + \left[ \begin{array}{cccc} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{array} \right] \right\|_{pr} \\
 &\leq \left\| \left[ \begin{array}{cccc} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{array} \right] \right\|_{pr} + \left\| \left[ \begin{array}{cccc} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{array} \right] \right\|_{pr} \\
 (3.3) \quad &= \left\| \sum_{i=1}^n |A_i|^{pr} \right\|_1^{\frac{1}{pr}} + \left\| \sum_{i=1}^n |B_i|^{pr} \right\|_1^{\frac{1}{pr}}.
 \end{aligned}$$

Now, by using (1.3), (2.3), (3.3), and (2.2), respectively, we have

$$\begin{aligned}
 \left\| \sum_{i=1}^n |A_i + B_i|^p \right\|_r^{\frac{1}{p}} &= \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^r \right\|_1^{\frac{1}{pr}} \\
 &\leq n^{(r-1)/pr} \left\| \sum_{i=1}^n |A_i + B_i|^{pr} \right\|_1^{\frac{1}{pr}} \\
 &\leq n^{(1-1/r)/p} \left( \left\| \sum_{i=1}^n |A_i|^{pr} \right\|_1^{\frac{1}{pr}} + \left\| \sum_{i=1}^n |B_i|^{pr} \right\|_1^{\frac{1}{pr}} \right) \\
 &= n^{(1-1/r)/p} \left( \left\| \left( \sum_{i=1}^n |A_i|^{pr} \right)^{\frac{1}{r}} \right\|_r^{\frac{1}{p}} + \left\| \left( \sum_{i=1}^n |B_i|^{pr} \right)^{\frac{1}{r}} \right\|_r^{\frac{1}{p}} \right)
 \end{aligned}$$



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$$\leq n^{(1-1/r)/p} \left( \left\| \sum_{i=1}^n |A_i|^p \right\|_r^{\frac{1}{p}} + \left\| \sum_{i=1}^n |B_i|^p \right\|_r^{\frac{1}{p}} \right).$$

This proves inequality (3.1). The proof of inequality (3.2) follows from (3.1) by a proof similar to that given for inequality (2.6) in Theorem 2.3. The proof is complete.  $\square$

The following is our final result.

**Theorem 3.2.** Let  $A_i, B_i \in B(H)$  ( $i = 1, 2, \dots, n$ ) and  $1 \leq p, r < \infty$ . Then

$$(3.4) \quad n^{-|1/p-1/r|} \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{\frac{1}{p}} \right\|_r \leq \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{\frac{1}{p}} \right\|_r + \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{\frac{1}{p}} \right\|_r$$

and

$$(3.5) \quad \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{\frac{1}{p}} \right\|_r + \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{\frac{1}{p}} \right\|_r \\ \leq n^{|1/p-1/r|} \left( \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{\frac{1}{p}} \right\|_r + \left\| \left( \sum_{i=1}^n |A_i - B_i|^p \right)^{\frac{1}{p}} \right\|_r \right).$$

*Proof.* First suppose that  $r \leq p$ . By using (1.3), (2.2), (3.3), and (2.4), respectively,



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we have

$$\begin{aligned}
 \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{\frac{1}{p}} \right\|_r &= \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{\frac{r}{p}} \right\|_1^{\frac{1}{r}} \\
 &\leq \left\| \sum_{i=1}^n |A_i + B_i|^r \right\|_1^{\frac{1}{r}} \\
 &\leq \left\| \sum_{i=1}^n |A_i|^r \right\|_1^{\frac{1}{r}} + \left\| \sum_{i=1}^n |B_i|^r \right\|_1^{\frac{1}{r}} \\
 &\leq n^{1/r-1/p} \left( \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{\frac{r}{p}} \right\|_1^{\frac{1}{r}} + \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{\frac{r}{p}} \right\|_1^{\frac{1}{r}} \right) \\
 &= n^{1/r-1/p} \left( \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{\frac{1}{p}} \right\|_r + \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{\frac{1}{p}} \right\|_r \right).
 \end{aligned}$$

Next, for  $p < r$ , by using (1.3) and (3.1), we have

$$\begin{aligned}
 \left\| \left( \sum_{i=1}^n |A_i + B_i|^p \right)^{\frac{1}{p}} \right\|_r &= \left\| \sum_{i=1}^n |A_i + B_i|^p \right\|_{\frac{r}{p}}^{\frac{1}{p}} \\
 &\leq n^{1/p(1-p/r)} \left( \left\| \sum_{i=1}^n |A_i|^p \right\|_{\frac{r}{p}}^{\frac{1}{p}} + \left\| \sum_{i=1}^n |B_i|^p \right\|_{\frac{r}{p}}^{\frac{1}{p}} \right)
 \end{aligned}$$

$$= n^{1/p-1/r} \left( \left\| \left( \sum_{i=1}^n |A_i|^p \right)^{\frac{1}{p}} \right\|_r + \left\| \left( \sum_{i=1}^n |B_i|^p \right)^{\frac{1}{p}} \right\|_r \right).$$

This proves inequality (3.4). The proof of inequality (3.5) follows from (3.4) by a proof similar to that given for inequality (2.6) in Theorem 2.3. The proof is complete.  $\square$

*Remark 3.* For the Schatten  $p$ -norm, (3.4) is better than (2.21), and if  $rp \leq 2$  or  $r(4-p) \leq 2$ , then (3.1) is better than (2.5).



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