Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 5, Article 175, 2006

GRÜNBAUM-TYPE INEQUALITIES FOR SPECIAL FUNCTIONS

ÁRPÁD BARICZ

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
"BABEŞ-BOLYAI" UNIVERSITY
STR. M. KOGĂLNICEANU NR. 1
RO-400084 CLUJ-NAPOCA, ROMANIA
bariczocsi@yahoo.com

Received 03 November, 2006; accepted 08 December, 2006
Communicated by T. Mills

Dedicated to Professor József Sándor on the occasion of his 50^{th} birthday

ABSTRACT. In this short note our aim is to establish some Grünbaum-type inequalities for the complementary error function, the incomplete gamma function and for Mills' ratio of the standard normal distribution, and of the gamma distribution, respectively.

Key words and phrases: Error function, Mills' ratio, functional inequalities.

2000 *Mathematics Subject Classification.* 33B20, 26D15.

1. INTRODUCTION

The Bessel function of the first kind of order μ, denoted usually by J_μ, is defined as a particular solution of the following second-order differential equation [18, p. 38]

$$x^2y''(x) + xy'(x) + (x^2 - \mu^2)y(x) = 0.$$

In 1973 F.A. Grünbaum [12] established the following interesting inequality for the function J_0, i.e.

$$1 + J_0(z) \geq J_0(x) + J_0(y),$$

where $x, y \geq 0$ and $z^2 = x^2 + y^2$. In fact this result on the Bessel function J_0 arose first in the context of a problem involving the Boltzmann equation, see [10, 11]. In this case one needs an inequality for the Legendre polynomials, which was proved in [9]. The Bessel case is proved

ISSN (electronic): 1443-5756

Research partially supported by the Institute of Mathematics, University of Debrecen, Hungary. The author is grateful to the referee for his/her helpful suggestions.

282-06

in [12] by using the well known fact, see [17], that the spherical functions on a sphere (i.e. the Legendre polynomials) approach the spherical functions on the plane (the Bessel functions) as the radius approaches infinity.

In the same year R. Askey [3] extended the above Grünbaum inequality for the function

$$\mathcal{J}_\mu(x) = 2^\mu \Gamma(\mu+1) x^{-\mu} J_\mu(x)$$

by showing that

$$1 + \mathcal{J}_\mu(z) \geq \mathcal{J}_\mu(x) + \mathcal{J}_\mu(y),$$

where $x, y, \mu \geq 0$ and $z^2 = x^2 + y^2$. It is worth mentioning here that in 1975 and 1977 A. Mcd. Mercer [14], [15] deduced and extended too the above inequality using a different approach. In 2005 Á. Baricz and E. Neuman [5] showed that the Grünbaum-type inequality

$$1 + I_0(z) \geq I_0(x) + I_0(y),$$

and the Askey-type inequality

$$1 + \mathcal{I}_\mu(z) \geq \mathcal{I}_\mu(x) + \mathcal{I}_\mu(y) \tag{1.1}$$

also holds for all $x, y, \mu \geq 0$ and $z^2 = x^2 + y^2$, where I_μ is the modified Bessel function of the first kind of order μ, and

$$\mathcal{I}_\mu(x) = 2^\mu \Gamma(\mu+1) x^{-\mu} I_\mu(x).$$

Recently, in 2005 H. Alzer [2] asked "whether there exist other special functions which satisfy inequalities of Grünbaum-type" and proved that for x, y, z positive real numbers such that $x^q + y^q = z^q$, and $n = 1, 2, \dots$

$$\Delta_n(x) = \frac{x^{n+1}}{n!} |\psi^{(n)}(x)|$$

we have the following Grünbaum-type inequality

$$1 + \Delta_n(z) > \Delta_n(x) + \Delta_n(y)$$

if and only if $q \in (0, 1]$. Moreover H. Alzer showed that the reverse of the above inequality is true if and only if $q < 0$ or $q \geq n+1$. Here ψ denotes the digamma function, i.e. the logarithmic derivative of the Euler gamma function.

In this paper our aim is to continue studies in [5] and [2] by showing that in fact every normalized power series with positive coefficients satisfies the Grünbaum-type inequality. Moreover we deduce some other Grünbaum-type inequalities for functions which frequently occur in mathematical statistics: for the complementary error function, for the gamma distribution function and finally for Mills' ratio of the standard normal distribution.

2. Grünbaum-type Inequality for General Power Series

Our first main result reads as follows.

Lemma 2.1. *Let us consider the function* $f : (a, \infty) \to \mathbb{R}$, *where* $a \geq 0$. *If the function* g, *defined by*

$$g(x) = \frac{f(x) - 1}{x}$$

is increasing on (a, ∞), *then for the function* h, *defined by* $h(x) = f(x^2)$, *we have the following Grünbaum-type inequality*

$$1 + h(z) \geq h(x) + h(y), \tag{2.1}$$

where $x, y \geq a$ *and* $z^2 = x^2 + y^2$. *If the function* g *is decreasing, then inequality (2.1) is reversed.*

Proof. Let us consider the function $\alpha : (a, \infty) \to \mathbb{R}$, defined by $\alpha(x) = f(x) - 1$. Then from the hypothesis we have for all $x, y \geq a$ the following inequality

$$\begin{aligned} \alpha(x + y) =& \frac{x}{x + y}\alpha(x + y) + \frac{y}{x + y}\alpha(x + y) \\ =& xg(x + y) + yg(x + y) \\ \geq& xg(x) + yg(y) = \alpha(x) + \alpha(y), \end{aligned}$$

i.e. the function α is super-additive on (a, ∞). From this we immediately get that

$$1 + f(x + y) \geq f(x) + f(y)$$

holds. Thus changing x with x^2 and y with y^2 the inequality (2.1) is proved. Similarly, when g is decreasing, the function α is sub-additive, which implies the converse of inequality (2.1). With this the proof is complete. □

Theorem 2.2. *Let us consider the power series* $f(x) = \sum_{n \geq 0} a_n x^n$, *which has a radius of convergence* $\rho \in [0, \infty]$ *and suppose that* $a_0 \in [0, 1]$ *and* $a_n \geq 0$ *for all* $n \geq 1$. *Then for all* $x, y, z \in [0, \rho)$ *such that* $z^2 = x^2 + y^2$ *the power series*

$$h(x) = f(x^2) = \sum_{n \geq 0} a_n x^{2n}$$

satisfies the inequality (2.1).

Proof. Let us consider the function g defined as in Lemma 2.1. Then it is easy to see that

$$x^2 g'(x) = 1 - a_0 + \sum_{n \geq 1} (n - 1) a_n x^n \geq 0,$$

i.e. the function g is increasing on $[0, \rho]$. Thus applying Lemma 2.1 the result follows. □

The next result shows that the Askey-type inequality (1.1) is valid for $\mu \in (-1, 0)$ too.

Corollary 2.3. *For all* $x, y \geq 0$, $z^2 = x^2 + y^2$ *and* $\mu > -1$ *we have the following inequality*

$$1 + \mathcal{I}_\mu(z) \geq \mathcal{I}_\mu(x) + \mathcal{I}_\mu(y).$$

Proof. Since the modified Bessel function of the first kind for $\mu > -1$ and for all $x \in \mathbb{R}$ is defined by the formula [18, p. 77]

$$I_\mu(x) = \sum_{n\geq 0} \frac{1}{n!\Gamma(\mu+n+1)}\left(\frac{x}{2}\right)^{2n+\mu},$$

by definition we easily obtain

$$\mathcal{I}_\mu(x) = \sum_{n\geq 0} a_n x^{2n}, \quad \text{where} \quad a_n = \frac{1}{2^{2n}}\frac{\Gamma(\mu+1)}{\Gamma(\mu+n+1)}.$$

Thus using Theorem 2.2 the asserted result follows. □

For $a, b, c \in \mathbb{C}$ and $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric series (function) is defined by

$${}_2F_1(a,b,c,x) := \sum_{n\geq 0} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, \qquad |x| < 1,$$

where $(a)_0 = 1$ and $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. Applying Theorem 2.2 we have the following result for this function, which we state without proof.

Corollary 2.4. *If* $a, b, c > 0$ *and* $x, y, z \in [0, 1)$ *such that* $z^2 = x^2 + y^2$, *then*

$$1 + {}_2F_1(a,b,c,z^2) \geq {}_2F_1(a,b,c,x^2) + {}_2F_1(a,b,c,y^2).$$

In particular the complete elliptic integral of the first kind, defined by

$$\mathcal{K}(r) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - r^2\sin^2\theta}} = \frac{\pi}{2}F\left(\frac{1}{2}, \frac{1}{2}, 1, r^2\right)$$

satisfies the following Grünbaum-type inequality

$$1 + \frac{2}{\pi}\mathcal{K}(z) \geq \frac{2}{\pi}\mathcal{K}(x) + \frac{2}{\pi}\mathcal{K}(y).$$

3. Grünbaum-type Inequality for Mills' Ratio

Let

(3.1) $$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-t^2/2}\,\mathrm{dt}, \qquad \mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{dt}$$

and

(3.2) $$\mathrm{erfc}(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,\mathrm{dt}$$

denote, as usual, the distribution function [1, 26.2.2, p. 931] of the standard normal law, the error function [1, 7.1.1, p. 297] and the complementary error function [1, 7.1.2, p. 297]. The tail function $\overline{\Phi} : \mathbb{R} \to (0, 1)$ of the standard normal law is defined by the relation $\overline{\Phi}(x) = 1 - \Phi(x)$. Now the ratio

(3.3) $$r(x) := \frac{\overline{\Phi}(x)}{\varphi(x)} = \frac{1-\Phi(x)}{\Phi'(x)} = e^{x^2/2}\int_x^\infty e^{-t^2/2}\,\mathrm{dt},$$

where

$$\varphi(x) := \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \text{ denotes the density function,}$$

is known in the literature as Mills' ratio [16, Sect. 2.26], while its reciprocal, $1/r(x) = \varphi(x)/\overline{\Phi}(x)$ is the so-called failure rate for the standard normal law. The Mills ratio is frequently used in mathematical statistics and in difraction theory. Various lower and upper bounds are known for this ratio [16, Sect. 2.26] and functions of the form (3.3) have been defined for some other distributions. For example let us consider the incomplete gamma function [1, 6.5.2, p. 260]

$$\gamma(p,x) = \int_0^x t^{p-1}e^{-t}\,\mathrm{dt},$$

where $p > 0$ and $x \geq 0$. This function plays also an important role in mathematical statistics, i.e. the function [1, 6.5.1, p. 260] $F(x) := \gamma(p,x)/\Gamma(p)$ is the so-called gamma distribution function. The Mills ratio for the gamma distribution is defined as follows

$$R(x) := \frac{1-F(x)}{F'(x)} = e^x x^{1-p}\Gamma(p,x),$$

where

$$\Gamma(p,x) = \Gamma(p) - \gamma(p,x) = \int_x^\infty t^{p-1}e^{-t}\,\mathrm{dt}$$

and

$$\Gamma(p,0) = \int_0^\infty t^{p-1}e^{-t}\,\mathrm{dt} = \Gamma(p).$$

In the following theorem our aim is to deduce some Grünbaum-type inequalities for the complementary error function, for the incomplete gamma function, and finally for the functions r and R.

Theorem 3.1. *Let us suppose that $z^2 = x^2 + y^2$. Then the following assertions are true:*

a. *For the complementary error function for all $x, y \geq 0$ we have the following inequality*

$$1 + \operatorname{erfc}(z^2) \geq \operatorname{erfc}(x^2) + \operatorname{erfc}(y^2). \tag{3.4}$$

b. *The Mills' ratio of the standard normal distribution satisfies the next inequality*

$$1 + r(z^2) \geq r(x^2) + r(y^2) \tag{3.5}$$

for all $x, y \geq 1$. Moreover when $x, y \in [0, 1]$, the inequality (3.5) is reversed.

c. *For a fixed $p > 0$, the function $f(x) := \Gamma(p,x)/\Gamma(p)$ satisfies the Grünbaum-type inequality*

$$1 + f(z^2) \geq f(x^2) + f(y^2), \tag{3.6}$$

where $x, y \geq 0$ and $p \leq 1$ or $x, y \geq p - 1 \geq 0$. When $p \geq 1$ and $x, y \in [0, p-1]$ the inequality (3.6) is reversed.

d. *For a fixed $p > 0$, the Mills' ratio of the gamma distribution satisfies the following inequality*

$$1 + R(z^2) \geq R(x^2) + R(y^2), \tag{3.7}$$

where $p \leq 1$ and $x, y \in [p, 1]$. Moreover if $p \geq 1$, then for all $x, y \in [1, p]$ the inequality (3.7) is reversed.

Proof. **a.** In view of Lemma 2.1 clearly it is enough to show that

$$x \mapsto \frac{1-\operatorname{erfc}(x)}{x} = \frac{\operatorname{erf}(x)}{x}$$

is decreasing on $[0,\infty)$, which was proved recently by the author [4]. For the reader's convenience we reproduce the proof here. Due to M. Gromov [7, p. 42] we know that if $f_1, f_2 : \mathbb{R} \to [0,\infty)$ are integrable functions and the ratio f_1/f_2 is decreasing, then the function

$$x \mapsto \int_0^x f_1(t)\,\mathrm{dt} \Bigg/ \int_0^x f_2(t)\,\mathrm{dt}$$

is decreasing too. For all $t \in \mathbb{R}$ let us consider $f_1(t) = 2e^{-t^2}/\sqrt{\pi}$ and $f_2(t) = 1$, then clearly $f_1/f_2 = f_1$ is decreasing on $[0,\infty)$ and consequently

$$\int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2}\,\mathrm{dt} \Bigg/ \int_0^x 1\,\mathrm{dt} = \operatorname{erf}(x)/x$$

is decreasing too in $[0,\infty)$.

b. Let us consider the function $g_1 : (0,\infty) \to \mathbb{R}$, defined by

$$g_1(x) = \frac{r(x)-1}{x}.$$

Using the relation $r'(x) = xr(x) - 1$, it is easy to verify that

$$\frac{x^2}{x+1} g_1'(x) = (x-1)\left[r(x) - \frac{1}{x+1}\right].$$

On the other hand it is known that due to R.D. Gordon [8] for all $x > 0$ we have

$$r(x) \geq \frac{x}{x^2+1}, \tag{3.8}$$

and this lower bound was improved by Z.W. Birnbaum [6] and Y. Komatu [13] by showing that for all $x > 0$, we have

$$r(x) > \frac{2}{\sqrt{x^2+4}+x}. \tag{3.9}$$

If $x \geq 1$, then using inequality (3.8) we easily get

$$\frac{x^2}{x+1} g_1'(x) \geq (x-1)\left[\frac{x}{x^2+1} - \frac{1}{x+1}\right] \geq 0,$$

i.e. the function g_1 is increasing on $[1,\infty)$. Now suppose that $x \in (0,1]$. From (3.9) it follows that g_1 is decreasing on $(0,1]$, since

$$r(x) - \frac{1}{x+1} \geq \frac{2}{\sqrt{x^2+4}+x} - \frac{1}{x+1} > 0.$$

Finally using Lemma 2.1 again, the proof of this part is complete.

c. Since $\gamma(p,x) + \Gamma(p,x) = \Gamma(p)$, it is enough to prove that the function

$$x \mapsto \frac{1-f(x)}{x} = \frac{F(x)}{x} = \frac{1}{x}\frac{\gamma(p,x)}{\Gamma(p)}$$

is decreasing on $[0,\infty)$ when $p \le 1$ and is decreasing on $[p-1,\infty)$ when $p \ge 1$. Let us consider the functions $f_1(t) = t^{p-1}e^{-t}/\Gamma(p)$ and $f_2(t) = 1$ for all $t \in \mathbb{R}$. Since

$$tf_1'(t) = (p-1-t)f_1(t),$$

it follows that $f_1/f_2 = f_1$ is decreasing on $[0,\infty)$ for $p \le 1$ and on $[p-1,\infty)$ for $p \ge 1$. Consequently using again the result of M. Gromov [7, p. 42], the function

$$x \mapsto \int_0^x \frac{t^{p-1}e^{-t}}{\Gamma(p)}\,\mathrm{dt} \bigg/ \int_0^x 1\,\mathrm{dt} = \frac{1}{x}\frac{\gamma(p,x)}{\Gamma(p)}$$

is decreasing on the mentioned intervals. For the reversed inequality from Lemma 2.1 it is enough to show that the function

$$x \mapsto \frac{F(x)}{x} = \frac{1}{x}\frac{\gamma(p,x)}{\Gamma(p)}$$

is increasing on $[0,p-1]$ when $p \ge 1$. Easy computation yields

$$x^2\frac{\partial}{\partial x}\left[\frac{1}{x}\frac{\gamma(p,x)}{\Gamma(p)}\right] = \frac{1}{\Gamma(p)}\left[x^pe^{-x} - \gamma(p,x)\right].$$

Now consider the function $f_3 : [0,\infty) \to \mathbb{R}$, defined by $f_3(x) = x^pe^{-x} - \gamma(p,x)$. Since for $x \in [0,p-1]$ we have $f_3'(x) = (p-1-x)e^{-x}x^{p-1} \ge 0$, it follows that $f_3(x) \ge f_3(0) = 0$, thus the required result follows.

d. Let us consider the function $g_2 : (0,\infty) \to \mathbb{R}$, defined by

$$g_2(x) = \frac{R(x)-1}{x} = \frac{1-F(x)-F'(x)}{xF'(x)}.$$

From simple computations we have

$$x^2[F'(x)]^2g_2'(x) = (1-x)[F'(x)]^2 + [F(x)-1][F'(x)+xF''(x)].$$

First observe that since F is the gamma distribution function, for all $x \ge 0$ we have $F(x) \in [0,1]$, i.e. $F(x)-1 \le 0$. On the other hand

$$\Gamma(p)[F'(x)+xF''(x)] = (p-x)e^{-x}x^{p-1},$$

thus if we suppose that $0 < p \le x \le 1$, then we have that the function g_2 is increasing on $[p,1]$. Moreover if $1 \le x \le p$, then clearly the function g_2 is decreasing on $[1,p]$. Using again Lemma 2.1 the inequality (3.7) and its reverse follows. □

Remark 3.2. Observe that the inequality (3.6) is equivalent to the inequality

$$1 + \frac{\Gamma(p,z^2)}{\Gamma(p)} \ge \frac{\Gamma(p,x^2)}{\Gamma(p)} + \frac{\Gamma(p,y^2)}{\Gamma(p)},$$

where $z^2 = x^2 + y^2$ and x,y,p are as in part **c** of the above theorem. Using the relation [1, 6.5.17, p. 262]

$$\Gamma\left(\frac{1}{2},x^2\right) = \Gamma\left(\frac{1}{2}\right)\mathrm{erfc}(x) = \sqrt{\pi}\,\mathrm{erfc}(x),$$

we immediately get the following inequality:

$$1 + \operatorname{erfc}(\sqrt{x^2 + y^2}) \geq \operatorname{erfc}(x) + \operatorname{erfc}(y),$$

where $x, y \geq 0$. But this inequality is weaker than (3.4), because (3.4) is equivalent to the inequality $1+\operatorname{erfc}(x + y) \geq \operatorname{erfc}(x)+\operatorname{erfc}(y)$, and the complementary error function is decreasing on $[0, \infty)$, i.e. for all $x, y \geq 0$ we have that $\operatorname{erfc}(x + y) \leq \operatorname{erfc}\left(\sqrt{x^2 + y^2}\right)$.

REFERENCES

[1] M. ABRAMOVITZ AND I.A. STEGUN (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1965.

[2] H. ALZER, A functional inequality for the polygamma functions, *Bull. Austral. Math. Soc.*, **72** (2005), 455–459.

[3] R. ASKEY, Grünbaum's inequality for Bessel functions, *J. Math. Anal. Appl.*, **41** (1973), 122–124.

[4] Á. BARICZ, Some functional inequalities for the error function, *Proc. Edinb. Math. Soc.*, submitted.

[5] Á. BARICZ AND E. NEUMAN, Inequalities involving generalized Bessel functions, *J. Ineq. Pure and Appl. Math.*, **6**(4) (2005), Article 126. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=600`].

[6] Z.W. BIRNBAUM, An inequality for Mills' ratio, *Ann. Math. Statistics*, **13** (1942), 245–246.

[7] J. CHEEGER, M. GROMOV AND M. TAYLOR, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, *J. Differential Geom.*, **17** (1982), 15–53.

[8] R.D. GORDON, Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument, *Ann. Math. Statistics*, **12** (1941), 364–366.

[9] F.A. GRÜNBAUM, A property of Legendre polynomials, *Proc. Nat. Acad. Sci., USA*, **67** (1970), 959–960.

[10] F.A. GRÜNBAUM, Linearization for the Boltzmann equation, *Trans. Amer. Math Soc.*, **165** (1972), 425–449.

[11] F.A. GRÜNBAUM, On the existence of a "wave operator" for the Boltzmann equation, *Bull. Amer. Math. Soc.*, **78**(5) (1972), 759–762.

[12] F.A. GRÜNBAUM, A new kind of inequality for Bessel functions, *J. Math. Anal. Appl.*, **41** (1973), 115–121.

[13] Y. KOMATU, Elementary inequalities for Mills' ratio, *Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs.*, **4** (1955), 69–70.

[14] A.McD. MERCER, Grünbaum's inequality for Bessel functions and its extensions, *SIAM J. Math. Anal.*, **6**(6) (1975), 1021–1023.

[15] A.McD. MERCER, Integral representations and inequalities for Bessel functions, *SIAM J. Math. Anal.*, **8**(3) (1977), 486–490.

[16] D.S. MITRINOVIĆ, *Analytic inequalities*, Springer-Verlag, Berlin, 1970.

[17] G. SZEGŐ, *Orthogonal polynomials, in* Colloquium Publications, Vol. 23, 4th ed., *American Mathematical Society*, Providence, RI, 1975.

[18] G.N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1962.