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# ON A CERTAIN VOLTERRA-FREDHOLM TYPE INTEGRAL EQUATION

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ABSTRACT. The aim of this paper is to study the existence, uniqueness and other properties of solutions of a certain Volterra-Fredholm type integral equation. The main tools employed in the analysis are based on applications of the Banach fixed point theorem and a certain integral inequality with explicit estimate.

*Key words and phrases:* Volterra-Fredholm type, integral equation, Banach fixed point theorem, integral inequality, Bielecki type norm, existence and uniqueness, continuous dependence.

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## 1. INTRODUCTION

Consider the following Volterra-Fredholm type integral equation

$$x(t) = f(t) + \int_a^t g(t, s, x(s), x'(s))\, ds + \int_a^b h(t, s, x(s), x'(s))\, ds, \tag{1.1}$$

for $-\infty < a \le t \le b < \infty$, where $x, f, g, h$ are in $\mathbb{R}^n$, the $n$-dimensional Euclidean space with appropriate norm denoted by $|\cdot|$. Let $\mathbb{R}$ and $'$ denote the set of real numbers and the derivative of a function. We denote by $I = [a, b]$, $\mathbb{R}_+ = [0, \infty)$ the given subsets of $\mathbb{R}$ and assume that $f \in C(I, \mathbb{R}^n)$, $g, h \in C(I^2 \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and are continuously differentiable with respect to $t$ on the respective domains of their definitions.

The literature provides a good deal of information related to the special versions of equation (1.1), see [3, 5, 6, 8, 12] and the references cited therein. Recently, in [1] the authors studied a Fredholm type equation similar to equation (1.1) for $g = 0$ using Perov's fixed point theorem, the method of successive approximations and the trapezoidal quadature rule. The purpose of this paper is to study the existence, uniqueness and other properties of solutions of equation (1.1) under various assumptions on the functions involved and their derivatives. The well known Banach fixed point theorem (see [5, p. 37]) coupled with a Bielecki type norm (see [2]) and an integral inequality with an explicit estimate given in [10, p. 44] are used to establish the results.

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## 2. EXISTENCE AND UNIQUENESS

By a solution of equation (1.1) we mean a continuous function $x(t)$ for $t \in I$ which is continuously differentiable with respect to $t$ and satisfies the equation (1.1). For every continuous function $u(t)$ in $\mathbb{R}^n$ together with its continuous first derivative $u'(t)$ for $t \in I$ we denote by $|u(t)|_1 = |u(t)| + |u'(t)|$. Let $S$ be a space of those continuous functions $u(t)$ in $\mathbb{R}^n$ together with the continuous first derivative $u'(t)$ in $\mathbb{R}^n$ for $t \in I$ which fulfil the condition

$$|u(t)|_1 = O(\exp(\lambda(t-a))), \tag{2.1}$$

for $t \in I$, where $\lambda$ is a positive constant. In the space $S$ we define the norm (see [2, 4, 7, 9, 11])

$$|u|_S = \sup_{t \in I} \{|u(t)|_1 \exp(\lambda(t-a))\}. \tag{2.2}$$

It is easy to see that $S$ with its norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a nonnegative constant $N$ such that

$$|u(t)|_1 \le N \exp(\lambda(t-a)).$$

Using this fact in (2.2) we observe that

$$|u|_S \le N. \tag{2.3}$$

We need the following special version of the integral inequality given in [10, Theorem 1.5.2, part $(b_2)$, p. 44]. We shall state it in the following lemma for completeness.

**Lemma 2.1.** *Let $u(t) \in C(I, \mathbb{R}_+)$, $k(t,s), r(t,s) \in C(I^2, \mathbb{R}_+)$ be nondecreasing in $t \in I$ for each $s \in I$ and*

$$u(t) \le c + \int_a^t k(t,s)\, u(s)\, ds + \int_a^b r(t,s)\, u(s)\, ds,$$

*for $t \in I$ where $c \ge 0$ is a constant. If*

$$d(t) = \int_a^b r(t,s) \exp\left(\int_a^s k(s,\sigma)\, d\sigma\right) ds < 1,$$

*for $t \in I$, then*

$$u(t) \le \frac{c}{1 - d(t)} \exp\left(\int_a^t k(t,s)\, ds\right),$$

*for $t \in I$.*

The following theorem ensures the existence of a unique solution to equation (1.1).

**Theorem 2.2.** *Assume that*

$(i)$ *the functions $g, h$ in equation (1.1) and their derivatives with respect to $t$ satisfy the conditions*

$$|g(t,s,u,v) - g(t,s,\bar{u},\bar{v})| \le p_1(t,s)\,[|u - \bar{u}| + |v - \bar{v}|], \tag{2.4}$$

$$\left|\frac{\partial}{\partial t} g(t,s,u,v) - \frac{\partial}{\partial t} g(t,s,\bar{u},\bar{v})\right| \le p_2(t,s)\,[|u - \bar{u}| + |v - \bar{v}|], \tag{2.5}$$

$$|h(t,s,u,v) - h(t,s,\bar{u},\bar{v})| \le q_1(t,s)\,[|u - \bar{u}| + |v - \bar{v}|], \tag{2.6}$$

$$\left|\frac{\partial}{\partial t} h(t,s,u,v) - \frac{\partial}{\partial t} h(t,s,\bar{u},\bar{v})\right| \le q_2(t,s)\,[|u - \bar{u}| + |v - \bar{v}|], \tag{2.7}$$

*where* $p_i(t,s), q_i(t,s) \in C(I^2, \mathbb{R}_+)$ $(i = 1,2)$,

$(ii)$ *for* $\lambda$ *as in (2.1)*

(a) *there exists a nonnegative constant* $\alpha$ *such that* $\alpha < 1$ *and*

$$p_1(t,t)\exp(\lambda(t-a)) + \int_a^t p(t,s)\exp(\lambda(s-a))\,ds + \int_a^b q(t,s)\exp(\lambda(s-a))\,ds \le \alpha\exp(\lambda(t-a)), \tag{2.8}$$

*for* $t \in I$, *where* $p(t,s) = p_1(t,s) + p_2(t,s)$, $q(t,s) = q_1(t,s) + q_2(t,s)$,

(b) *there exists a nonnegative constant* $\beta$ *such that*

$$|f(t)| + |f'(t)| + |g(t,t,0)| + \int_a^t \left[|g(t,s,0,0)| + \left|\frac{\partial}{\partial t}g(t,s,0,0)\right|\right]ds + \int_a^b \left[|h(t,s,0,0)| + \left|\frac{\partial}{\partial t}h(t,s,0,0)\right|\right]ds \le \beta\exp(\lambda(t-a)), \tag{2.9}$$

*where* $f, g, h$ *are the functions given in equation (1.1).*

*Then the equation (1.1) has a unique solution* $x(t)$ *in* $S$ *on* $I$ *.*

*Proof.* Let $x(t) \in S$ and define the operator

$$(Tx)(t) = f(t) + \int_a^t g(t,s,x(s),x'(s))\,ds + \int_a^b h(t,s,x(s),x'(s))ds. \tag{2.10}$$

Differentiating both sides of (2.10) with respect to $t$ we have

$$(Tx)'(t) = f'(t) + g(t,t,x(t),x'(t)) + \int_a^t \frac{\partial}{\partial t}g(t,s,x(s),x'(s))\,ds + \int_a^b \frac{\partial}{\partial t}h(t,s,x(s),x'(s))ds. \tag{2.11}$$

Now we show that $Tx$ maps $S$ into itself. Evidently, $Tx, (Tx)'$ are continuous on $I$ and $Tx, (Tx)' \in \mathbb{R}^n$. We verify that (2.1) is fulfilled. From (2.10), (2.11), using the hypotheses and (2.3) we have

$$\begin{aligned} &|(Tx)(t)|_1 \\ &= |(Tx)(t)| + |(Tx)'(t)| \\ &\le |f(t)| + |f'(t)| + |g(t,t,x(t),x'(t)) - g(t,t,0,0)| + |g(t,t,0,0)| \\ &\quad + \int_a^t |g(t,s,x(s),x'(s)) - g(t,s,0,0)|\,ds + \int_a^t |g(t,s,0,0)|\,ds \\ &\quad + \int_a^t \left|\frac{\partial}{\partial t}g(t,s,x(s),x'(s)) - \frac{\partial}{\partial t}g(t,s,0,0)\right|ds + \int_a^t \left|\frac{\partial}{\partial t}g(t,s,0,0)\right|ds \\ &\quad + \int_a^b |h(t,s,x(s),x'(s)) - h(t,s,0,0)|\,ds + \int_a^b |h(t,s,0,0)|\,ds \\ &\quad + \int_a^b \left|\frac{\partial}{\partial t}h(t,s,x(s),x'(s)) - \frac{\partial}{\partial t}h(t,s,0,0)\right|ds + \int_a^b \left|\frac{\partial}{\partial t}h(t,s,0,0)\right|ds \end{aligned} \tag{2.12}$$

$$
\begin{aligned}
&\le \beta \exp(\lambda(t-a)) + p_1(t,t)\,|x(t)|_1 + \int_a^t p(t,s)\,|x(s)|_1\,ds + \int_a^b q(t,s)\,|x(s)|_1\,ds \\
&\le \beta \exp(\lambda(t-a)) + |x|_S \left\{ p_1(t,t)\exp(\lambda(t-a)) + \int_a^t p(t,s)\exp(\lambda(s-a))\,ds \right. \\
&\qquad \left. + \int_a^b q(t,s)\exp(\lambda(s-a))\,ds \right\} \\
&\le \beta \exp(\lambda(t-a)) + |x|_S\,\alpha \exp(\lambda(t-a)) \\
&\le [\beta + N\alpha]\exp(\lambda(t-a)).
\end{aligned}
$$

From (2.12) it follows that $Tx \in S$. This proves that $T$ maps $S$ into itself.

Now, we verify that the operator $T$ is a contraction map. Let $x(t), y(t) \in S$. From (2.10), (2.11) and using the hypotheses we have

$$
\begin{aligned}
(2.13)\qquad & |(Tx)(t) - (Ty)(t)|_1 \\
&= |(Tx)(t) - (Ty)(t)| + \left|(Tx)'(t) - (Ty)'(t)\right| \\
&\le |g(t,t,x(t),x'(t)) - g(t,t,y(t),y'(t))| \\
&\qquad + \int_a^t |g(t,s,x(s),x'(s)) - g(t,s,y(s),y'(s))|\,ds \\
&\qquad + \int_a^t \left| \frac{\partial}{\partial t} g(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} g(t,s,y(s),y'(s)) \right| ds \\
&\qquad + \int_a^b |h(t,s,x(s),x'(s)) - h(t,s,y(s),y'(s))|\,ds \\
&\qquad + \int_a^b \left| \frac{\partial}{\partial t} h(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} h(t,s,y(s),y'(s)) \right| ds \\
&\le p_1(t,t)\,|x(t) - y(t)|_1 + \int_a^t p(t,s)\,|x(s) - y(s)|_1\,ds \\
&\qquad + \int_a^b q(t,s)\,|x(s) - y(s)|_1\,ds \\
&\le |x-y|_S \left\{ p_1(t,t)\exp(\lambda(t-a)) + \int_a^t p(t,s)\exp(\lambda(s-a))\,ds \right. \\
&\qquad \left. + \int_a^b q(t,s)\exp(\lambda(s-a))\,ds \right\} \\
&\le |x-y|_S\,\alpha \exp(\lambda(t-a)).
\end{aligned}
$$

From (2.13) we obtain

$$
|Tx - Ty|_S \le \alpha\,|x-y|_S.
$$

Since $\alpha < 1$, it follows from the Banach fixed point theorem (see [5, p. 37]) that $T$ has a unique fixed point in $S$. The fixed point of $T$ is however a solution of equation (1.1). The proof is complete. □

**Remark 1.** We note that in 1956 Bielecki [2] first used the norm defined in (2.2) for proving global existence and uniqueness of solutions of ordinary differential equations. It is now used very frequently to obtain global existence and uniqueness results for wide classes of differential and integral equations. For developments related to the topic, see [4] and the references cited therein.

The following theorem holds concerning the uniqueness of solutions of equation (1.1) in $\mathbb{R}^n$ without the existence part.

**Theorem 2.3.** *Assume that the functions $g, h$ in equation (1.1) and their derivatives with respect to $t$ satisfy the conditions (2.4) – (2.7). Further assume that the functions $p_i(t,s), q_i(t,s)$ $(i = 1, 2)$ in (2.4) – (2.7) are nondecreasing in $t \in I$ for each $s \in I$,*

$$p_1(t,t) \le d, \tag{2.14}$$

*for $t \in I$, where $d \ge 0$ is a constant such that $d < 1$,*

$$e(t) = \int_a^b \frac{1}{1-d} q(t,s) \exp\left(\int_a^s \frac{1}{1-d} p(s,\sigma)\, d\sigma\right) ds < 1, \tag{2.15}$$

*where*

$$p(t,s) = p_1(t,s) + p_2(t,s), \qquad q(t,s) = q_1(t,s) + q_2(t,s).$$

*Then the equation (1.1) has at most one solution on $I$.*

*Proof.* Let $x(t)$ and $y(t)$ be two solutions of equation (1.1) and

$$w(t) = |x(t) - y(t)| + |x'(t) - y'(t)|.$$

Then by hypotheses we have

$$\begin{aligned} w(t) \le & \int_a^t |g(t,s,x(s),x'(s)) - g(t,s,y(s),y'(s))|\, ds \\ & + \int_a^b |h(t,s,x(s),x'(s)) - h(t,s,y(s),y'(s))|\, ds \\ & + |g(t,t,x(t),x'(t)) - g(t,t,y(t),y'(t))| \\ & + \int_a^t \left| \frac{\partial}{\partial t} g(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} g(t,s,y(s),y'(s)) \right| ds \\ & + \int_a^b \left| \frac{\partial}{\partial t} h(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} h(t,s,y(s),y'(s)) \right| ds \\ \le & \int_a^t p_1(t,s)\left[|x(s) - y(s)| + |x'(s) - y'(s)|\right] ds \\ & + \int_a^b q_1(t,s)\left[|x(s) - y(s)| + |x'(s) - y'(s)|\right] ds \\ & + p_1(t,t)\left[|x(t) - y(t)| + |x'(t) - y'(t)|\right] \\ & + \int_a^t p_2(t,s)\left[|x(s) - y(s)| + |x'(s) - y'(s)|\right] ds \\ & + \int_a^b q_2(t,s)\left[|x(s) - y(s)| + |x'(s) - y'(s)|\right] ds. \end{aligned} \tag{2.16}$$

Using (2.14) in (2.16) we observe that

$$w(t) \le \frac{1}{1-d} \int_a^t p(t,s)\, w(s)\, ds + \frac{1}{1-d} \int_a^b q(t,s)\, w(s)\, ds. \tag{2.17}$$

Now a suitable application of Lemma 2.1 to (2.17) yields

$$|x(t) - y(t)| + |x'(t) - y'(t)| \le 0,$$

and hence $x(t) = y(t)$, which proves the uniqueness of solutions of equation (1.1) on $I$. □

## 3. Bounds on Solutions

In this section we obtain estimates on the solutions of equation (1.1) under some suitable conditions on the functions involved and their derivatives.

The following theorem concerning an estimate on the solution of equation (1.1) holds.

**Theorem 3.1.** *Assume that the functions $f, g, h$ in equation (1.1) and their derivatives with respect to $t$ satisfy the conditions*

$$|f(t)| + |f'(t)| \leq \bar{c}, \tag{3.1}$$

$$|g(t,s,u,v)| \leq m_1(t,s)[|u| + |v|], \tag{3.2}$$

$$\left|\frac{\partial}{\partial t} g(t,s,u,v)\right| \leq m_2(t,s)[|u| + |v|], \tag{3.3}$$

$$|h(t,s,u,v)| \leq n_1(t,s)[|u| + |v|], \tag{3.4}$$

$$\left|\frac{\partial}{\partial t} h(t,s,u,v)\right| \leq n_2(t,s)[|u| + |v|], \tag{3.5}$$

*where $\bar{c} \geq 0$ is a constant and for $i = 1, 2$, $m_i(t,s), n_i(t,s) \in C(I^2, \mathbb{R}_+)$ and they are nondecreasing in $t \in I$ for each $s \in I$. Further assume that*

$$m_1(t,t) \leq \bar{d}, \tag{3.6}$$

$$\bar{e}(t) = \int_a^b \frac{1}{1-\bar{d}} n(t,s) \exp\left(\int_a^s \frac{1}{1-\bar{d}} m(s,\sigma)\,d\sigma\right) ds < 1, \tag{3.7}$$

*for $t \in I$ where $\bar{d} \geq 0$ is a constant such that $\bar{d} < 1$ and*

$$m(t,s) = m_1(t,s) + m_2(t,s), \qquad n(t,s) = n_1(t,s) + n_2(t,s).$$

*If $x(t)$, $t \in I$ is any solution of equation (1.1), then*

$$|x(t)| + |x'(t)| \leq \left(\frac{\bar{c}}{1-\bar{d}}\right)\left(\frac{1}{1-\bar{e}(t)}\right) \exp\left(\int_a^t m(t,s)\,ds\right), \tag{3.8}$$

*for $t \in I$.*

*Proof.* Let $u(t) = |x(t)| + |x'(t)|$ for $t \in I$. Using the fact that $x(t)$ is a solution of equation (1.1) and the hypotheses we have

$$\begin{aligned} u(t) &\leq |f(t)| + |f'(t)| + \int_a^t |g(t,s,x(s),x'(s))|\,ds \\ &\quad + \int_a^b |h(t,s,x(s),x'(s))|\,ds + |g(t,t,x(t),x'(t))| \\ &\quad + \int_a^t \left|\frac{\partial}{\partial t} g(t,s,x(s),x'(s))\right| ds + \int_a^b \left|\frac{\partial}{\partial t} h(t,s,x(s),x'(s))\right| ds \\ &\leq \bar{c} + \int_a^t m_1(t,s)u(s)\,ds + \int_a^b n_1(t,s)u(s)\,ds \\ &\quad + m_1(t,t)u(t) + \int_a^t m_2(t,s)u(s)\,ds + \int_a^b n_2(t,s)u(s)\,ds. \end{aligned} \tag{3.9}$$

Using (3.6) in (3.9) we observe that

$$u(t) \le \frac{\bar{c}}{1-\bar{d}} + \frac{1}{1-\bar{d}}\int_a^t m(t,s)\,u(s)\,ds + \frac{1}{1-\bar{d}}\int_a^b n(t,s)\,u(s)\,ds. \tag{3.10}$$

Now an application of Lemma 2.1 to (3.10) yields (3.8). □

**Remark 2.** We note that the estimate obtained in (3.8) yields not only the bound on the solution of equation (1.1) but also the bound on its derivative. If the estimate on the right hand side in (3.8) is bounded, then the solution of equation (1.1) and its derivative is bounded on $I$ .

Now we shall obtain an estimate on the solution of equation (1.1) assuming that the functions $g, h$ and their derivatives with respect to $t$ satisfy Lipschitz type conditions.

**Theorem 3.2.** *Assume that the hypotheses of Theorem 2.3 hold. Suppose that*

$$\int_a^t |g(t,s,f(s)),f'(s)|\,ds + \int_a^b |h(t,s,f(s),f'(s))|\,ds + |g(t,t,f(t),f'(t))| + \int_a^t \left|\frac{\partial}{\partial t} g(t,s,f(s),f'(s))\right| ds + \int_a^b \left|\frac{\partial}{\partial t} h(t,s,f(s),f'(s))\right| ds \le D,$$

*for $t \in I$, where $D \ge 0$ is a constant. If $x(t), t \in I$ is any solution of equation (1.1), then*

$$|x(t)-f(t)| + |x'(t)-f'(t)| \le \left(\frac{D}{1-d}\right)\left(\frac{1}{1-e(t)}\right)\exp\left(\int_a^t p(t,s)\,ds\right), \tag{3.11}$$

*for $t \in I$.*

*Proof.* Let $u(t) = |x(t)-f(t)|+|x'(t)-f'(t)|$ for $t \in I$. Using the fact that $x(t)$ is a solution of equation (1.1) and the hypotheses we have

$$\begin{aligned} u(t) \le & \int_a^t |g(t,s,x(s),x'(s)) - g(t,s,f(s),f'(s))| ds \\ & + \int_a^t |g(t,s,f(s),f'(s))| ds \\ & + \int_a^b |h(t,s,x(s),x'(s)) - h(t,s,f(s),f'(s))| ds \\ & + \int_a^b |h(t,s,f(s),f'(s))| ds \\ & + |g(t,t,x(t),x'(t)) - g(t,t,f(t),f'(t))| + |g(t,t,f(t),f'(t))| \\ & + \int_a^t \left|\frac{\partial}{\partial t} g(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} g(t,s,f(s),f'(s))\right| ds \\ & + \int_a^t \left|\frac{\partial}{\partial t} g(t,s,f(s),f'(s))\right| ds \\ & + \int_a^b \left|\frac{\partial}{\partial t} h(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} h(t,s,f(s),f'(s))\right| ds \\ & + \int_a^b \left|\frac{\partial}{\partial t} h(t,s,f(s),f'(s))\right| ds \end{aligned} \tag{3.12}$$

$$\leq D+\int_a^t p_1(t,s)\,u(s)\,ds+\int_a^b q_1(t,s)\,u(s)\,ds$$
$$+p_1(t,t)\,u(t)+\int_a^t p_2(t,s)\,u(s)\,ds+\int_a^b q_2(t,s)\,u(s)\,ds.$$

Using (2.14) in (3.12) we observe that

$$u(t)\leq \frac{D}{1-d}+\frac{1}{1-d}\int_a^t p(t,s)\,u(s)\,ds+\frac{1}{1-d}\int_a^b q(t,s)\,u(s)\,ds. \tag{3.13}$$

Now an application of Lemma 2.1 to (3.13) yields (3.11). □

## 4. CONTINUOUS DEPENDENCE

In this section we shall deal with continuous dependence of solutions of equation (1.1) on the functions involved therein and also the continuous dependence of solutions of equations of the form (1.1) on parameters.

Consider the equation (1.1) and the following Volterra-Fredholm type integral equation

$$y(t)=F(t)+\int_a^t G(t,s,y(s),y'(s))\,ds+\int_a^b H(t,s,y(s),y'(s))\,ds, \tag{4.1}$$

for $t\in I$, where $y,F,G,H$ are in $\mathbb{R}^n$. We assume that $F\in C(I,\mathbb{R}^n)$, $G,H\in C(I^2\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ and are continuously differentiable with respect to $t$ on the respective domains of their definitions.

The following theorem deals with the continuous dependence of solutions of equation (1.1) on the functions involved therein.

**Theorem 4.1.** *Assume that the hypotheses of Theorem 2.3 hold. Suppose that*

$$\begin{aligned} &|f(t)-F(t)|+|f'(t)-F'(t)|+|g(t,t,y(t),y'(t))-G(t,t,y(t),y'(t))| \\ &\quad+\int_a^t |g(t,s,y(s),y'(s))-G(t,s,y(s),y'(s))|\,ds \\ &\quad+\int_a^t \left|\frac{\partial}{\partial t}g(t,s,y(s),y'(s))-\frac{\partial}{\partial t}G(t,s,y(s),y'(s))\right| ds \\ &\quad+\int_a^b |h(t,s,y(s),y'(s))-H(t,s,y(s),y'(s))|\,ds \\ &\quad+\int_a^b \left|\frac{\partial}{\partial t}h(t,s,y(s),y'(s))-\frac{\partial}{\partial t}H(t,s,y(s),y'(s))\right| ds \\ &\leq \varepsilon, \end{aligned} \tag{4.2}$$

*where $f,g,h$ and $F,G,H$ are the functions involved in equations (1.1) and (4.1), $y(t)$ is a solution of equation (4.1) and $\varepsilon>0$ is an arbitrary small constant. Then the solution $x(t)$, $t\in I$ of equation (1.1) depends continuously on the functions involved on the right hand side of equation (1.1).*

*Proof.* Let $z(t) = |x(t) - y(t)| + |x'(t) - y'(t)|$ for $t \in I$. Using the facts that $x(t)$ and $y(t)$ are the solutions of equations (1.1) and (4.1) and the hypotheses we have

$$\begin{aligned}
z(t) \leq{} & |f(t) - F(t)| + |f'(t) - F'(t)| \\
& + |g(t,t,x(t),x'(t)) - g(t,t,y(t),y'(t))| \\
& + |g(t,t,y(t),y'(t)) - G(t,t,y(t),y'(t))| \\
& + \int_a^t |g(t,s,x(s),x'(s)) - g(t,s,y(s),y'(s))|\,ds \\
& + \int_a^t |g(t,s,y(s),y'(s)) - G(t,s,y(s),y'(s))|\,ds \\
& + \int_a^b |h(t,s,x(s),x'(s)) - h(t,s,y(s),y'(s))|\,ds \\
& + \int_a^b |h(t,s,y(s),y'(s)) - H(t,s,y(s),y'(s))|\,ds \\
& + \int_a^t \left|\frac{\partial}{\partial t} g(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} g(t,s,y(s),y'(s))\right| ds \\
& + \int_a^t \left|\frac{\partial}{\partial t} g(t,s,y(s),y'(s)) - \frac{\partial}{\partial t} G(t,s,y(s),y'(s))\right| ds \\
& + \int_a^b \left|\frac{\partial}{\partial t} h(t,s,x(s),x'(s)) - \frac{\partial}{\partial t} h(t,s,y(s),y'(s))\right| ds \\
& + \int_a^b \left|\frac{\partial}{\partial t} h(t,s,y(s),y'(s)) - \frac{\partial}{\partial t} H(t,s,y(s),y'(s))\right| ds \\
\leq{} & \varepsilon + p_1(t,t)\,z(t) + \int_a^t p(t,s)\,z(s)\,ds + \int_a^b q(t,s)\,z(s)\,ds.
\end{aligned} \tag{4.3}$$

Using (2.14) in (4.3) we observe that

$$z(t) \leq \frac{\varepsilon}{1-d} + \frac{1}{1-d}\int_a^t p(t,s)\,z(s)\,ds + \frac{1}{1-d}\int_a^b q(t,s)\,z(s)\,ds. \tag{4.4}$$

Now an application of Lemma 2.1 to (4.4) yields

$$|x(t) - y(t)| + |x'(t) - y'(t)| \leq \left(\frac{\varepsilon}{1-d}\right)\left(\frac{1}{1-e(t)}\right)\exp\left(\int_a^t p(t,s)\,ds\right), \tag{4.5}$$

for $t \in I$. From (4.5) it follows that the solutions of equation (1.1) depend continuously on the functions involved on the right hand side of equation (1.1). □

Next, we consider the following Volterra-Fredholm type integral equations

$$z(t) = f(t) + \int_a^t g(t,s,z(s),z'(s),\mu)\,ds + \int_a^b h(t,s,z(s),z'(s),\mu)\,ds, \tag{4.6}$$

and

$$z(t) = f(t) + \int_a^t g(t,s,z(s),z'(s),\mu_0)\,ds + \int_a^b h(t,s,z(s),z'(s),\mu_0)\,ds, \tag{4.7}$$

for $t \in I$, where $z, f, g, h$ are in $\mathbb{R}^n$ and $\mu, \mu_0$ are real parameters. We assume that $f \in C(I, \mathbb{R}^n)$; $g, h \in C(I^2 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and are continuously differentiable with respect to $t$ on the respective domains of their definitions.

Finally, we present the following theorem which deals with the continuous dependency of solutions of equations (4.6) and (4.7) on parameters.

**Theorem 4.2.** *Assume that the functions $g, h$ in equations (4.6) and (4.7) and their derivatives with respect to $t$ satisfy the conditions*

$$|g(t,s,u,v,\mu) - g(t,s,\bar{u},\bar{v},\mu)| \leq k_1(t,s)[|u-\bar{u}| + |v-\bar{v}|], \tag{4.8}$$

$$|g(t,s,u,v,\mu) - g(t,s,u,v,\mu_0)| \leq \delta_1(t,s)|\mu-\mu_0|, \tag{4.9}$$

$$|h(t,s,u,v,\mu) - h(t,s,\bar{u},\bar{v},\mu)| \leq r_1(t,s)[|u-\bar{u}| + |v-\bar{v}|], \tag{4.10}$$

$$|h(t,s,u,v,\mu) - h(t,s,u,v,\mu_0)| \leq \gamma_1(t,s)|\mu-\mu_0|, \tag{4.11}$$

$$\left|\frac{\partial}{\partial t}g(t,s,u,v,\mu) - \frac{\partial}{\partial t}g(t,s,\bar{u},\bar{v},\mu)\right| \leq k_2(t,s)[|u-\bar{u}| + |v-\bar{v}|], \tag{4.12}$$

$$\left|\frac{\partial}{\partial t}g(t,s,u,v,\mu) - \frac{\partial}{\partial t}g(t,s,u,v,\mu_0)\right| \leq \delta_2(t,s)|\mu-\mu_0|, \tag{4.13}$$

$$\left|\frac{\partial}{\partial t}h(t,s,u,v,\mu) - \frac{\partial}{\partial t}h(t,s,\bar{u},\bar{v},\mu)\right| \leq r_2(t,s)[|u-\bar{u}| + |v-\bar{v}|], \tag{4.14}$$

$$\left|\frac{\partial}{\partial t}h(t,s,u,v,\mu) - \frac{\partial}{\partial t}h(t,s,u,v,\mu_0)\right| \leq \gamma_2(t,s)|\mu-\mu_0|, \tag{4.15}$$

*where $k_i(t,s), r_i(t,s) \in C(I^2, \mathbb{R}_+)$ $(i = 1, 2)$ are nondecreasing in $t \in I$, for each $s \in I$ and $\delta_i(t,s), \gamma_i(t,s) \in C(I^2, \mathbb{R}_+)$ $(i = 1, 2)$. Further, assume that*

$$k_1(t,t) \leq \lambda, \tag{4.16}$$

$$e_0(t) = \int_a^b \frac{1}{1-\lambda}\bar{r}(t,s)\exp\left(\int_a^s \frac{1}{1-\lambda}\bar{k}(s,\sigma)d\sigma\right)ds < 1, \tag{4.17}$$

$$\delta_1(t,t) + \int_a^t [\delta_1(t,s) + \delta_2(t,s)]ds + \int_a^b [\gamma_1(t,s) + \gamma_2(t,s)]ds \leq M, \tag{4.18}$$

*for $t \in I$ where $\lambda, M$ are nonnegative constants such that $\lambda < 1$ and*

$$\bar{k}(t,s) = k_1(t,s) + k_2(t,s), \qquad \bar{r}(t,s) = r_1(t,s) + r_2(t,s).$$

*Let $z_1(t)$ and $z_2(t)$ be the solutions of equations (4.6) and (4.7) respectively. Then*

$$\begin{aligned} &|z_1(t) - z_2(t)| + |z_1'(t) - z_2'(t)| \\ &\qquad \leq \left(\frac{|\mu-\mu_0| M}{1-\lambda}\right)\left(\frac{1}{1-e_0(t)}\right)\exp\left(\int_a^t \bar{k}(t,s)\,ds\right), \end{aligned} \tag{4.19}$$

*for $t \in I$.*

*Proof.* Let $u(t) = |z_1(t) - z_2(t)| + |z_1'(t) - z_2'(t)|$ for $t \in I$. Using the facts that $z_1(t)$ and $z_2(t)$ are the solutions of the equations (4.6) and (4.7) and the hypotheses we have

$$
\begin{aligned}
(4.20)\qquad u(t) \le & \int_a^t |g(t,s,z_1(s),z_1'(s),\mu) - g(t,s,z_2(s),z_2'(s),\mu)|\,ds \\
&+ \int_a^t |g(t,s,z_2(s),z_2'(s),\mu) - g(t,s,z_2(s),z_2'(s),\mu_0)|\,ds \\
&+ \int_a^b |h(t,s,z_1(s),z_1'(s),\mu) - h(t,s,z_2(s),z_2'(s),\mu)|\,ds \\
&+ \int_a^b |h(t,s,z_2(s),z_2'(s),\mu) - h(t,s,z_2(s),z_2'(s),\mu_0)|\,ds \\
&+ |g(t,t,z_1(t),z_1'(t),\mu) - g(t,t,z_2(t),z_2'(t),\mu)| \\
&+ |g(t,t,z_2(t),z_2'(t),\mu) - g(t,t,z_2(t),z_2'(t),\mu_0)| \\
&+ \int_a^t \left|\frac{\partial}{\partial t}g(t,s,z_1(s),z_1'(s),\mu) - \frac{\partial}{\partial t}g(t,s,z_2(s),z_2'(s),\mu)\right| ds \\
&+ \int_a^t \left|\frac{\partial}{\partial t}g(t,s,z_2(s),z_2'(s),\mu) - \frac{\partial}{\partial t}g(t,s,z_2(s),z_2'(s),\mu_0)\right| ds \\
&+ \int_a^b \left|\frac{\partial}{\partial t}h(t,s,z_1(s),z_1'(s),\mu) - \frac{\partial}{\partial t}h(t,s,z_2(s),z_2'(s),\mu)\right| ds \\
&+ \int_a^b \left|\frac{\partial}{\partial t}h(t,s,z_2(s),z_2'(s),\mu) - \frac{\partial}{\partial t}h(t,s,z_2(s),z_2'(s),\mu_0)\right| ds \\
\le & \int_a^t k_1(t,s)\,u(s)\,ds + \int_a^t \delta_1(t,s)\,|\mu-\mu_0|\,ds \\
&+ \int_a^b r_1(t,s)\,u(s)\,ds + \int_a^b \gamma_1(t,s)\,|\mu-\mu_0|\,ds \\
&+ k_1(t,t)\,u(t) + \delta_1(t,t)\,|\mu-\mu_0| \\
&+ \int_a^t k_2(t,s)\,u(s)\,ds + \int_a^t \delta_2(t,s)\,|\mu-\mu_0|\,ds \\
&+ \int_a^b r_2(t,s)\,u(s)\,ds + \int_a^b \gamma_2(t,s)\,|\mu-\mu_0|\,ds.
\end{aligned}
$$

Using (4.16), (4.18) in (4.20) we observe that

$$
(4.21)\qquad u(t) \le \frac{|\mu-\mu_0|\,M}{1-\lambda} + \frac{1}{1-\lambda}\int_a^t \bar{k}(t,s)\,u(s)\,ds + \frac{1}{1-\lambda}\int_a^b \bar{r}(t,s)\,u(s)\,ds.
$$

Now an application of Lemma 2.1 to (4.21) yields (4.19), which shows the dependency of solutions to equations (4.6) and (4.7) on parameters. □

**Remark 3.** We note that our approach to the study of the more general equation (1.1) is different from those used in [1] and we believe that the results obtained here are of independent interest.

## REFERENCES

[1] A.M. BICA, V.A. CĂUŞ AND S. MUREŞAN, Application of a trapezoid inequality to neutral Fredholm integro-differential equations in Banach spaces, *J. Inequal. Pure and Appl. Math.*, **7**(5) (2006), Art. 173. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=791`].

[2] A. BIELECKI, Un remarque sur l'application de la méthode de Banach-Cacciopoli-Tikhonov dans la théorie des équations differentilles ordinaires, *Bull. Acad. Polon. Sci. Sér. Sci. Math.Phys. Astr.*, **4** (1956), 261–264.

[3] T.A. BURTON, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[4] C. CORDUNEANU, Bielecki's method in the theory of integral equations, *Ann. Univ.Mariae Curie-Sklodowska, Section A*, **38** (1984), 23–40.

[5] C. CORDUNEANU, *Integral Equations and Applications*, Cambridge University Press, 1991.

[6] M.A. KRASNOSELSKII, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford, 1964.

[7] M. KWAPISZ, Bieleck's method, existence and uniqueness results for Volterra integral equations in $L^p$ spaces, *J. Math. Anal. Appl.*, **154** (1991), 403–416.

[8] R.K. MILLER, *Nonlinear Volterra Integral Equations*, W.A. Benjamin, Menlo Park CA, 1971.

[9] B.G. PACHPATTE, On a nonlinear Volterra Integral-Functional equation, *Funkcialaj Ekvacioj*, **26** (1983), 1–9.

[10] B.G. PACHPATTE, *Integral and Finite Difference Inequalities and Applications*, North-Holland Mathematics Studies, Vol. 205 Elsevier Science, Amsterdam, 2006.

[11] B.G. PACHPATTE, On higher order Volterra-Fredholm integrodifferential equation, *Fasciculi Mathematici*, **37** (2007), 35–48.

[12] W. WALTER, *Differential and Integral Inequalities*, Springer-Verlag, Berlin, New York, 1970.