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SECOND ORDER DIFFERENTIAL SUBORDINATIONS OF HOLOMORPHIC MAPPINGS ON BOUNDED CONVEX BALANCED DOMAINS IN $\mathbb{C}^n$

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ABSTRACT. In this paper, we obtain some second order differential subordinations of holomorphic mappings on a bounded convex balanced domain Ω in $\mathbb{C}^n$. These results imply some first order differential subordinations of holomorphic mappings on a bounded convex balanced domain Ω in $\mathbb{C}^n$. When Ω is the unit disc in the complex plane $\mathbb{C}$, these results are just ones of Miller and Mocanu et al. about differential subordinations of analytic functions on the unit disc in the complex plane $\mathbb{C}$.

Key words and phrases: Differential subordination, biholomorphic convex mapping, convex balanced domain, Minkowski functional.

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1. INTRODUCTION

Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, z_2, \dots, z_n)$ with the Euclidian inner product $\langle z, w\rangle = \sum_{j=1}^n z_j\overline{w}_j$ and the norm $\|z\| = \sqrt{\langle z, z\rangle}$. A domain Ω is called a balanced domain in $\mathbb{C}^n$ if $\lambda z \in \Omega$ for all $z \in \Omega$ and $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$. The Minkowski functional of the balanced domain Ω is

$$\rho(z) = \inf\left\{t > 0, \frac{z}{t} \in \Omega\right\}, \quad z \in \mathbb{C}^n.$$

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Suppose that Ω is a bounded convex balanced domain in $\mathbb{C}^n$, and $\rho(z)$ is the Minkowski functional of Ω. Then $\rho(\cdot)$ is a norm of $\mathbb{C}^n$ such that

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 1\}, \quad \rho(\lambda z) = |\lambda| \rho(z)$$

for $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$ (see [20]).

Let $p_j > 1$ $(j = 1, 2, \ldots, n)$. Then

$$D_p = \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{p_j} < 1 \right\}$$

is a bounded convex balanced domain, and the Minkowski functional $\rho(z)$ of D_p satisfies

$$\sum_{j=1}^{n} \left| \frac{z_j}{\rho(z)} \right|^{p_j} = 1. \tag{1.1}$$

$\rho(z) = \left(\sum_{j=1}^{n} |z_j|^p \right)^{1/p}$ is the Minkowski functional of domain $B_p = \left\{ z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^p < 1 \right\}$, where $p > 1$.

Let $Df(z)$ and $D^2 f(z)(\cdot, \cdot)$ denote the first Fréchet derivative and the second Fréchet derivative for a holomorphic mapping $f : \Omega \to \mathbb{C}^n$ respectively. Then they have the matrix representation

$$Df(z) = \left(\frac{\partial f_j(z)}{\partial z_k} \right)_{1 \le j,\, k \le n}, \quad D^2 f(z)(b, \cdot) = \left(\sum_{l=1}^{n} \frac{\partial^2 f_j(z)}{\partial z_k \partial z_l} b_l \right)_{1 \le j,\, k \le n},$$

where $b = (b_1, b_2, \ldots, b_n) \in \mathbb{C}^n$. The mapping $f : \Omega \to \mathbb{C}^n$ is called locally biholomorphic if the matrix $Df(z)$ is nonsingular at each point z in Ω.

The class of all holomorphic mappings $f : \Omega \to \mathbb{C}^n$ is denoted by $H(\Omega, \mathbb{C}^n)$. Assume $f, g \in H(\Omega, \mathbb{C}^n)$. Then we say that the mapping f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a holomorphic mapping $w : \Omega \to \Omega$ with $w(0) = 0$ such that $f(z) \equiv g(w(z))$ for all $z \in \Omega$. If g is a biholomorphic mapping, then $f(z) \prec g(z)$ if and only if $f(\Omega) \subset g(\Omega)$ and $f(0) = g(0)$.

In classical results of geometric function theory, differential subordinations provide some simple proofs. They play a key role in the study of some integral operators, differential equations, and properties of subclasses of univalent functions, etc. S.S. Miller and P.T. Mocanu et al. have obtained some deep results for differential subordinations [10, 11, 12, 13, 16, 14]. There is a excellent text *Differential Subordinations Theory and Applications*, by S.S. Miller and P.T. Mocanu [15].

The geometric function theory of several complex variables has been studied by many authors. Many important results for biholomorphic convex or starlike mappings in $\mathbb{C}^n$ have been obtained (see [2, 3]). Some differential subordinations of analytic functions in the complex plane are also extended to $\mathbb{C}^n$ [4, 6, 8, 15, 22]. But there are very few results on second order differential subordinations of holomorphic mappings in $\mathbb{C}^n$.

In this paper, we obtain some second order differential subordinations of holomorphic mappings on a bounded convex balanced domain Ω in $\mathbb{C}^n$. These results imply some first order differential subordinations of holomorphic mappings on a bounded convex balanced domain Ω in $\mathbb{C}^n$. When Ω is the unit disc in the complex plane $\mathbb{C}$, these results are just those of Miller and Mocanu et al. about differential subordinations of analytic functions on the unit disc in the complex plane $\mathbb{C}$.

2. MAIN RESULTS AND THEIR PROOFS

In the following, we always assume that the domain Ω is a bounded convex balanced domain in $\mathbb{C}^n$ and $\rho(z)$ is the Minkowski functional of Ω. Then $\rho(\cdot)$ is a norm of $\mathbb{C}^n$ such that

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 1\}, \quad \rho(\lambda z) = |\lambda|\rho(z)$$

for $\lambda \in \mathbb{C}, z \in \mathbb{C}^n$.

In order to derive our main results, we need the following lemmas.

Lemma 2.1. *Suppose that $\rho(z)$ is twice differentiable in $\Omega - \{0\}$, and let $w \in H(\Omega, \mathbb{C}^n)$ with $w(z) \not\equiv 0$ and $w(0) = 0$. If $z_0 \in \Omega - \{0\}$ satisfies*

$$\rho(w(z_0)) = \max_{\rho(z)\le\rho(z_0)} \rho(w(z)),$$

then there exists a real number $t \geq 1/2$ such that

$$\left\langle Dw(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0) \right\rangle = t\rho(w_0), \tag{2.1}$$

and

$$\begin{aligned} \operatorname{Re}\left\langle D^2w(z_0)(z_0,z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \\ \geq \operatorname{Re}\left\{\sum_{j,l=1}^n \frac{\partial^2\rho}{\partial z_j \partial \overline{z}_l}(w_0) b_j \overline{b}_l - \sum_{j,l=1}^n \frac{\partial^2\rho}{\partial z_j \partial z_l}(w_0) b_j b_l\right\} - t\rho(w_0), \end{aligned} \tag{2.2}$$

where $w_0 = w(z_0), Dw(z_0)(z_0) = (b_1, b_2, \ldots, b_n)$.

Proof. Since $\rho(w_0) = \max_{\rho(z)\le\rho(z_0)} \rho(w(z))$, then we have $w_0 \neq 0$. Otherwise, there is $w(z) \equiv 0$, which contradicts the hypothesis of Lemma 2.1.

Let $w(z) = (w_1(z), w_2(z), \ldots, w_n(z))$, $\gamma(t) = w(e^{it}z_0) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$. Then we have $\gamma_j(t) = w_j(e^{it}z_0)$ $(j = 1, 2, \ldots, n)$, $\gamma(0) = w(z_0) = w_0$ and

$$\frac{d\gamma_j(t)}{dt} = ie^{it}\sum_{k=1}^n \frac{\partial w_j(e^{it}z_0)}{\partial z_k} z_k^0, \qquad \frac{d\overline{\gamma_j(t)}}{dt} = -ie^{-it}\sum_{k=1}^n \left(\overline{\frac{\partial w_j(e^{it}z_0)}{\partial z_k} z_k^0}\right),$$

where $z_0 = (z_1^0, z_2^0, \ldots, z_n^0)$. Set $L(t) = \rho(\gamma(t))$ $(-\pi \leq t \leq \pi)$. Some straightforward calculations yield

$$\begin{aligned} L'(t) &= \sum_{j=1}^n \frac{\partial\rho}{\partial z_j}(\gamma(t)) \cdot \frac{d\gamma_j(t)}{dt} + \sum_{j=1}^n \frac{\partial\rho}{\partial \overline{z}_j}(\gamma(t)) \cdot \frac{d\overline{\gamma_j(t)}}{dt} \\ &= -2\operatorname{Im}\left[e^{it}\sum_{j,k=1}^n \frac{\partial\rho}{\partial z_j}(\gamma(t)) \cdot \frac{\partial w_j(e^{it}z_0)}{\partial z_k} z_k^0\right] \\ &= -2\operatorname{Im}\left\langle Dw(e^{it}z_0)(e^{it}z_0), \overline{\frac{\partial\rho}{\partial z}}(\gamma(t))\right\rangle, \end{aligned}$$

$$L''(t) = -2\,\mathrm{Im}\left[ie^{it}\sum_{j,k=1}^{n}\frac{\partial\rho}{\partial z_j}(\gamma(t))\cdot\frac{\partial w_j}{\partial z_k} + ie^{2it}\sum_{j,k=1}^{n}\frac{\partial\rho}{\partial z_j}(\gamma(t))\sum_{l=1}^{n}\frac{\partial^2 w_j}{\partial z_k\partial z_l}z_k^0 z_l^0\right]$$
$$-2\,\mathrm{Im}\left[i\sum_{j,k=1}^{n}\sum_{l,m=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial z_l}(\gamma(t))\cdot\frac{\partial w_l}{\partial z_m}\cdot(e^{it}z_m^0)\frac{\partial w_j}{\partial z_k}\cdot(e^{it}z_k^0)\right]$$
$$+2\,\mathrm{Im}\left[i\sum_{j,k=1}^{n}\sum_{l,m=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_l}(\gamma(t))\cdot\overline{\left(\frac{\partial w_l}{\partial z_m}\cdot(e^{it}z_m^0)\right)}\frac{\partial w_j}{\partial z_k}\cdot(e^{it}z_k^0)\right].$$

Noting $L(0) = \max_{-\pi\le t\le\pi} L(t)$, we have $L'(0) = 0$ and $L''(0) \le 0$. It follows that

$$\mathrm{Im}\left\langle Dw(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle = 0, \tag{2.3}$$

and

$$\mathrm{Re}\left\langle D^2w(z_0)(z_0,z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle + \mathrm{Re}\left\langle Dw(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle + \mathrm{Re}\left[\sum_{j,l=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial z_l}(w_0)b_jb_l - \frac{\partial^2\rho}{\partial z_j\partial\overline{z_l}}(w_0)b_j\overline{b}_l\right] \ge 0. \tag{2.4}$$

On the other hand, by Schwarz's Lemma in $\mathbb{C}^n$ [19], we have

$$\frac{\rho(w(z))}{\rho(z)} \le \frac{\rho(w_0)}{\rho(z_0)} \quad \text{for } 0<\rho(z)\le\rho(z_0).$$

Let

$$\varphi(r) = \frac{\rho(w(rz_0))}{\rho(rz_0)} = \frac{\rho(w(rz_0))}{r\rho(z_0)}.$$

Then $\varphi(1) = \max_{0<r\le 1}\varphi(r)$. It follows that

$$\varphi'(1) = \lim_{r\to 1^-}\frac{\varphi(r)-\varphi(1)}{r-1} \ge 0.$$

By a simple calculation, we obtain

$$\varphi'(1) = -\frac{\rho(w_0)}{\rho(z_0)} + \frac{2}{\rho(z_0)}\mathrm{Re}\left\langle Dw(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \ge 0.$$

If we let

$$t = \frac{1}{\rho(w_0)}\mathrm{Re}\left\langle Dw(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle,$$

then we have $t \ge 1/2$, therefore (2.1) of Lemma 2.1 holds, and (2.2) follows from (2.3) and (2.4). This completes the proof. □

Remark 2.2. Since $\rho(tz) = t\rho(z)$ for $t>0$, then for $z\in\mathbb{C}^n - \{0\}$, we have

$$\rho(z) = \left.\frac{d\rho(tz)}{dt}\right|_{t=1} = \sum_{j=1}^{n}\frac{\partial\rho}{\partial z_j}z_j + \sum_{j=1}^{n}\frac{\partial\rho}{\partial\overline{z}_j}\overline{z}_j = 2\,\mathrm{Re}\left\langle z, \overline{\frac{\partial\rho}{\partial z}}(z)\right\rangle. \tag{2.5}$$

For any $z \in \mathbb{C}^n - \{0\}$, we have $\rho(\frac{z}{\rho(z)}) = 1$. Letting $w(z) \equiv z$ in (2.1), we obtain that there exists a real number $t \geq \frac{1}{2}$ such that

$$\left\langle z, \overline{\frac{\partial \rho}{\partial z}}(z) \right\rangle = t\rho(z) \geq 0, \quad z \in \mathbb{C}^n - \{0\}.$$

Hence it follows from (2.5) that

$$\rho(z) = 2\left\langle z, \overline{\frac{\partial \rho}{\partial z}}(z) \right\rangle, \quad z \in \mathbb{C}^n - \{0\}.$$

Lemma 2.3 ([10])**.** *Let* $g(\xi) = a + b_1\xi + b_2\xi^2 + \cdots$ *be analytic in* $|\xi| < 1$ *with* $g(\xi) \not\equiv 0$. *If* $\xi_0 = r_0e^{i\theta_0}$ $(0 < r_0 < 1)$ *and* $\operatorname{Re} g(\xi_0) = \min_{|\xi|\leq r_0} \operatorname{Re} g(\xi)$, *then*

$$\xi_0 g'(\xi_0) \leq -\frac{|\, a - g(\xi_0)\, |^2}{2\operatorname{Re}(a - g(\xi_0))}, \tag{2.6}$$

and

$$\operatorname{Re}\{\xi_0^2 g''(\xi_0) + \xi_0 g'(\xi_0)\} \leq 0. \tag{2.7}$$

Lemma 2.4. *Suppose that* $\rho(z)$ *is differentiable in* $\Omega - \{0\}$. *Let* $h : \Omega \to \mathbb{C}^n$ *be a biholomorphic convex mapping with* $h(0) = 0$. *Then for every* $z \in \Omega - \{0\}$, *we have*

$$\left| 2\left\langle Dh(z)^{-1}h(z), \overline{\frac{\partial \rho}{\partial z}}(z) \right\rangle - \rho(z) \right| \leq \rho(z).$$

Proof. For each $z \in \Omega - \{0\}$, we let $g(\xi) = \langle Dh(z)^{-1}(h(z) - h(\xi z)), \overline{\frac{\partial \rho}{\partial z}}(z)\rangle$ for $|\xi| \leq 1$. Then $g(\xi)$ is analytic in $|\xi| \leq 1$ and

$$g(\xi) = \left\langle Dh(z)^{-1}h(z), \overline{\frac{\partial \rho}{\partial z}}(z) \right\rangle + b_1\xi + \cdots .$$

From the result in [3, 7], we have $\operatorname{Re} g(\xi) > 0$ for all $|\xi| < 1$. Hence we obtain

$$0 = \operatorname{Re} g(1) = \min_{|\xi|\leq 1} \operatorname{Re} g(\xi).$$

By a simple calculation, we may obtain

$$g'(1) = -\left\langle Dh(z)^{-1}Dh(z)(z), \overline{\frac{\partial \rho}{\partial z}}(z) \right\rangle = -\left\langle z, \overline{\frac{\partial \rho}{\partial z}}(z) \right\rangle = -\frac{\rho(z)}{2}.$$

By (2.6), we have

$$-\rho(z)\operatorname{Re} a + |a|^2 \leq 0,$$

where $a = \left\langle (Dh(z)^{-1}h(z), \overline{\frac{\partial \rho}{\partial z}}(z) \right\rangle$. It follows that

$$\left| 2\left\langle Dh(z)^{-1}h(z), \overline{\frac{\partial \rho}{\partial z}}(z) \right\rangle - \rho(z) \right| \leq \rho(z).$$

□

Lemma 2.5 ([23])**.** *Suppose that* $\rho(z)$ *is twice differentiable in* $\Omega - \{0\}$. *If* $f : \Omega \to \mathbb{C}^n$ *is a biholomorphic convex mapping, then we have*

$$\operatorname{Re}\left\{ \sum_{l,m=1}^{n} \frac{\partial^2 \rho}{\partial z_l \partial z_m} b_l b_m + \sum_{l,m=1}^{n} \frac{\partial^2 \rho}{\partial z_l \partial \overline{z}_m} b_l \overline{b}_m - \left\langle Df(z)^{-1}D^2 f(z)(b,b), \overline{\frac{\partial \rho}{\partial z}} \right\rangle \right\} \geq 0 \tag{2.8}$$

for every $z = (z_1, z_2, \dots, z_n) \in \Omega - \{0\}$, $b = (b_1, b_2, \dots, b_n) \in \mathbb{C}^n$ *with Re*$\langle b, \overline{\frac{\partial \rho}{\partial z}}\rangle = 0$.

Lemma 2.6. *Assume that $\rho(z)$ is differentiable in $\Omega - \{0\}$. Then*

$$\rho(z) = 2\left\langle z, \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle, \quad z \in \mathbb{C}^n - \{0\}, \tag{2.9}$$

and

$$\left|2\left\langle w, \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle\right| \le \rho(w), \quad z \in \mathbb{C}^n - \{0\},\ w \in \mathbb{C}^n. \tag{2.10}$$

Proof. From Remark 2.2, we only need to prove (2.10). Let $z \in \mathbb{C}^n - \{0\}$ and

$$\Omega_z = \{w \in \mathbb{C}^n : \rho(w) < \rho(z)\}.$$

Then Ω_z is a convex domain in $\mathbb{C}^n$, and $\overline{\frac{\partial \rho}{\partial z}}(z)$ is the normal vector of $\partial\Omega_z$ at z. For every $z, w \in \mathbb{C}^n$ with $\rho(z) = 1, \rho(w) = 1$, we have $\operatorname{Re}\left\langle z - w, \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle \ge 0$. It follows that

$$2\operatorname{Re}\left\langle w, \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle \le 2\operatorname{Re}\left\langle z, \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle = \rho(z) = 1. \tag{2.11}$$

When $\left\langle w, \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle = 0$, it is obvious that (2.10) holds.

When $\left\langle w, \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle \ne 0$, then $\rho(w) \ne 0$. Using $\frac{z}{\rho(z)}$ to substitute for z and $\frac{w}{\rho(w)}e^{-i\theta}$ to substitute for w in (2.11), we obtain

$$\left|2\operatorname{Re}\left\langle w, \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle\right| \le \rho(w), \quad z \in \mathbb{C}^n - \{0\},\ w \in \mathbb{C}^n,$$

where $\theta = \arg\left\langle w, \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle$ and $\frac{\partial \rho}{\partial z}(\lambda z) = \frac{\partial \rho}{\partial z}(z)$ for all $\lambda \in (0, +\infty)$ and $z \in \Omega - \{0\}$. This completes the proof. □

Lemma 2.7. *Suppose that $\rho(z)$ is differentiable in $\Omega - \{0\}$, and let $h : \Omega \to \mathbb{C}^n$ be a biholomorphic convex mapping with $h(0) = 0$. Then for every $z \in \Omega - \{0\}$ and vector $\xi \in \mathbb{C}^n$, the inequality*

$$\left|2\left\langle Dh(z)^{-1}(\xi), \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle\right| \le (1 + \rho(z))^2 \rho(Dh(0)^{-1}(\xi))$$

holds.

Proof. Without loss of generality, we may assume that h is a biholomorphic convex mapping on $\overline{\Omega}$. If not, then we can replace $h(z)$ by $h_r(z) = h(rz)$, where $0 < r < 1$.

For any fixed $z \in \Omega - \{0\}$, from the proof of Theorem 2.1 in [5, 9], there exist $\widetilde{z} \in \partial\Omega$ and $\mu \in (0, 1)$ such that $h(z) = \mu h(\widetilde{z})$ and

$$1 - \mu \ge \frac{1 - \rho(z)}{1 + \rho(z)}.$$

Let $g(w) = h^{-1}[(1 - \mu)h(w) + \mu h(\widetilde{z})]$. Since h is a biholomorphic convex mapping on Ω, then $g \in H(\Omega, \mathbb{C}^n)$ with $g(\Omega) \subset \Omega$ and $g(0) = z$. For every $\xi \in \mathbb{C}^n - \{0\}$, we set

$$\psi(\lambda) = 2\left\langle g\left(\lambda \frac{\xi}{\rho(\xi)}\right), \overline{\frac{\partial \rho}{\partial z}}(z)\right\rangle,$$

then $\psi(\lambda)$ is an analytic function in $|\lambda| < 1$. By Lemma 2.6, we obtain

$$|\psi(\lambda)| \le \rho\left(g\left(\lambda \frac{\xi}{\rho(\xi)}\right)\right) < 1$$

for all $|\lambda| < 1$, and

$$\psi(\lambda) = \rho(z) + 2\left\langle Dg(0)\left(\frac{\xi}{\rho(\xi)}\right), \overline{\frac{\partial\rho}{\partial z}}(z)\right\rangle \lambda + \cdots .$$

From the classical result in [1], we have $|\psi'(0)| \leq 1 - |\psi(0)|^2$. It follows that

$$\left|2\left\langle Dg(0)(\xi), \overline{\frac{\partial\rho}{\partial z}}(z)\right\rangle\right| \leq (1-\rho(z)^2)\rho(\xi).$$

Since $Dh(z)Dg(0) = (1-\mu)Dh(0)$, then

$$\left|2\left\langle Dh(z)^{-1}Dh(0)(\xi), \overline{\frac{\partial\rho}{\partial z}}(z)\right\rangle\right| \leq \frac{1}{1-\mu}(1-\rho(z)^2)\rho(\xi) \leq (1+\rho(z))^2\rho(\xi)$$

for all $\xi \in \mathbb{C}^n$, $z \in \Omega - \{0\}$.

Set $\zeta = Dh(0)(\xi)$, then $\xi = Dh(0)^{-1}\zeta$ and

$$\left|2\left\langle Dh(z)^{-1}(\zeta), \overline{\frac{\partial\rho}{\partial z}}(z)\right\rangle\right| \leq (1+\rho(z))^2\rho(Dh(0)^{-1}(\zeta)),$$

which completes the proof. □

Theorem 2.8. *Let $f, g \in H(\Omega, \mathbb{C}^n)$ with $f(0) = g(0)$, and let g be biholomorphic convex on $\overline{\Omega}$. Suppose that $\rho(z)$ is twice differentiable in $\Omega - \{0\}$. If f is not subordinate to g, then there exist points $z_0 \in \Omega - \{0\}, w_0 \in \partial\Omega$ with $0 < \rho(z_0) < 1, \rho(w_0) = 1$ and there is a real number $t \geq 1/2$ such that*

(1) $f(z_0) = g(w_0)$,

(2) $\left\langle Dg(w_0)^{-1}Df(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle = t$, *and*

(3) $\operatorname{Re}\left\langle Dg(w_0)^{-1}D^2f(z_0)(z_0,z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \geq -t$.

Proof. If f is not subordinate to g, then there exist points $z_0 \in \Omega - \{0\}, w_0 \in \partial\Omega$ with $0 < \rho(z_0) < 1, \rho(w_0) = 1$ such that $f(z_0) = g(w_0)$ and $f(D_r) \subset g(\Omega)$, where $D_r = \{z \in \mathbb{C}^n : \rho(z) < r\}$ and $r = \rho(z_0)$.

Let $w(z) = g^{-1}(f(z))$. Then $w : D_r \to \Omega$ is a holomorphic mapping with $w(z) \not\equiv 0$ and $w(0) = 0$ satisfying $f(z) = g(w(z))$ for $z \in D_r$. Hence

$$1 = \rho(w_0) = \max_{\rho(z)\leq\rho(z_0)} \rho(w(z)).$$

By a simple calculation, we have

$$\begin{aligned}
Dw(z_0)(z_0) &= Dg(w_0)^{-1}Df(z_0)(z_0),\\
Dg(w_0)^{-1}D^2f(z_0)(z_0,z_0) &= Dg(w_0)^{-1}D^2g(w_0)(Dw(z_0)(z_0), Dw(z_0)(z_0))\\
&\quad + D^2w(z_0)(z_0,z_0).
\end{aligned}$$

From (2.1), there is a real number $t \geq 1/2$ such that

$$\left\langle Dw(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle = t\rho(w_0) = t.$$

So we obtain

$$\left\langle Dg(w_0)^{-1}Df(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle = t,$$

and

$$\begin{aligned}
&\operatorname{Re}\left\langle Dg(w_0)^{-1}D^2f(z_0)(z_0,z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \\
&= \operatorname{Re}\left\langle Dg(w_0)^{-1}D^2g(w_0)(Dw(z_0)(z_0), Dw(z_0)(z_0)), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \\
&\qquad + \operatorname{Re}\left\langle D^2w(z_0)(z_0,z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \\
&\geq \operatorname{Re}\left\langle Dg(w_0)^{-1}D^2g(w_0)(a,a), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \\
&\qquad + \operatorname{Re}\left\{\sum_{j,l=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_l}(w_0)a_j\overline{a_l} - \sum_{j,l=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial z_l}(w_0)a_ja_l\right\} - t,
\end{aligned}$$

where $a = Dw(z_0)(z_0) = (a_1, a_2, \ldots, a_n)$. If we let $b = (b_1, b_2, \ldots, b_n)$ with $b_j = ia_j$, then we have

$$\operatorname{Re}\left\langle b, \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle = \operatorname{Re}\left\{i\left\langle Dw(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle\right\} = \operatorname{Re}\{it\} = 0.$$

From Lemma 2.5, we obtain

$$\begin{aligned}
&\operatorname{Re}\left\langle Dg(w_0)^{-1}D^2f(z_0)(z_0,z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \\
&\geq -\operatorname{Re}\left\langle Dg(w_0)^{-1}D^2g(w_0)(b,b), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \\
&\qquad + \operatorname{Re}\left\{\sum_{j,l=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial\overline{z}_l}(w_0)b_j\overline{b_l} + \sum_{j,l=1}^{n}\frac{\partial^2\rho}{\partial z_j\partial z_l}(w_0)b_jb_l\right\} - t \\
&\geq -t.
\end{aligned}$$

This completes the proof. □

Remark 2.9. When $n = 1$, Ω is the unit disc in the complex plane $\mathbb{C}$ and $\rho(z) = |z| (z \in \mathbb{C})$, we may obtain Lemma 1 in [14] from Theorem 2.8. Theorem 2.8 will play a key role in studying some second order differential subordinations of holomorphic mappings on a bounded convex balanced domain Ω in $\mathbb{C}^n$.

Let Ω_1 be a set of $\mathbb{C}^n$, and let h be a biholomorphic convex mapping on $\overline{\Omega}$. Suppose that $\rho(z)$ is twice differentiable in $\Omega - \{0\}$. We define $\Psi(\Omega_1, h)$ to be the class of maps $\psi : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \times \Omega \to \mathbb{C}^n$ that satisfy the following conditions:

(1) $\psi(h(0), 0, 0, 0) \in \Omega_1$, and
(2) $\psi(\alpha, \beta, \gamma, z) \notin \Omega_1$ for $\alpha = h(w)$, $\left\langle Dh(w)^{-1}(\beta), \overline{\frac{\partial\rho}{\partial z}}(w)\right\rangle = t$, $\operatorname{Re}\left\langle Dh(w)^{-1}(\gamma), \overline{\frac{\partial\rho}{\partial z}}(w)\right\rangle \geq -t$, and $z \in \Omega$, where $\rho(w) = 1$ and $t \geq 1/2$.

Theorem 2.10. *Let $\psi \in \Psi(\Omega_1, h)$. If $f \in H(\Omega, \mathbb{C}^n)$ with $f(0) = h(0)$ satisfies*

$$\psi(f(z), Df(z)(z), D^2f(z)(z,z), z) \in \Omega_1 \tag{2.12}$$

for all $z \in \Omega$, then $f(z) \prec h(z)$.

Proof. If f is not subordinate to h, then by Theorem 2.8, there exist points $z_0 \in \Omega - \{0\}$, $w_0 \in \partial\Omega$ with $0 < \rho(z_0) < 1$, $\rho(w_0) = 1$ and there is a real number $t \geq 1/2$ such that

$$f(z_0) = h(w_0), \qquad \left\langle Dh(w_0)^{-1} Df(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0) \right\rangle = t,$$

and

$$\operatorname{Re}\left\langle Dh(w_0)^{-1} D^2 f(z_0)(z_0, z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0) \right\rangle \geq -t.$$

Set $\alpha = f(z_0)$, $\beta = Df(z_0)(z_0)$, $\gamma = D^2 f(z_0)(z_0, z_0)$, then according to the definition of $\Psi(\Omega_1, h)$, we have

$$\psi(f(z_0), Df(z_0)(z_0), D^2 f(z_0)(z_0, z_0), z_0) \notin \Omega_1$$

which contradicts (2.12). Hence $f(z) \prec h(z)$, and the proof of Theorem 2.10 is complete. □

Theorem 2.11. *Let $A \geq 0$, $h \in H(\Omega, \mathbb{C}^n)$ be biholomorphic convex with $h(0) = 0$, and let $\psi(z) \in H(\Omega, \mathbb{C}^n)$ with $\psi(0) = 0$. Suppose that $\rho(z)$ is twice differentiable in $\Omega - \{0\}$, $k > 4\|Dh(0)^{-1}\|$, and $\varphi, \phi : \Omega_1 \times \Omega \to \mathbb{C}$ are holomorphic such that*

$$\operatorname{Re}\phi(\alpha, z) \geq A + |\varphi(\alpha, z) - 1| - \operatorname{Re}[\varphi(\alpha, z) - 1] + k\rho(\psi(z))$$

for all $(\alpha, z) \in h(\Omega) \times \Omega$, where Ω_1 is a domain of $\mathbb{C}^n$ with $h(\Omega) \subset \Omega_1$ and $\|Dh(0)^{-1}\| = \sup_{\rho(\xi)\leq 1} \rho(Dh(0)^{-1}(\xi))$. If $f \in H(\Omega, \mathbb{C}^n)$ with $f(0) = 0$ satisfies

$$AD^2 f(z)(z, z) + \phi(f(z), z)Df(z)(z) + \varphi(f(z), z)f(z) + \psi(z) \prec h(z),$$

then $f(z) \prec h(z)$.

Proof. Without loss of generality, we may assume that f and g satisfy the conditions of Theorem 2.11 on $\overline{\Omega}$. If not, then we can replace $f(z)$ by $f_r(z) = f(rz)$, $\psi(z)$ by $\psi_r(z) = \psi(rz)$, and $h(z)$ by $h_r(z) = h(rz)$, where $0 < r < 1$. We would then prove $f_r(z) \prec h_r(z)$ for all $0 < r < 1$. By letting $r \to 1^-$, we obtain $f(z) \prec h(z)$.

Let

$$\psi(\alpha, \beta, \gamma, z) = A\gamma + \phi(\alpha, z)\beta + \varphi(\alpha, z)\alpha + \psi(z),$$

and let $\alpha = h(w)$, $\left\langle Dh(w)^{-1}(\beta), \overline{\frac{\partial\rho}{\partial z}}(w) \right\rangle = t$, $\operatorname{Re}\left\langle Dh(w)^{-1}(\gamma), \overline{\frac{\partial\rho}{\partial z}}(w) \right\rangle \geq -t$, where $\rho(w) = 1$, $t \geq 1/2$. If we set

$$\psi(\alpha, \beta, \gamma, z) = h(w) + \lambda Dh(w)(w),$$

then we have

$$\lambda w = ADh(w)^{-1}(\gamma) + \phi(\alpha, z)Dh(w)^{-1}(\beta) + [\varphi(\alpha, z) - 1]Dh(w)^{-1}h(w) + Dh(w)^{-1}(\psi(z)).$$

Since $2\left\langle w, \overline{\frac{\partial\rho}{\partial z}}(w) \right\rangle = \rho(w) = 1$ from Remark 2.2, we obtain

$$\lambda = 2A\left\langle Dh(w)^{-1}(\gamma), \overline{\frac{\partial\rho}{\partial z}}(w) \right\rangle + 2\phi(\alpha, z)\left\langle Dh(w)^{-1}(\beta), \overline{\frac{\partial\rho}{\partial z}}(w) \right\rangle + 2[\varphi(\alpha, z) - 1]\left\langle Dh(w)^{-1}h(w), \overline{\frac{\partial\rho}{\partial z}}(w) \right\rangle + 2\left\langle Dh(w)^{-1}(\psi(z)), \overline{\frac{\partial\rho}{\partial z}}(w) \right\rangle. \tag{2.13}$$

By Lemma 2.4, we have

$$\left| 2\left\langle Dh(w)^{-1}h(w), \overline{\frac{\partial\rho}{\partial z}}(w) \right\rangle - 1 \right| \leq 1.$$

By Lemma 2.7, we obtain

$$\begin{aligned}\operatorname{Re}\lambda &\geq -2At + 2t\operatorname{Re}\phi(\alpha,z) + \operatorname{Re}[\varphi(\alpha,z)-1] \\ &\quad - |\varphi(\alpha,z)-1| - 4\|Dh(0)^{-1}\|\rho(\psi(z)) \\ &\geq (2t-1)\{|\varphi(\alpha,z)-1| - \operatorname{Re}[\varphi(\alpha,z)-1]\} \\ &\quad + (k - 4\|Dh(0)^{-1}\|)\rho(\psi(z)) \geq 0. \end{aligned} \tag{2.14}$$

Now we verify that $\psi(\alpha,\beta,\gamma,z) \notin h(\Omega)$. Suppose not, then there exists $w_1 \in \Omega$ such that $\psi(\alpha,\beta,\gamma,z) = h(w_1)$. From the result in [3, 7, 18, 19], we have

$$-\operatorname{Re}\lambda = 2\operatorname{Re}\left\langle Dh(w)^{-1}(h(w)-h(w_1)), \overline{\frac{\partial\rho}{\partial z}}(w)\right\rangle > 0,$$

which contradicts (2.14), hence $\psi(\alpha,\beta,\gamma,z) \notin h(\Omega)$. By Theorem 2.10, we obtain $f(z) \prec h(z)$, and the proof is complete. □

Corollary 2.12. *Let $A \geq 0, h \in H(\Omega,\mathbb{C}^n)$ be biholomorphic convex with $h(0) = 0$, and let $\psi(z) \in H(\Omega,\mathbb{C}^n)$ with $\psi(0) = 0$. Suppose that $k > 4\|Dh(0)^{-1}\|$, $\rho(z)$ is twice differentiable in $\Omega - \{0\}$, and $B(z), C(z) \in H(\Omega,\mathbb{C})$ satisfy*

$$\operatorname{Re} B(z) \geq A + |C(z)-1| - \operatorname{Re}[C(z)-1] + k\rho(\psi(z))$$

for all $z \in \Omega$, where $\|Dh(0)^{-1}\| = \sup\limits_{\rho(\xi)\leq 1} \rho(Dh(0)^{-1}(\xi))$. If $f \in H(\Omega,\mathbb{C}^n)$ with $f(0) = 0$ satisfies

$$AD^2f(z)(z,z) + B(z)Df(z)(z) + C(z)f(z) + \psi(z) \prec h(z),$$

then $f(z) \prec h(z)$.

Corollary 2.13. *Let $A \geq 0, h \in H(\Omega,\mathbb{C}^n)$ be biholomorphic convex. Suppose that $\rho(z)$ is twice differentiable in $\Omega - \{0\}$, and $B(z) \in H(\Omega,\mathbb{C})$ with $\operatorname{Re} B(z) \geq A$ for all $z \in \Omega$. If $f \in H(\Omega,\mathbb{C}^n)$ with $f(0) = h(0)$ satisfies*

$$AD^2f(z)(z,z) + B(z)Df(z)(z) + f(z) \prec h(z),$$

then $f(z) \prec h(z)$.

Corollary 2.14. *Let $h \in H(\Omega,\mathbb{C}^n)$ be biholomorphic convex with $h(0) = 0$. Suppose that $\rho(z)$ is twice differentiable in $\Omega - \{0\}$ and $\phi : \Omega_1 \to \mathbb{C}$ is holomorphic such that $\operatorname{Re}\phi(h(z)) \geq 0$ for all $z \in \Omega$. If $f \in H(\Omega,\mathbb{C}^n)$ with $f(0) = 0$ satisfies*

$$f(z) + \phi(f(z))Df(z)(z) \prec h(z),$$

then $f(z) \prec h(z)$.

Remark 2.15. When $n = 1$, we have $Df(z)(z) = zf'(z)$ and $D^2f(z)(z,z) = z^2f''(z)$. From Corollary 2.12, we may obtain Theorem 2 in [13], Theorem 3.1a in [15], Theorem 1 for case 1 in [12] and Theorem 1 in [14]. From Corollary 2.13, we may obtain Corollary 2.1 in [13].

Example 2.1. Let $\beta > 0$ and $\gamma \in \mathbb{C}$ with $2\operatorname{Re}\gamma \geq \beta$. The unit ball in $\mathbb{C}^n$ is denoted by $B = \{z \in \mathbb{C}^n : \|z\| < 1\}$. If $u \in \mathbb{C}^n$ with $\|u\| = 1$, then $h(z) = \frac{z}{1-\langle z,u\rangle}$ is a biholomorphic convex mapping on B (see [17]). By a simple calculation, we have

$$\operatorname{Re}\left[\beta\left\langle\frac{z}{1-\langle z,u\rangle},u\right\rangle+\gamma\right]=\frac{\operatorname{Re}\gamma|1-\langle z,u\rangle|^2+\beta[\operatorname{Re}\langle z,u\rangle-|\langle z,u\rangle|^2]}{|1-\langle z,u\rangle|^2} \geq \frac{\beta|1-\langle z,u\rangle|^2+2\beta[\operatorname{Re}\langle z,u\rangle-|\langle z,u\rangle|^2]}{2|1-\langle z,u\rangle|^2} = \frac{\beta(1-|\langle z,u\rangle|^2)}{2|1-\langle z,u\rangle|^2}>0 \tag{2.15}$$

for all $z\in B$. If $f\in H(B,\mathbb{C}^n)$ with $f(0)=0$, then by Corollary 2.14, we have

$$f(z)+\frac{Df(z)(z)}{\beta\langle f(z),u\rangle+\gamma}\prec\frac{z}{1-\langle z,u\rangle}\Longrightarrow f(z)\prec\frac{z}{1-\langle z,u\rangle}.$$

Example 2.2. Let $A\geq 0,\beta\geq 0$ and $\gamma\in\mathbb{C}$ with $\operatorname{Re}\gamma\geq\beta/2+A$, $u\in\mathbb{C}^n$ with $\|u\|=1$. If $f\in H(B,\mathbb{C}^n)$ with $f(0)=0$, then by Theorem 2.11, Corollary 2.13 and (2.15), we have

$$AD^2f(z)(z,z)+\left[\beta\frac{\langle z,u\rangle}{1-\langle z,u\rangle}+\gamma\right]Df(z)(z)+f(z)\prec\frac{z}{1-\langle z,u\rangle}\Longrightarrow f(z)\prec\frac{z}{1-\langle z,u\rangle},$$

and

$$AD^2f(z)(z,z)+[\beta\langle f(z),u\rangle+\gamma]Df(z)(z)+f(z)\prec\frac{z}{1-\langle z,u\rangle}\Longrightarrow f(z)\prec\frac{z}{1-\langle z,u\rangle}.$$

Let $\rho(z)$ be differentiable in $\Omega-\{0\}$. For $M>0$, we define $\Psi(M)$ to be the class of maps $\psi:\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^n\times\Omega\to\mathbb{C}^n$ that satisfy the following conditions:

(1) $\rho(\psi(0,0,0,0))<M$ and
(2) $\rho(\psi(\alpha,\beta,\gamma,z))\geq M$ for all $\rho(\alpha)=M,\ 2\left\langle\beta,\overline{\frac{\partial\rho}{\partial z}}(\alpha)\right\rangle=tM,\ 2\operatorname{Re}\left\langle\gamma,\overline{\frac{\partial\rho}{\partial z}}(\alpha)\right\rangle\geq(t^2-t)M$, and $t\geq 1$.

Theorem 2.16. *Let $\psi\in\Psi(M)$. If $w\in H(\Omega,\mathbb{C}^n)$ with $w(0)=0$ satisfies*

$$\rho(\psi(w(z),Dw(z)(z),D^2w(z)(z,z),z))<M \tag{2.16}$$

for all $z\in\Omega-\{0\}$, then $\rho(w(z))<M$ for $z\in\Omega$.

Proof. Suppose that the conclusion of Theorem 2.16 is false. Then there exists a point $z_0\in\Omega-\{0\}$ such that $\rho(w(z_0))=M$ and $\rho(w(z))\leq M$ for $\rho(z)\leq\rho(z_0)$. It implies $w_0=w(z_0)\neq 0$.

Let

$$\varphi(\xi)=2\left\langle w\left(\frac{z_0}{\rho(z_0)}\xi\right),\overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle,\qquad\xi\in\mathbb{C}.$$

Then $\varphi(\xi)$ is an analytic function in $|\xi|<1$. By Lemma 2.6, we have

$$|\varphi(\xi)|\leq\left|2\left\langle w\left(\frac{z_0}{\rho(z_0)}\xi\right),\overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle\right|\leq\rho\left(w\left(\frac{z_0}{\rho(z_0)}\xi\right)\right)\leq\rho(w(z_0))$$

for all $|\xi|\leq\rho(z_0)$, and

$$\varphi(\rho(z_0))=2\left\langle w(z_0),\overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle=\rho(w(z_0))=\max_{|\xi|\leq\rho(z_0)}|\varphi(\xi)|.$$

By a simple calculation, we have

$$\rho(z_0)\varphi'(\rho(z_0))=2\left\langle Dw(z_0)(z_0),\overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle,$$

$$\rho(z_0)^2\varphi''(\rho(z_0)) = 2\left\langle D^2w(z_0)(z_0,z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle.$$

Using Lemma A in [10] (also see [15, p. 19]), there exists a real number $t \geq 1$ such that

$$2\left\langle Dw(z_0)(z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle = tM,$$

$$2\,\mathrm{Re}\left\langle D^2w(z_0)(z_0,z_0), \overline{\frac{\partial\rho}{\partial z}}(w_0)\right\rangle \geq (t^2-t)M.$$

By the definition of ψ, we have

$$\rho(\psi(w(z_0), Dw(z_0)(z_0), D^2w(z_0)(z_0,z_0), z_0)) \geq M,$$

which contradicts (2.16). Hence $\rho(w(z)) < M$ for $z \in \Omega$, and the proof is complete. □

Theorem 2.17. *Let $\rho(z)$ be differentiable in $\Omega - \{0\}$. Suppose that $A(z), B(z), C(z) \in H(\Omega,\mathbb{C})$ with $A(z) \neq 0$ for all $z \in \Omega$ satisfy*

$$\mathrm{Re}\,\frac{B(z)}{A(z)} \geq \max\left\{-1, \frac{1}{|A(z)|} - \mathrm{Re}\,\frac{C(z)}{A(z)} + \frac{\rho(\varphi(z))}{|A(z)|}\right\}, \tag{2.17}$$

or

$$\mathrm{Re}\,\frac{C(z)}{A(z)} \geq \frac{1}{|A(z)|} + \frac{\rho(\varphi(z))}{|A(z)|} + 1 \tag{2.18}$$

$$\text{and}\quad 1 - 2\sqrt{\mathrm{Re}\,\frac{C(z)}{A(z)} - \frac{1}{|A(z)|} - \frac{\rho(\varphi(z))}{|A(z)|}} \leq \mathrm{Re}\,\frac{B(z)}{A(z)} \leq -1$$

for all $z \in \Omega$. If $w(z) \in H(\Omega,\mathbb{C}^n)$ with $w(0) = 0$ satisfies

$$\rho(A(z)D^2w(z)(z,z) + B(z)Dw(z)(z) + C(z)w(z) + \varphi(z)) < 1$$

for all $z \in \Omega$, then $\rho(w(z)) < 1$ for $z \in \Omega$.

Proof. Let

$$\psi(\alpha,\beta,\gamma,z) = A(z)\gamma + B(z)\beta + C(z)\alpha + \varphi(z),$$

where $\rho(\alpha) = 1$, $2\left\langle \beta, \overline{\frac{\partial\rho}{\partial z}}(\alpha)\right\rangle = t$, $2\,\mathrm{Re}\left\langle \gamma, \overline{\frac{\partial\rho}{\partial z}}(\alpha)\right\rangle \geq (t^2-t)$ and $t \geq 1$. From (2.9) and (2.10), we have

$$\begin{aligned}\rho(\psi(\alpha,\beta,\gamma,z)) &\geq \left|2\left\langle \psi(\alpha,\beta,\gamma,z)e^{-i\theta}, \overline{\frac{\partial\rho}{\partial z}}(\alpha)\right\rangle\right| \\ &= \Bigg||A(z)|2\left\langle \gamma, \overline{\frac{\partial\rho}{\partial z}}(\alpha)\right\rangle + B(z)e^{-i\theta}2\left\langle \beta, \overline{\frac{\partial\rho}{\partial z}}(\alpha)\right\rangle \\ &\qquad + C(z)e^{-i\theta}2\left\langle \alpha, \overline{\frac{\partial\rho}{\partial z}}(\alpha)\right\rangle + 2e^{-i\theta}\left\langle \varphi(z), \overline{\frac{\partial\rho}{\partial z}}(\alpha)\right\rangle\Bigg| \\ &\geq |A(z)|\left\{t^2 + t\left[\mathrm{Re}\,\frac{B(z)}{A(z)} - 1\right] + \mathrm{Re}\,\frac{C(z)}{A(z)} - \frac{\rho(\varphi(z))}{|A(z)|}\right\},\end{aligned}$$

where $\theta = \arg A(z)$. Let

$$L(t) = t^2 + t\left[\mathrm{Re}\,\frac{B(z)}{A(z)} - 1\right] + \mathrm{Re}\,\frac{C(z)}{A(z)} - \frac{\rho(\varphi(z))}{|A(z)|}$$

for $t \geq 1$. Then we have

$$L'(t) = 2t + \operatorname{Re} \frac{B(z)}{A(z)} - 1.$$

If $\operatorname{Re} \frac{B(z)}{A(z)} \geq -1$ for $z \in \Omega$, then $L'(t) \geq \operatorname{Re} \frac{B(z)}{A(z)} + 1 \geq 0$. Hence we obtain

$$\min_{t \geq 1} L(t) = L(1) = \operatorname{Re} \frac{B(z)}{A(z)} + \operatorname{Re} \frac{C(z)}{A(z)} - \frac{\rho(\varphi(z))}{|A(z)|} \geq \frac{1}{|A(z)|}.$$

It follows that $\rho(\psi(\alpha, \beta, \gamma, z)) \geq 1$.

If $\operatorname{Re} \frac{B(z)}{A(z)} \leq -1$ for $z \in \Omega$, then

$$\begin{aligned} \min_{t \geq 1} L(t) &= L\left(\frac{1}{2}\left(1 - \operatorname{Re} \frac{B(z)}{A(z)}\right)\right) \\ &= -\frac{1}{4}\left(\operatorname{Re} \frac{B(z)}{A(z)} - 1\right)^2 + \operatorname{Re} \frac{C(z)}{A(z)} - \frac{\rho(\varphi(z))}{|A(z)|} \\ &\geq \frac{1}{|A(z)|}. \end{aligned}$$

It also follows that $\rho(\psi(\alpha, \beta, \gamma, z)) \geq 1$.

Hence we have $\psi \in \Psi(1)$. From Theorem 2.16, we obtain $\rho(w(z)) < 1$ for $z \in \Omega$. □

Remark 2.18. Setting $n = 1$, $\varphi(z) \equiv 0$, $A(z) \equiv A$ and $C(z) = 1 - B(z)$ in Theorem 2.17, we get Theorem 4 in [13].

Corollary 2.19. *Suppose that* $B(z) \in H(B, \mathbb{C})$ *and* $A \geq 0$ *satisfy* $\operatorname{Re} B(z) \geq 0$ *for all* $z \in B$. *If* $w(z) \in H(B, \mathbb{C}^m)$ *with* $w(0) = 0$ *satisfy*

$$\|AD^2 w(z)(z, z) + B(z)Dw(z)(z) + w(z)\| < 1$$

for all $z \in B$*, then* $\|w(z)\| < 1$ *for* $z \in B$.

Example 2.3. Let k and n_1 be positive integers and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{C}^m$. We define $\alpha^k = (\alpha_1^k, \alpha_2^k, \ldots, \alpha_m^k)$. Suppose $A_1, A_2, \ldots, A_{n_1} \in H(B, \mathbb{C})$ $(n_1 \geq 2)$ satisfy

$$\operatorname{Re} A_1(z) \geq \sum_{k=2}^{n_1} |A_k(z)|$$

for $z \in B$, where B is the unit ball in $\mathbb{C}^n$. If $w(z) = (w_1(z), w_2(z), \ldots, w_m(z)) \in H(B, \mathbb{C}^m)$ with $w(0) = 0$ satisfies

$$\sum_{v=1}^{m} \left| \sum_{j,l=1}^{n} \frac{\partial^2 w_v(z)}{\partial z_j \partial z_l} z_j z_l + \sum_{j=1}^{n} \frac{\partial w_v(z)}{\partial z_j} z_j + \sum_{k=1}^{n_1} A_k(z)[w_v(z)]^k \right|^2 < 1$$

for all $z \in B$, then $\sum_{v=1}^{m} |w_v(z)|^2 < 1$ for all $z \in B$.

In fact, if we let

$$\psi(\alpha, \beta, \gamma, z) = \gamma + \beta + \sum_{k=1}^{n_1} A_k(z) \alpha^k$$

for $\|\alpha\| = 1$, $\langle\beta,\alpha\rangle = t$, $\mathrm{Re}\langle\gamma,\alpha\rangle \geq t^2 - t$ and $t \geq 1$, then we have

$$\begin{aligned}\|\psi(\alpha,\beta,\gamma,z)\| &\geq \left|\langle\gamma,\alpha\rangle + \langle\beta,\alpha\rangle + A_1(z) + \sum_{k=2}^{n_1} A_k(z)\langle\alpha^k,\alpha\rangle\right| \\ &\geq \mathrm{Re}\{\langle\gamma,\alpha\rangle + \langle\beta,\alpha\rangle + A_1(z)\} - \sum_{k=2}^{n_1} |A_k(z)| \left(\sum_{j=1}^{m} |\alpha_j|^{k+1}\right) \\ &\geq t^2 + \mathrm{Re}\, A_1(z) - \sum_{k=2}^{n_1} |A_k(z)| \geq 1.\end{aligned}$$

Hence $\psi \in \Psi(1)$ for $\rho(z) = \sqrt{\sum_{j=1}^{n} |z_j|^2}$. According to Theorem 2.16, we have $\sum_{v=1}^{m} |w_v(z)|^2 < 1$ for all $z \in B$.

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