journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 2, Article 37, 5 pp.
© 2007 Victoria University. All rights reserved.

NOTE ON AN OPEN PROBLEM

GHOLAMREZA ZABANDAN

DEPARTMENT OF MATHEMATICS
FACULTY OF MATHEMATICAL SCIENCE AND COMPUTER ENGINEERING
TEACHER TRAINING UNIVERSITY
599 TALEGHANI AVENUE
TEHRAN 15618, IRAN
Zabandan@saba.tmu.ac.ir

Received 23 August, 2007; accepted 13 March, 2008
Communicated by F. Qi

ABSTRACT. In this paper we give an affirmative answer to an open problem proposed by Quôc Anh Ngô, Du Duc Thang, Tran Tat Dat, and Dang Anh Tuan [6].

Key words and phrases: Integral inequality.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

In [6] the authors proved some integral inequalities and proposed the following question: Let f be a continuous function on $[0,1]$ satisfying

$$\int_x^1 f(t)dt \geq \frac{1-x^2}{2}, \qquad (0 \leq x \leq 1). \tag{1.1}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx$$

hold for α, β?

In [1] the author has given an answer to this open problem, but there is a clear gap in the proof of Lemma 1.1, so that the other results of the paper break down too. In this paper we give an affirmative answer to this problem by presenting stronger results. First we prove the following two essential lemmas.

Throughout this paper, we always assume that f is a non-negative continuous function on $[0,1]$, satisfying (1.1).

I am grateful to the referee for his comments, especially for Theorem 2.3.
275-07

Lemma 1.1. *If (1.1) holds, then for each* $x \in [0,1]$ *we have*

$$\int_x^1 t^k f(t)dt \geq \frac{1-x^{k+2}}{k+2} \quad (k \in \mathbb{N}).$$

Proof. By our assumptions, we have

$$\begin{aligned}\int_x^1 y^{k-1}\left(\int_y^1 f(t)dt\right)dy &\geq \int_x^1 y^{k-1}\frac{1-y^2}{2}dy \\ &= \frac{1}{2}\int_x^1 (y^{k-1}-y^{k+1})dy \\ &= \frac{1}{k(k+2)} - \frac{1}{2k}x^k + \frac{1}{2(k+2)}x^{k+2}.\end{aligned}$$

On the other hand, integrating by parts, we also obtain

$$\begin{aligned}\int_x^1 y^{k-1}\left(\int_y^1 f(t)dt\right)dy &= \frac{1}{k}y^k \int_y^1 f(t)dt\Big|_x^1 + \frac{1}{k}\int_x^1 y^k f(y)dy \\ &= -\frac{1}{k}x^k\int_x^1 f(t)dt + \frac{1}{k}\int_x^1 y^k f(y)dy.\end{aligned}$$

Thus

$$\begin{aligned} -\frac{1}{k}x^k\int_x^1 f(t)dt + \frac{1}{k}\int_x^1 y^k f(y)dy &\geq \frac{1}{k(k+2)} - \frac{1}{2k}x^k + \frac{1}{2(k+2)}x^{k+2} \\ \Longrightarrow \int_x^1 y^k f(y)dy &\geq x^k\int_x^1 f(t)dt + \frac{1}{k+2} - \frac{1}{2}x^k + \frac{k}{2(k+2)}x^{k+2} \\ &\geq x^k\left(\frac{1}{2}-\frac{1}{2}x^2\right) + \frac{1}{k+2} - \frac{1}{2}x^k + \frac{k}{2(k+2)}x^{k+2} \\ &= \frac{1-x^{k+2}}{k+2}.\end{aligned}$$

□

Remark 1. By a similar argument, we can show that Lemma 1.1 also holds when k is a real number in $[1,\infty)$. That is

$$\int_x^1 t^\alpha f(t)dt \geq \frac{1-x^{\alpha+2}}{\alpha+2} \quad (\forall \alpha \geq 1).$$

It is also interesting to note that the result of [5, Lemma 1.3] holds if we take $x = 0$ in Lemma 1.1.

Lemma 1.2. *Let* f *be a non-negative continuous function on* $[0,1]$ *such that* $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$ $(0 \leq x \leq 1)$*. Then for each* $x \in [0,1]$ *and* $k \in \mathbb{N}$*, we have*

$$\int_x^1 f^k(t)dt \geq \frac{1-x^{k+1}}{k+1}.$$

Proof. Since

$$\begin{aligned}0 &\leq \int_x^1 (f(t)-t)(f^k(t)-t^k)dt \\ &= \int_x^1 f^{k+1}(t)dt - \int_x^1 t^k f(t)dt - \int_0^1 tf^k(t)dt + \int_x^1 t^{k+1}dt\end{aligned}$$

it follows that

$$\int_x^1 f^{k+1}(t)dt \geq \int_x^1 t^k f(t)dt + \int_x^1 tf^k(t)dt - \frac{1}{k+2}(1-x^{k+2}).$$

By using Lemma 1.1, we get

$$\int_x^1 f^{k+1}(t)dt \geq \int_x^1 tf^k(t)dt. \tag{1.2}$$

We continue the proof by mathematical induction. The assertion is obvious for $k = 1$. Let $\int_x^1 f^k(t)dt \geq \frac{1-x^{k+1}}{k+1}$, we show that $\int_x^1 f^{k+1}(t)dt \geq \frac{1-x^{k+2}}{k+2}$. We have

$$\begin{aligned}\int_x^1 \left(\int_y^1 f^k(t)dt\right) dy &\geq \int_x^1 \frac{1-y^{k+1}}{k+1}dy \\ &= \frac{1}{k+1}\left(y - \frac{1}{k+2}y^{k+2}\right)\Bigg|_x^1 \\ &= \frac{1}{k+2} - \frac{1}{k+1}x + \frac{1}{(k+1)(k+2)}x^{k+2}.\end{aligned}$$

On the other hand, integrating by parts, we also obtain

$$\begin{aligned}\int_x^1 \left(\int_y^1 f^k(t)dt\right) dy &= y\int_y^1 f^k(t)dt\Bigg|_x^1 + \int_x^1 yf^k(y)dy \\ &= -x\int_x^1 f^k(t)dt + \int_x^1 yf^k(y)dy.\end{aligned}$$

Thus

$$-x\int_x^1 f^k(t)dt + \int_x^1 yf^k(y)dy \geq \frac{1}{k+2} - \frac{1}{k+1}x + \frac{1}{(k+1)(k+2)}x^{k+2}$$

and hence

$$\begin{aligned}\int_x^1 yf^k(y)dy &\geq x\int_x^1 f^k(t)dt + \frac{1}{k+2} - \frac{1}{k+1}x + \frac{1}{(k+1)(k+2)}x^{k+2} \\ &\geq x\frac{1-x^{k+1}}{k+1} + \frac{1}{k+2} - \frac{1}{k+1}x + \frac{1}{(k+1)(k+2)}x^{k+2} \\ &= \frac{1-x^{k+2}}{k+2}.\end{aligned}$$

So by (1.2) we get

$$\int_x^1 f^{k+1}(t)dt \geq \int_x^1 tf^k(t)dt \geq \frac{1-x^{k+2}}{k+2},$$

which completes the proof. □

2. MAIN RESULTS

Theorem 2.1. *Let f be a non-negative and continuous function on $[0,1]$. If $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$ $(0 \leq x \leq 1)$, then for each $m,n \in \mathbb{N}$,*

$$\int_0^1 f^{m+n}(x)dx \geq \int_0^1 x^m f^n(x)dx.$$

Proof. By using the general Cauchy inequality [5, Theorem 3.1], we have

$$\frac{n}{m+n}f^{m+n}(x)+\frac{m}{m+n}x^{m+n}\geq x^m f^n(x),$$

which implies

$$\frac{n}{m+n}\int_0^1 f^{m+n}(x)dx+\frac{m}{m+n}\int_0^1 x^{m+n}dx\geq\int_0^1 x^m f^n(x)dx.$$

Hence

$$\begin{aligned}\int_0^1 f^{m+n}(x)dx &\geq \int_0^1 x^m f^n(x)dx+\frac{m}{m+n}\int_0^1 f^{m+n}(x)dx-\frac{m}{(m+n)(m+n+1)}\\ &=\int_0^1 x^m f^n(x)dx+\frac{m}{m+n}\left(\int_0^1 f^{m+n}(x)dx-\frac{1}{m+n+1}\right).\end{aligned}$$

By Lemma 1.2, we have $\int_0^1 f^{m+n}(x)dx\geq\frac{1}{m+n+1}$. Therefore

$$\int_0^1 f^{m+n}(x)dx\geq\int_0^1 x^m f^n(x)dx.$$

□

Theorem 2.2. *Let f be a continuous function such that $f(x)\geq 1$ $(0\leq x\leq 1)$. If $\int_x^1 f(t)dt\geq\frac{1-x^2}{2}$, then for each $\alpha,\beta>0$,*

$$\int_0^1 f^{\alpha+\beta}(x)dx\geq\int_0^1 x^\alpha f^\beta(x)dx. \tag{2.1}$$

Proof. By a similar method to that used in the proof of Theorem 2.1 the inequality (2.1) holds if $\int_0^1 f^{\alpha+\beta}(x)dx\geq\frac{1}{\alpha+\beta+1}$. So it is enough to prove that $\int_0^1 f^\gamma(x)dx\geq\frac{1}{\gamma+1}$ $(\gamma>0)$. Since $f(x)\geq 1$ $(0\leq x\leq 1)$ and $[\gamma]\leq\gamma<[\gamma]+1$, we have

$$\int_0^1 f^\gamma(x)dx>\int_0^1 f^{[\gamma]}(x)dx.$$

By Lemma 1.2 we obtain

$$\int_0^1 f^\gamma(x)dx\geq\int_0^1 f^{[\gamma]}(x)dx\geq\frac{1}{[\gamma]+1}\geq\frac{1}{\gamma+1}.$$

□

Remark 2. The condition $f(x)\geq 1$ $(0\leq x\leq 1)$ in Theorem 2.2 is necessary for $\int_0^1 f^\gamma(x)dx\geq\frac{1}{\gamma+1}$ $(\gamma>0)$. For example, let

$$f(x)=\begin{cases}0 & 0\leq x\leq\frac{1}{2}\\ 2(2x-1) & \frac{1}{2}<x\leq 1\end{cases}$$

and $\gamma=\frac{1}{2}$, then f is continuous on $[0,1]$ and $\int_x^1 f(t)dt\geq\frac{1-x^2}{2}$, but $\int_0^1 f^{\frac{1}{2}}(x)dx=\frac{\sqrt{2}}{3}<\frac{2}{3}$.

In the following theorem, we show that the condition $f(x)\geq 1$ $(0\leq x\leq 1)$ in Theorem 2.2 can be removed if we assume that $\alpha+\beta\geq 1$.

Theorem 2.3. *Let f be a non-negative continuous function on $[0,1]$. If $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$ $(0 \leq x \leq 1)$, then for each $\alpha, \beta > 0$ such that $\alpha + \beta \geq 1$, we have*

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \frac{1}{\alpha+\beta+1}.$$

Proof. By using Theorem A of [5] for $g(t) = t$, $\alpha = 1$, $a = 0$, and $b = 1$, the assertion is obvious. □

REFERENCES

[1] L. BOUGOFFA, Note on an open problem, *J. Ineq. Pure and Appl. Math.*, **8**(2) (2007), Art. 58. [ONLINE: http://jipam.vu.edu.au/article.php?sid=871].

[2] K. BOUKERRIOUA AND A. GUEZANE-LAKOUD, On an open question regarding an integral inequality, *J. Ineq. Pure and Appl. Math.*, **8**(3) (2007), Art. 77. [ONLINE: http://jipam.vu.edu.au/article.php?sid=885].

[3] W.J. LIU, C.C. LI AND J.W. DONG, On an open problem concerning an integral inequality, *J. Ineq. Pure and Appl. Math.*, **8**(3) (2007), Art. 74. [ONLINE: http://jipam.vu.edu.au/article.php?sid=882].

[4] F. QI, Several integral inequalities, *J. Ineq. Pure and Appl. Math.*, **1**(2) (2000), Art. 19. [ONLINE: http://jipam.vu.edu.au/article.php?sid=113].

[5] QUÔC ANH NGÔ, On the inverse of an integral inequality, *RGMIA Research Report Collection*, **10**(4) (2007), Art. 10. [ONLINE: http://www.staff.vu.edu.au/RGMIA/v10n4.asp].

[6] QUÔC ANH NGÔ, DU DUC THANG, TRAN TAT DAT, AND DANG ANH TUAN, Notes on an integral inequality, *J. Ineq. Pure and Appl. Math.*, **7**(4) (2006), Art. 120. [ONLINE: http://jipam.vu.edu.au/article.php?sid=737].