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AN EQUIVALENT FORM OF THE FUNDAMENTAL TRIANGLE INEQUALITY AND ITS APPLICATIONS

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ABSTRACT. An equivalent form of the fundamental triangle inequality is given. The result is then used to obtain an improvement of the Leuenberger's inequality and a new proof of the Garfunkel-Bankoff inequality.

Key words and phrases: Fundamental triangle inequality, Equivalent form, Garfunkel-Bankoff inequality, Leuenberger's inequality.

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1. INTRODUCTION AND MAIN RESULTS

In what follows, we denote by A, B, C the angles of triangle ABC, let a, b, c denote the lengths of its corresponding sides, and let s, R and r denote respectively the semi-perimeter, circumradius and inradius of a triangle. We will customarily use the symbol of cyclic sums:

$$\sum f(a) = f(a) + f(b) + f(c), \quad \sum f(a,b) = f(a,b) + f(b,c) + f(c,a).$$

The fundamental triangle inequality is one of the cornerstones of geometric inequalities for triangles. It reads as follows:

$$2R^2 + 10Rr - r^2 - 2(R-2r)\sqrt{R^2-2Rr} \leqslant s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2-2Rr}. \tag{1.1}$$

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The equality holds in the left (or right) inequality of (1.1) if and only if the triangle is isosceles.

As is well known, the inequality (1.1) is a necessary and sufficient condition for the existence of a triangle with elements R, r and s. This classical inequality has many important applications in the theory of geometric inequalities and has received much attention. There exist a large number of papers that have been written about applying the inequality (1.1) to establish and prove the geometric inequalities for triangles, e.g., see [1] to [10] and the references cited therein.

In a recent paper [11], we presented a sharpened version of the fundamental triangle, as follows:

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\cos\phi \leqslant s^2 \leqslant 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}\cos\phi, \tag{1.2}$$

where $\phi=\min\{|A-B|,|B-C|,|C-A|\}$.

The objective of this paper is to give an equivalent form of the fundamental triangle inequality. As applications, we shall apply our results to a new proof of the Garfunkel-Bankoff inequality and an improvement of the Leuenberger inequality. It will be shown that our new inequality can efficaciously reduce the computational complexity in the proof of certain inequalities for triangles. We state the main result in the following theorem:

Theorem 1.1. *For any triangle ABC the following inequalities hold true:*

$$\frac{1}{4}\delta(4-\delta)^3 \leqslant \frac{s^2}{R^2} \leqslant \frac{1}{4}(2-\delta)(2+\delta)^3, \tag{1.3}$$

where $\delta=1-\sqrt{1-(2r/R)}$. Furthermore, the equality holds in the left (or right) inequality of (1.3) if and only if the triangle is isosceles.

2. PROOF OF THEOREM 1.1

We rewrite the fundamental triangle inequality (1.1) as:

$$2+\frac{10r}{R}-\frac{r^2}{R^2}-2\left(1-\frac{2r}{R}\right)\sqrt{1-\frac{2r}{R}} \leqslant \frac{s^2}{R^2} \leqslant 2+\frac{10r}{R}-\frac{r^2}{R^2}+2\left(1-\frac{2r}{R}\right)\sqrt{1-\frac{2r}{R}}. \tag{2.1}$$

By the Euler's inequality $R\geqslant 2r$ (see [1]), we observe that

$$0\leqslant 1-\frac{2r}{R}<1.$$

Let

$$\sqrt{1-\frac{2r}{R}}=1-\delta, \quad 0<\delta\leqslant 1. \tag{2.2}$$

Also, the identity (2.2) is equivalent to the following identity:

$$\frac{r}{R}=\delta-\frac{1}{2}\delta^2. \tag{2.3}$$

By applying the identities (2.2) and (2.3) to the inequality (2.1), we obtain that

$$\begin{aligned}2+10\left(\delta-\frac{1}{2}\delta^2\right)-\left(\delta-\frac{1}{2}\delta^2\right)^2-2\left[1-2\left(\delta-\frac{1}{2}\delta^2\right)\right](1-\delta)\\ \leqslant \frac{s^2}{R^2}\leqslant 2+10\left(\delta-\frac{1}{2}\delta^2\right)-\left(\delta-\frac{1}{2}\delta^2\right)^2+2\left[1-2\left(\delta-\frac{1}{2}\delta^2\right)\right](1-\delta),\end{aligned}$$

that is

$$16\delta-12\delta^2+3\delta^3-\frac{1}{4}\delta^4\leqslant\frac{s^2}{R^2}\leqslant 4+4\delta-\delta^3-\frac{1}{4}\delta^4.$$

After factoring out common factors, the above inequality can be transformed into the desired inequality (1.3). This completes the proof of Theorem 1.1. □

3. APPLICATION TO A NEW PROOF OF THE GARFUNKEL-BANKOFF INEQUALITY

Theorem 3.1. *If $A,\ B,\ C$ are angles of an arbitrary triangle, then we have the inequality*

$$\sum\tan^2\frac{A}{2}\ \geqslant\ 2-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \tag{3.1}$$

The equality holds in (3.1) if and only if the triangle ABC is equilateral.

Inequality (3.1) was proposed by Garfunkel as a conjecture in [12], and it was first proved by Bankoff in [13]. In this section, we give a simplified proof of this Garfunkel-Bankoff inequality by means of the equivalent form of the fundamental triangle inequality.

Proof. From the identities for an arbitrary triangle (see [2]):

$$\sum\tan^2\frac{A}{2}=\frac{(4R+r)^2}{s^2}-2, \tag{3.2}$$

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}, \tag{3.3}$$

it is easy to see that the Garfunkel-Bankoff inequality is equivalent to the following inequality:

$$\left(4-\frac{2r}{R}\right)\frac{s^2}{R^2}-\left(4+\frac{r}{R}\right)^2\leqslant 0. \tag{3.4}$$

Using the inequality (1.3) and the identity (2.3), we have

$$\begin{aligned}\left(4-\frac{2r}{R}\right)\frac{s^2}{R^2}-\left(4+\frac{r}{R}\right)^2&\leqslant\frac{1}{4}(4-2\delta+\delta^2)(2-\delta)(2+\delta)^3-\frac{1}{4}(4-\delta)^2(2+\delta)^2\\&=-\frac{1}{4}\delta^2(2+\delta)^2(1-\delta)^2\\&\leqslant 0,\end{aligned}$$

which implies the required inequality (3.4), hence the Garfunkel-Bankoff inequality is proved. □

4. APPLICATION TO AN IMPROVEMENT OF LEUENBERGER'S INEQUALITY

In 1960, Leuenberger presented the following inequality concerning the sides and the circumradius of a triangle (see [1])

$$\sum \frac{1}{a} \geqslant \frac{\sqrt{3}}{R}. \tag{4.1}$$

Three years later, Steinig sharpened the inequality (4.1) to the following form ([14], see also [1])

$$\sum \frac{1}{a} \geqslant \frac{3\sqrt{3}}{2(R+r)}. \tag{4.2}$$

Mitrinović et al. [2, p. 173] showed another sharpened form of (4.1), as follows:

$$\sum \frac{1}{a} \geqslant \frac{5R-r}{2R^2+(3\sqrt{3}-4)Rr}. \tag{4.3}$$

Recently, a unified improvement of the inequalities (4.2) and (4.3) was given by Wu [15], that is,

$$\sum \frac{1}{a} \geqslant \frac{11\sqrt{3}}{5R+12r+k_0(2r-R)}, \tag{4.4}$$

where $k_0 = 0.02206078402\ldots$. It is the root on the interval $(0, 1/15)$ of the following equation

$$405k^5 + 6705k^4 + 129586k^3 + 1050976k^2 + 2795373k - 62181 = 0.$$

We show here a new improvement of the inequalities (4.2) and (4.3), which is stated in Theorem 4.1 below.

Theorem 4.1. *For any triangle ABC the following inequality holds true:*

$$\sum \frac{1}{a} \geqslant \frac{\sqrt{25Rr-2r^2}}{4Rr}, \tag{4.5}$$

with equality holding if and only if the triangle ABC is equilateral.

Proof. By using the identity (2.3) and the identities for an arbitrary triangle (see [2]):

$$\sum ab = s^2+4Rr+r^2, \quad abc = 4sRr, \tag{4.6}$$

we have

$$\begin{aligned} &\left(\sum \frac{1}{a}\right)^2 - \frac{25Rr-2r^2}{16R^2r^2} \\ &= \frac{(s^2+4Rr+r^2)^2}{16s^2R^2r^2} - \frac{25Rr-2r^2}{16R^2r^2} \\ &= \frac{R^2}{16s^2r^2}\left[\frac{s^4}{R^4} - \left(\frac{17r}{R} - \frac{4r^2}{R^2}\right)\frac{s^2}{R^2} + \left(\frac{4r}{R}+\frac{r^2}{R^2}\right)^2\right] \\ &= \frac{R^2}{16s^2r^2}\left[\frac{s^4}{R^4} - \left(-\delta^4+4\delta^3-\frac{25}{2}\delta^2+17\delta\right)\frac{s^2}{R^2} + \frac{1}{16}(4-\delta)^2(4-\delta^2)^2\delta^2\right]. \end{aligned} \tag{4.7}$$

Let

$$f\left(\frac{s^2}{R^2}\right) = \left(\frac{s^2}{R^2}\right)^2 - \left(-\delta^4+4\delta^3-\frac{25}{2}\delta^2+17\delta\right)\left(\frac{s^2}{R^2}\right) + \frac{1}{16}(4-\delta)^2(4-\delta^2)^2\delta^2.$$

It is easy to see that the quadratic function

$$f(x) = x^2 - \left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right)x + \frac{1}{16}(4-\delta)^2(4-\delta^2)^2\delta^2$$

is increasing on the interval $\left[\frac{1}{2}\left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right), +\infty\right)$.

Now, from the inequalities

$$\frac{s^2}{R^2} \geqslant \frac{1}{4}\delta(4-\delta)^3$$

and

$$\frac{1}{4}\delta(4-\delta)^3 - \frac{1}{2}\left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right) = \delta\left(\frac{15}{2} - \frac{23}{4}\delta\right) + \delta^3 + \frac{1}{4}\delta^4 > 0,$$

we deduce that

$$\begin{aligned} f\left(\frac{s^2}{R^2}\right) &\geqslant f\left(\frac{\delta(4-\delta)^3}{4}\right) \\ &= \frac{1}{16}\delta^2(4-\delta)^6 - \frac{1}{4}\delta(4-\delta)^3\left(-\delta^4 + 4\delta^3 - \frac{25}{2}\delta^2 + 17\delta\right) \\ &\qquad + \frac{1}{16}(4-\delta)^2(4-\delta^2)^2\delta^2 \\ &= \frac{1}{8}(4-\delta)^2(1-\delta)^2(6-\delta)\delta^3 \\ &\geqslant 0. \end{aligned}$$

The above inequality with the identity (4.7) lead us to

$$\left(\sum \frac{1}{a}\right)^2 - \frac{25Rr - 2r^2}{16R^2r^2} \geqslant 0.$$

Theorem 4.1 is thus proved. □

Remark 1. The inequality (4.5) is stronger than the inequalities (4.1), (4.2) and (4.3) because from the Euler inequality $R \geqslant 2r$ it is easy to verify that the following inequalities hold for any triangle.

(4.8) $$\sum \frac{1}{a} \geqslant \frac{\sqrt{25Rr - 2r^2}}{4Rr} \geqslant \frac{5R - r}{2R^2 + (3\sqrt{3} - 4)Rr} \geqslant \frac{\sqrt{3}}{R},$$

(4.9) $$\sum \frac{1}{a} \geqslant \frac{\sqrt{25Rr - 2r^2}}{4Rr} \geqslant \frac{3\sqrt{3}}{2(R+r)} \geqslant \frac{\sqrt{3}}{R}.$$

In addition, it is worth noticing that inequalities (4.4) and (4.5) are incomparable in general, which can be observed from the following fact.

Letting $a = \sqrt{3},\ b = 1,\ c = 1$, then $R = 1,\ r = \sqrt{3} - \frac{3}{2}$, direct calculation gives

$$\begin{aligned} \frac{\sqrt{25Rr - 2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R + 12r + k_0(2r - R)} &> \frac{\sqrt{25Rr - 2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R + 12r + 0.023(2r - R)} \\ &= 0.11934\cdots > 0. \end{aligned}$$

Letting $a = 2,\ b = \sqrt{2},\ c = \sqrt{2}$, then $R = 1,\ r = \sqrt{2} - 1$, direct calculation gives

$$\begin{aligned} \frac{\sqrt{25Rr - 2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R + 12r + k_0(2r - R)} &< \frac{\sqrt{25Rr - 2r^2}}{4Rr} - \frac{11\sqrt{3}}{5R + 12r + 0.022(2r - R)} \\ &= -0.00183\cdots < 0. \end{aligned}$$

As a further improvement of the inequality (4.5), we propose the following interesting problem:

Open Problem 4.3. *Determine the best constant k for which the inequality below holds*

$$\sum \frac{1}{a} \geqslant \frac{1}{4Rr}\sqrt{25Rr - 2r^2 + k\left(1 - \frac{2r}{R}\right)\frac{r^3}{R}}. \tag{4.10}$$

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