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# FOUR INEQUALITIES SIMILAR TO HARDY-HILBERT'S INTEGRAL INEQUALITY

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Abstract

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## Abstract

Four new different types of inequalities similar to Hardy-Hilbert's inequality are given.

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## Contents

# 1. Introduction

Suppose that $f$ and $g$ are real functions, such that $0 < \int_0^\infty f^2(t)dt < \infty$ and $0 < \int_0^\infty g^2(t)dt < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} < \pi \left( \int_0^\infty f^2(t)dt \int_0^\infty g^2(t)dt \right)^{\frac{1}{2}}, \tag{1.1}$$

where $\pi$ is best possible. If $(a_n)$ and $(b_n)$ are sequences of real numbers such that $0 < \sum_{n=1}^\infty a_n^2 < \infty$ and $0 < \sum_{n=1}^\infty b_n^2 < \infty$, then

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi \left( \sum_{n=1}^\infty a_n^2 \sum_{n=1}^\infty b_n^2 \right)^{\frac{1}{2}}. \tag{1.2}$$

The inequalities (1.1) and (1.2) are called Hilbert's inequalities. These inequalities play an important role in analysis (cf. [1, Chap. 9]). In their recent papers Hu [5] and Gao [3] gave two distinct improvements of (1.1) and Gao [4] gave a strengthened version of (1.2).

The following definitions are given:

$$\varphi_\lambda(r) = \frac{r+\lambda-2}{r} \quad (r = p, q), \qquad k_\lambda(p) = B\left(\varphi_\lambda(p), \varphi_\lambda(q)\right),$$

and $B$ is the beta function.

Recently, by introducing some parameters, Yang and Debnath [2] gave the following extensions:

**Theorem A.** *If* $f, g \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, *such that*

$$0 < \int_0^\infty t^{1-\lambda} f^p(t)dt < \infty \quad \textit{and} \quad 0 < \int_0^\infty t^{1-\lambda} g^q(t)dt < \infty,$$

*then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(Ax+By)^\lambda} dxdy < \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left( \int_0^\infty x^{1-\lambda} f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{1-\lambda} g^q(y)dy \right)^{\frac{1}{q}}, \tag{1.3}$$

*where the constant factor* $[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}]$ *is the best possible.*

**Theorem B.** *If* $f \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $A, B > 0$ *such that* $0 < \int_0^\infty t^{1-\lambda} f^p(t)dt < \infty$, *then*

$$\int_0^\infty y^{(\lambda-1)(p-1)} \left( \int_0^\infty \frac{f(x)}{(Ax+By)^\lambda} dx \right)^p dy < \left( \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \right)^p \int_0^\infty x^{1-\lambda} f^p(x)dx, \tag{1.4}$$

*where the constant factor* $\left[k_\lambda(p)/A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}\right]^p$ *is the best possible. The inequalities (1.3) and (1.4) are equivalent.*

**Theorem C.** *If* $a_n, b_n > 0$ $(n \in \mathbb{N})$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda < 2$, $A, B > 0$ *such that*

$$0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty \quad \textit{and} \quad 0 < \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q < \infty,$$

*then*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(Am + Bn)^{\lambda}} < \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \left( \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} n^{1-\lambda} b_n^q \right)^{\frac{1}{q}}, \tag{1.5}$$

*where the constant factor* $\left[ k_\lambda(p) / A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)} \right]$ *is the best possible.*

**Theorem D.** *If* $a_n \geq 0$ $(n \in \mathbb{N})$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $2 - \min\{p, q\} < \lambda \leq 2$, $A, B > 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{(\lambda-1)(p-1)} \left( \sum_{m=1}^{\infty} \frac{a_m}{(Am + Bn)^{\lambda}} \right)^p < \left( \frac{k_\lambda(p)}{A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)}} \right)^p \sum_{n=1}^{\infty} n^{1-\lambda} a_n^p, \tag{1.6}$$

*where the constant factor* $\left[ k_\lambda(p) / A^{\varphi_\lambda(p)} B^{\varphi_\lambda(q)} \right]^p$ *is the best possible. The inequalities (1.5) and (1.6) are equivalent.*

## 2. New Inequalities

The aim of this paper is to give the following results:

**Theorem 2.1.** *Let* $\ln f$, $\ln g$ *be convex for nonnegative functions* $f$ *and* $g$ *such that* $f(0) = g(0) = 0$, $f(\infty) = g(\infty) = \infty$, $f'(s) \geq 0$, $g'(s) \geq 0$, $s \in \{x^p, y^q\}$. *Let* $\lambda > \max\{p, q\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Let*

$$0 < \int_0^\infty \frac{t^{-p^2/q^2}\left[f(t^p)\right]^{2-\lambda+p/q}}{\left[f'(t)\right]^{\frac{p}{q}}} dt < \infty,$$

$$0 < \int_0^\infty \frac{t^{-q^2/p^2}\left[g(t^q)\right]^{2-\lambda+q/p}}{\left[g'(t)\right]^{\frac{q}{p}}} dt < \infty,$$

*then we have*

$$\int_0^\infty \int_0^\infty \frac{f(xy)g(xy)}{\left(f(x^p) + g(y^q)\right)^\lambda} dx dy \tag{2.1}$$
$$\leq \frac{1}{\sqrt[p]{p}\sqrt[q]{q}} B^{\frac{1}{p}}(p, \lambda - p) B^{\frac{1}{q}}(q, \lambda - q)$$
$$\times \left(\int_0^\infty \frac{t^{-p^2/q^2}\left[f(t^p)\right]^{2-\lambda+p/q}}{\left[f'(t)\right]^{\frac{p}{q}}} dt\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^\infty \frac{t^{-q^2/p^2}\left[g(t^q)\right]^{2-\lambda+q/p}}{\left[g'(t)\right]^{\frac{q}{p}}} dt\right)^{\frac{1}{q}}.$$

*Proof.* Since $\ln f$ is convex and $xy \le \frac{x^p}{p} + \frac{y^q}{q}$, then

$$f(xy) = e^{\ln f(xy)} \le e^{\ln f\left(\frac{x^p}{p}+\frac{y^q}{q}\right)} \le e^{\frac{\ln f(x^p)}{p}+\frac{\ln f(y^q)}{q}} = f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q).$$

Therefore, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f(xy)g(xy)}{(f(x^p)+g(y^q))^\lambda} dxdy \\
&\le \int_0^\infty \int_0^\infty \frac{f^{\frac{1}{p}}(x^p) g^{\frac{1}{q}}(y^q) \frac{[g'(y^q)]^{\frac{1}{p}}}{[f'(x^p)]^{\frac{1}{q}}} \frac{y^{\frac{q-1}{p}}}{x^{\frac{p-1}{q}}}}{(f(x^p)+g(y^q))^{\frac{\lambda}{p}}} \frac{f^{\frac{1}{q}}(y^q) g^{\frac{1}{p}}(x^p) \frac{[f'(x^p)]^{\frac{1}{q}}}{[g'(y^q)]^{\frac{1}{p}}} \frac{x^{\frac{p-1}{q}}}{y^{\frac{q-1}{p}}}}{(f(x^p)+g(y^q))^{\frac{\lambda}{q}}} dxdy \\
&\le \left( \int_0^\infty \int_0^\infty \frac{f(x^p) g^{p/q}(y^q) g'(y^q) y^{q-1}}{x^{(p-1)p/q} [f'(x^p)]^{\frac{p}{q}} (f(x^p)+g(y^q))^\lambda} dxdy \right)^{\frac{1}{p}} \\
&\qquad \times \left( \int_0^\infty \int_0^\infty \frac{f(y^q) g^{q/p}(x^p) f'(x^p) x^{p-1}}{y^{(q-1)q/p} [g'(y^q)]^{\frac{q}{p}} (f(x^p)+g(y^q))^\lambda} dxdy \right)^{\frac{1}{q}} \\
&= M^{\frac{1}{p}} N^{\frac{1}{q}}, \quad \text{say.}
\end{aligned}$$

Then

$$M = \frac{1}{q} \int_0^\infty \frac{x^{(1-p)p/q} [f(x^p)]^{2-\lambda+p/q}}{[f'(x)]^{\frac{p}{q}}} dx \int_0^\infty \frac{\left(\frac{g(y^q)}{f(x^p)}\right)^{\frac{p}{q}} g'(y^q) \frac{qy^{q-1}}{f(x^p)}}{\left(1+\frac{g(y^q)}{f(x^p)}\right)^\lambda} dy$$

$$= \frac{1}{q} \int_0^\infty \frac{x^{-p^2/q^2} [f(x^p)]^{2-\lambda+p/q}}{[f'(x)]^{\frac{p}{q}}} dx \int_0^\infty \frac{u^{p/q}}{(1+u)^\lambda} du$$

$$= \frac{1}{q} B(p, \lambda - p) \int_0^\infty \frac{x^{-p^2/q^2} [f(x^p)]^{2-\lambda+p/q}}{[f'(x)]^{\frac{p}{q}}} dx.$$

Similarly,

$$N = \frac{1}{p} B(q, \lambda - q) \int_0^\infty \frac{y^{-q^2/p^2} [g(y^q)]^{2-\lambda+q/p}}{[g'(y)]^{\frac{q}{p}}} dy.$$

Therefore

$$\int_0^\infty \int_0^\infty \frac{f(xy)g(xy)}{(f(x^p) + g(y^q))^\lambda} dxdy$$
$$\leq \frac{1}{\sqrt[q]{p}\sqrt[p]{q}} B^{\frac{1}{p}}(p, \lambda - p) B^{\frac{1}{q}}(q, \lambda - q)$$
$$\times \left( \int_0^\infty t \frac{t^{-p^2/q^2} [f(t^p)]^{2-\lambda+p/q}}{[f'(t)]^{\frac{p}{q}}} dt \right)^{\frac{1}{p}}$$
$$\times \left( \int_0^\infty \frac{t^{-q^2/p^2} [g(t^q)]^{2-\lambda+q/p}}{[g'(t)]^{\frac{q}{p}}} dt \right)^{\frac{1}{q}}.$$

□

**Theorem 2.2.** *Let $f, g, h$ be nonnegative functions, $h(x, y)$ is homogeneous of order $n$ such that $h(0, 1) = h(1, 0) = 0$, $h(\infty, 1) = h(1, \infty) = \infty$. Let $p > 1$,*

$\frac{1}{p}+\frac{1}{q}=1,$

$$0<1+\mu-r<\lambda, \quad r\in\left\{\frac{p}{q},\frac{q}{p}\right\}, \quad h_x(x,y)\geq 0, \quad h_y(x,y)\geq 0,$$

*where* $h_x = dh/dx$, $0<\int_0^\infty t^{p/q}f^p(t)dt<\infty$, $0<\int_0^\infty t^{q/p}g^q(t)dt<\infty$, *then*

$$\begin{aligned}(2.2)\quad &\int_0^\infty\int_0^\infty \frac{f(x)g(y)h^\mu(x,y)}{(1+h(x,y))^\lambda}dxdy \\ &\leq \frac{1}{n}B^{\frac{1}{p}}\left(1+\mu-\frac{p}{q},\lambda-1-\mu+\frac{p}{q}\right)B^{\frac{1}{q}}\left(1+\mu-\frac{q}{p},\lambda-1-\mu-\frac{q}{p}\right) \\ &\qquad\times\left(\int_0^\infty t^{\frac{p}{q-1}}f^p(t)dt\right)^{\frac{1}{p}}\left(\int_0^\infty t^{\frac{q}{p-1}}g^q(t)dt\right)^{\frac{1}{q}}.\end{aligned}$$

*Proof.* Observe that

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)h^\mu(x,y)}{(1+h(x,y))^\lambda}dxdy \\ &\leq\left(\int_0^\infty\int_0^\infty \frac{f^p(x)h^\mu(x,y)h_y(x,y)}{h_x^{p/q}(x,y)\left(1+h(x,y)\right)^\lambda}dxdy\right)^{\frac{1}{p}} \\ &\qquad\times\left(\int_0^\infty\int_0^\infty \frac{g^q(y)h^\mu(x,y)h_x(x,y)}{h_y^{q/p}(x,y)\left(1+h(x,y)\right)^\lambda}dxdy\right)^{\frac{1}{q}} \\ &= M^{\frac{1}{p}}N^{\frac{1}{q}}, \quad \text{say.}\end{aligned}$$

Then

$$M = \int_0^\infty f^p(x)dx \int_0^\infty \frac{h^\mu(x,y)h_y(x,y)}{h_x^{p/q}(x,y)\left(1+h(x,y)\right)^\lambda}dy.$$

Let $y = xv$, $dy = xdv$ and hence

$$h_y(x,y) = \frac{dh(x,xv)}{dy} = x^n\frac{dh(1,v)}{dy} = x^{n-1}\frac{dh(1,v)}{dv} = x^{n-1}h_v(1,v),$$
$$h_x(x,y) = \frac{dh(x,xv)}{dx} = \frac{d}{dx}x^n h(1,v) = nx^{n-1}h(1,v),$$

therefore

$$M = \frac{1}{n^{p/q}}\int_0^\infty x^{p/q-1}f^p(x)dx\int_0^\infty \frac{[x^n h(1,v)]^{\mu-p/q}\, x^n h_v(1,v)}{(1+x^n h(1,v))^\lambda}dv$$
$$= \frac{1}{n^{p/q}}B\left(1+\mu-\frac{p}{q}, \lambda-1-\mu+\frac{p}{q}\right)\int_0^\infty x^{p/q-1}f^p(x)dx.$$

Similarly,

$$N = \frac{1}{n^{q/p}}B\left(1+\mu-\frac{q}{p}, \lambda-1-\mu+\frac{q}{p}\right)\int_0^\infty y^{q/p-1}g^q(y)dy.$$

This implies

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)h^\mu(x,y)}{(1+h(x,y))^\lambda}dxdy$$

$$\leq \frac{1}{n} B^{\frac{1}{p}} \left(1+\mu-\frac{p}{q}, \lambda-1-\mu+\frac{p}{q}\right) B^{\frac{1}{q}} \left(1+\mu-\frac{q}{p}, \lambda-1-\mu+\frac{q}{p}\right)$$
$$\times \left(\int_0^\infty t^{p/q-1} f^p(t)dt\right)^{\frac{1}{p}} \left(\int_0^\infty t^{q/p-1} g^q(t)dt\right)^{\frac{1}{q}}.$$

□

The following lemma is needed for the coming result.

**Lemma 2.3.** *Let* $s \geq 1$, $0 < 1+\mu \leq \min\{\alpha, \lambda\}$ *and define*

$$f(s) = s^{-\alpha} \int_0^s \frac{t^\mu}{(1+t)^\lambda} dt,$$

*then* $f(s) \leq f(1)$.

*Proof.* We have

$$\begin{aligned} f'(s) &= s^{-\alpha} \frac{s^\mu}{(1+s)^\lambda} + \int_0^s \frac{t^\mu}{(1+t)^\lambda} dt \, (-\alpha) \, s^{-\alpha-1} \\ &\leq \frac{s^{\mu-\alpha}}{(1+s)^\lambda} - \frac{\alpha s^{-\alpha-1}}{(1+s)^\lambda} \int_0^s t^\mu dt \\ &= \frac{s^{\mu-\alpha}}{(1+s)^\lambda} \left(1 - \frac{\alpha}{1+\mu}\right) \leq 0. \end{aligned}$$

This shows that $f$ is nonincreasing and hence $f(s) \leq f(1)$. □

**Theorem 2.4.** *Let* $f, g, F, G$ *be nonnegative functions such that* $F(s) = \int_0^s f(t)dt$, $G(s) = \int_0^s g(t)dt$, *let,* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $(1-\lambda/2)r \leq \lambda/2 + \alpha \leq 2\alpha$, $r \in \left\{\frac{p}{q}, \frac{q}{p}\right\}$,

$$0 < \int_0^x (x-t)t^{(1-\lambda/2)p/q-\lambda/2-\alpha}F^{p-1}(t)f(t)dt < \infty,$$
$$0 < \int_0^x (x-t)t^{(1-\lambda/2)q/p-\lambda/2-\alpha}G^{q-1}(t)g(t)dt < \infty,$$

*then*

$$\int_0^x \int_0^x \frac{F(s)G(t)}{(s+t)^\lambda} dsdt \leq \frac{\sqrt[p]{p}\sqrt[q]{q}}{2} x^\alpha B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \times \left(\int_0^x (x-t)t^{(1-\lambda/2)p/q-\lambda/2-\alpha}F^{p-1}(t)f(t)dt\right)^{\frac{1}{p}} \times \left(\int_0^x (x-t)t^{(1-\lambda/2)q/p-\lambda/2-\alpha}G^{q-1}(t)g(t)dt\right)^{\frac{1}{q}}. \tag{2.3}$$

*Proof.* Observe that

$$\int_0^x \int_0^x \frac{F(s)G(t)}{(s+t)^\lambda} dsdt = \int_0^x \int_0^x \frac{F(s)\left(\frac{t^{1/p}}{s^{1/q}}\right)^{\lambda/2-1}}{(s+t)^{\lambda/p}} \cdot \frac{G(t)\left(\frac{s^{1/q}}{t^{1/p}}\right)^{\lambda/2-1}}{(s+t)^{\lambda/q}} dsdt$$

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$$\leq \left(\int_0^x \int_0^x \frac{F^p(s)t^{\lambda/2-1}}{(s+t)^\lambda s^{(\lambda/2-1)p/q}} dsdt\right)^{\frac{1}{p}} \left(\int_0^x \int_0^x \frac{G^q(t)s^{\lambda/2-1}}{(s+t)^\lambda t^{(\lambda/2-1)q/p}} dsdt\right)^{\frac{1}{q}}$$
$$= M^{\frac{1}{p}} N^{\frac{1}{q}}, \qquad \text{say.}$$

Then

$$\begin{aligned} M &= \int_0^x s^{(1-\lambda/2)p/q-\lambda/2} F^p(s) ds \int_0^x \frac{\left(\frac{t}{s}\right)^{\lambda/2-1} \frac{1}{s}}{\left(1+\frac{t}{s}\right)^\lambda} dt \\ &= \int_0^x s^{(1-\lambda/2)p/q-\lambda/2} F^p(s) ds \left(\frac{x}{s}\right)^\alpha \left(\frac{x}{s}\right)^{-\alpha} \int_0^{x/s} \frac{u^{\lambda/2-1}}{(1+u)^\lambda} du \\ &\leq \int_0^x s^{(1-\lambda/2)p/q-\lambda/2} F^p(s) ds \left(\frac{x}{s}\right)^\alpha \int_0^1 \frac{u^{\lambda/2-1}}{(1+u)^\lambda} du \\ &= \frac{x^\alpha}{2} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x s^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^p(s) ds, \end{aligned}$$

by virtue of the lemma.

As

$$F^p(s) = p \int_0^s F^{p-1}(u) f(u) du,$$

then

$$M = \frac{p}{2} x^\alpha B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x s^{(1-\lambda/2)p/q-\lambda/2-\alpha} ds \int_0^s F^{p-1}(u) f(u) du$$

$$= \frac{p}{2} x^{\alpha} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x \int_0^s u^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^{p-1}(u) f(u) ds du$$
$$= \frac{p}{2} x^{\alpha} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x (x-s) s^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^{p-1}(s) f(s) ds.$$

Similarly,

$$N = \frac{q}{2} x^{\alpha} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \int_0^x (x-t) t^{(1-\lambda/2)q/p-\lambda/2-\alpha} G^{q-1}(t) g(t) dt.$$

Therefore, we have

$$\int_0^x \int_0^x \frac{F(s)G(t)}{(s+t)^{\lambda}} ds dt$$
$$\leq \frac{\sqrt[p]{p}\sqrt[q]{q}}{2} x^{\alpha} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_0^x (x-t) t^{(1-\lambda/2)p/q-\lambda/2-\alpha} F^{p-1}(t) f(t) dt\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^x (x-t) t^{(1-\lambda/2)q/p-\lambda/2-\alpha} G^{q-1}(t) g(t) dt\right)^{\frac{1}{q}}.$$

□

**Theorem 2.5.** *Let $f, g$ be nonnegative functions, $f$ is submultiplicative and $g$ is concave nonincreasing, $f'(x), g'(y) \geq 0$, $f(0) = g(0) = 0$, $f(\infty) = g(\infty) =$*

$\infty, p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < a+1 < \lambda, 0 < b+1 < \lambda,$

$$0 < \int_0^\infty \frac{[f(x)]^{\mu p - bp/q} [g(x)]^{1+a-\lambda}}{[f'(x)]^{\frac{p}{q}}} dx < \infty,$$

$$0 < \int_0^\infty \frac{[g(y)]^{\mu q - aq/p} [f(y)]^{1+b-\lambda}}{[g'(y)]^{\frac{q}{p}}} dy < \infty,$$

*then*

$$(2.4) \quad \int_0^\infty \int_0^\infty \frac{f^\mu(xy)}{g^{\lambda/2}(xy)} dx dy \leq 2^{\lambda/2} B^{\frac{1}{p}} (a+1, \lambda - a - 1) B^{\frac{1}{q}} (b+1, \lambda - b - 1) \times \left( \int_0^\infty \frac{[f(t)]^{\mu p} [g(t)]^{1+a-\lambda-bp/q}}{[g'(t)]^{\frac{p}{q}}} dt \right)^{\frac{1}{p}} \times \left( \int_0^\infty \frac{[f(t)]^{\mu q} [g(t)]^{1+b-\lambda-aq/p}}{[g'(t)]^{\frac{q}{p}}} dt \right)^{\frac{1}{q}}.$$

*Proof.* Since $\sqrt{xy} \leq \frac{x+y}{2}$, then

$$g(xy) = g\left( (\sqrt{xy})^2 \right) \geq (g(\sqrt{xy}))^2 \geq \left( g\left( \frac{x+y}{2} \right) \right)^2 \geq \left( \frac{g(x)+g(y)}{2} \right)^2,$$

and hence

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{f^\mu(xy)}{g^{\lambda/2}(xy)} dxdy \\
&\leq 2^{\lambda/2} \int_0^\infty \int_0^\infty \frac{f^\mu(x) f^\mu(y)}{(g(x)+g(y))^\lambda} dxdy \\
&= 2^{\lambda/2} \int_0^\infty \int_0^\infty \frac{f^\mu(x) \frac{[g(y)]^{a/p}}{[g(x)]^{b/q}} \frac{[g'(y)]^{1/p}}{[g'(x)]^{1/q}}}{(g(x)+g(y))^{\frac{\lambda}{p}}} \cdot \frac{f^\mu(y) \frac{[g(x)]^{b/q}}{[g(y)]^{a/p}} \cdot \frac{[g'(x)]^{1/q}}{[g'(y)]^{1/p}}}{(g(x)+g(y))^{\frac{\lambda}{q}}} \\
&\leq 2^{\lambda/2} \left( \int_0^\infty \int_0^\infty \frac{f^{\mu p}(x) g^a(y) g'(y)}{[g(x)]^{bp/q} [g'(x)]^{\frac{p}{q}} (g(x)+g(y))^\lambda} dxdy \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_0^\infty \int_0^\infty \frac{f^{\mu q}(y) g^b(x) g'(x)}{[g(y)]^{aq/p} [g'(y)]^{\frac{q}{p}} (g(x)+g(y))^\lambda} dxdy \right)^{\frac{1}{q}} \\
&= 2^{\lambda/2} M^{\frac{1}{p}} N^{\frac{1}{q}}, \qquad \text{say.}
\end{aligned}$$

Then

$$\begin{aligned}
M &= \int_0^\infty \frac{[f(x)]^{\mu p} [g(x)]^{1+a-\lambda-bp/q}}{[g'(x)]^{\frac{p}{q}}} dx \int_0^\infty \frac{\left(\frac{g(y)}{g(x)}\right)^a \frac{g'(y)}{g(x)}}{\left(1+\frac{g(y)}{g(x)}\right)^\lambda} dy \\
&= B(a+1, \lambda-a-1) \int_0^\infty \frac{[f(x)]^{\eta p} [g(x)]^{1+a-\lambda-bp/q}}{[g'(x)]^{\frac{p}{q}}} dx.
\end{aligned}$$

Similarly, we can show that

$$N = B(b+1, \lambda - b - 1) \int_0^\infty \frac{[f(y)]^{\mu q} [g(y)]^{1+b-\lambda-aq/p}}{[g'(y)]^{\frac{q}{p}}} dy.$$

The result follows. □

# References


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