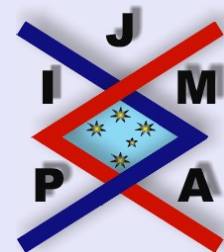


H. ÖZLEM GÜNEY, S. SÜMER EKER AND SHIGEYOSHI OWA

Department of Mathematics
Faculty of Science and Arts
University of Dicle
21280 - Diyarbakir, Turkey.
EMail: ozlemg@dicle.edu.tr

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577 - 8502
Japan.
EMail: owa@math.kindai.ac.jp



volume 7, issue 1, article 37,
2006.

*Received 29 September, 2005;
accepted 03 January, 2006.*

Communicated by: N.E. Cho

Abstract

Contents

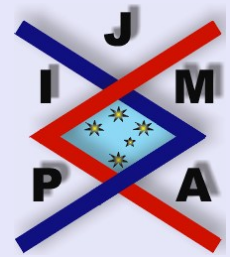


Home Page

Go Back

Close

Quit



Abstract

For analytic functions $f(z)$ and $g(z)$ which satisfy the subordination $f(z) \prec g(z)$, J. E. Littlewood (*Proc. London Math. Soc.*, **23** (1925), 481–519) has shown some interesting results for integral means of $f(z)$ and $g(z)$. The object of the present paper is to derive some applications of integral means by J.E. Littlewood and show interesting examples for our theorems. We also generalize the results of Owa and Sekine (*J. Math. Anal. Appl.*, **304** (2005), 772–782).

2000 Mathematics Subject Classification: Primary 30C45.

Key words: Integral means, Multivalent function, Subordination, Starlike, Convex.

Contents

1	Introduction	3
2	Integral Means Inequalities for $f(z)$ and $g(z)$	6
3	Integral Means Inequalities for $f(z)$ and $h(z)$	14
References		

Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 2 of 22

1. Introduction

Let $\mathcal{A}_{p,n}$ denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are *analytic* and *multivalent* in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ belonging to $\mathcal{A}_{p,n}$ is called to be *multivalently starlike of order α* in \mathbb{U} if it satisfies

$$(1.2) \quad \operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{U})$$

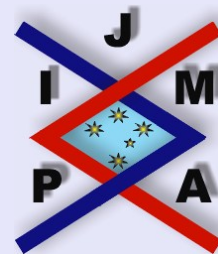
for some $\alpha (0 \leq \alpha < p)$. A function $f(z) \in \mathcal{A}_{p,n}$ is said to be *multivalently convex of order α* in \mathbb{U} if it satisfies

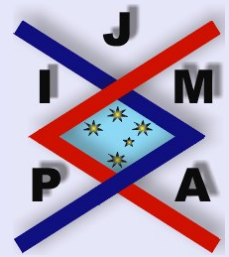
$$(1.3) \quad \operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \quad (z \in \mathbb{U})$$

for some $\alpha (0 \leq \alpha < p)$. We denote by $\mathcal{S}_{p,n}^*(\alpha)$ and $\mathcal{K}_{p,n}(\alpha)$ the classes of functions $f(z) \in \mathcal{A}_{p,n}$ which are multivalently starlike of order α in \mathbb{U} and multivalently convex of order α in \mathbb{U} , respectively. We note that

$$f(z) \in \mathcal{K}_{p,n}(\alpha) \Leftrightarrow \frac{z f'(z)}{p} \in \mathcal{S}_{p,n}^*(\alpha).$$

For functions $f(z)$ belonging to the classes $\mathcal{S}_{p,n}^*(\alpha)$ and $\mathcal{K}_{p,n}(\alpha)$, Owa [4] has shown the following coefficient inequalities.





Title Page

Contents



Go Back

Close

Quit

Page 4 of 22

Theorem 1.1. If a function $f(z) \in \mathcal{A}_{p,n}$ satisfies

$$(1.4) \quad \sum_{k=p+n}^{\infty} (k - \alpha)|a_k| \leq p - \alpha$$

for some α ($0 \leq \alpha < p$), then $f(z) \in \mathcal{S}_{p,n}^*(\alpha)$.

Theorem 1.2. If a function $f(z) \in \mathcal{A}_{p,n}$ satisfies

$$(1.5) \quad \sum_{k=p+n}^{\infty} k(k - \alpha)|a_k| \leq p - \alpha$$

for some α ($0 \leq \alpha < p$), then $f(z) \in \mathcal{K}_{p,n}(\alpha)$.

For analytic functions $f(z)$ and $g(z)$ in \mathbb{U} , $f(z)$ is said to be *subordinate* to $g(z)$ in \mathbb{U} if there exists an analytic function $w(z)$ in \mathbb{U} such that $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$), and $f(z) = g(w(z))$. We denote this subordination by

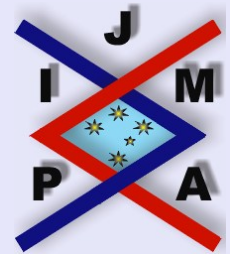
$$f(z) \prec g(z) \quad (\text{cf. Duren [2]}).$$

To discuss our problems for integral means of multivalent functions, we have to recall here the following result due to Littlewood [3].

Theorem 1.3. If $f(z)$ and $g(z)$ are analytic in \mathbb{U} with $f(z) \prec g(z)$, then for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$(1.6) \quad \int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta.$$

Applying Theorem 1.3 by Littlewood [3], Owa and Sekine [5] have considered some integral means inequalities for certain analytic functions. In the present paper, we discuss the integral means inequalities for multivalent functions which are the generalization of the paper by Owa and Sekine [5].



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 5 of 22

2. Integral Means Inequalities for $f(z)$ and $g(z)$

In this section, we discuss the integral means inequalities for $f(z) \in \mathcal{A}_{p,n}$ and $g(z)$ defined by

$$(2.1) \quad g(z) = z^p + b_j z^j + b_{2j-p} z^{2j-p} \quad (j \geq n+p).$$

We first derive

Theorem 2.1. *Let $f(z) \in \mathcal{A}_{p,n}$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies*

$$(2.2) \quad \sum_{k=p+n}^{\infty} |a_k| \leq |b_{2j-p}| - |b_j| \quad (|b_j| < |b_{2j-p}|)$$

and there exists an analytic function $w(z)$ such that

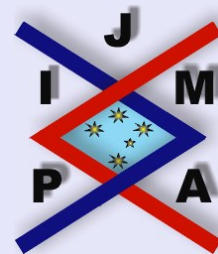
$$b_{2j-p} (w(z))^{2(j-p)} + b_j (w(z))^{j-p} - \sum_{k=p+n}^{\infty} a_k z^{k-p} = 0,$$

then for $\mu > 0$ and $z = r e^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |g(z)|^\mu d\theta.$$

Proof. By putting $z = r e^{i\theta}$ ($0 < r < 1$), we see that

$$\begin{aligned} \int_0^{2\pi} |f(z)|^\mu d\theta &= \int_0^{2\pi} \left| z^p + \sum_{k=p+n}^{\infty} a_k z^k \right|^\mu d\theta \\ &= r^{p\mu} \int_0^{2\pi} \left| 1 + \sum_{k=p+n}^{\infty} a_k z^{k-p} \right|^\mu d\theta \end{aligned}$$



Title Page

Contents



Go Back

Close

Quit

Page 6 of 22

and

$$\begin{aligned} \int_0^{2\pi} |g(z)|^\mu d\theta &= \int_0^{2\pi} |z^p + b_j z^j + b_{2j-p} z^{2j-p}|^\mu d\theta \\ &= r^{p\mu} \int_0^{2\pi} |1 + b_j z^{j-p} + b_{2j-p} z^{2j-2p}|^\mu d\theta. \end{aligned}$$

Applying Theorem 1.3, we have to show that

$$1 + \sum_{k=p+n}^{\infty} a_k z^{k-p} \prec 1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)}.$$

Let us define the function $w(z)$ by

$$1 + \sum_{k=p+n}^{\infty} a_k z^{k-p} = 1 + b_j (w(z))^{j-p} + b_{2j-p} (w(z))^{2(j-p)}$$

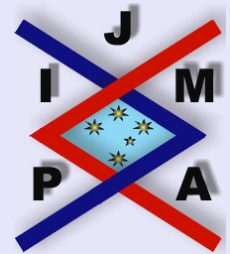
or by

$$(2.3) \quad b_{2j-p} (w(z))^{2(j-p)} + b_j (w(z))^{j-p} - \sum_{k=p+n}^{\infty} a_k z^{k-p} = 0.$$

Since, for $z = 0$,

$$(w(0))^{j-p} \left\{ b_{2j-p} (w(0))^{j-p} + b_j \right\} = 0,$$

there exists an analytic function $w(z)$ in \mathbb{U} such that $w(0) = 0$.



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 7 of 22

Next we prove the analytic function $w(z)$ satisfies $|w(z)| < 1$ ($z \in \mathbb{U}$) for

$$\sum_{k=p+n}^{\infty} |a_k| \leq |b_{2j-p}| - |b_j| \quad (|b_j| < |b_{2j-p}|).$$

By the inequality (2.3), we know that

$$\left| b_{2j-p} (w(z))^{2(j-p)} + b_j (w(z))^{j-p} \right| = \left| \sum_{k=p+n}^{\infty} a_k z^{k-p} \right| < \sum_{k=p+n}^{\infty} |a_k|$$

for $z \in \mathbb{U}$, hence

$$(2.4) \quad |b_{2j-p}| |w(z)|^{2(j-p)} - |b_j| |w(z)|^{j-p} - \sum_{k=p+n}^{\infty} |a_k| < 0.$$

Letting $t = |w(z)|^{j-p}$ ($t \geq 0$) in (2.4), we define the function $G(t)$ by

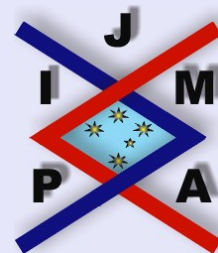
$$G(t) = |b_{2j-p}| t^2 - |b_j| t - \sum_{k=p+n}^{\infty} |a_k|.$$

If $G(1) \geq 0$, then we have $t < 1$ for $G(t) < 0$. Indeed we have

$$G(1) = |b_{2j-p}| - |b_j| - \sum_{k=p+n}^{\infty} |a_k| \geq 0.$$

that is,

$$\sum_{k=p+n}^{\infty} |a_k| \leq |b_{2j-p}| - |b_j|.$$



Title Page

Contents



Go Back

Close

Quit

Page 8 of 22

Consequently, if the inequality (2.2) holds true, there exists an analytic function $w(z)$ with $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$) such that $f(z) = g(w(z))$. This completes the proof of Theorem 2.1. \square

Theorem 2.1 gives us the following corollary.

Corollary 2.2. *Let $f(z) \in \mathcal{A}_{p,n}$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies the conditions of Theorem 2.1, then for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$)*

$$\begin{aligned} \int_0^{2\pi} |f(z)|^\mu d\theta &\leq 2\pi r^{p\mu} \{1 + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)}\}^{\frac{\mu}{2}} \\ &< 2\pi \{1 + |b_j|^2 + |b_{2j-p}|^2\}^{\frac{\mu}{2}}. \end{aligned}$$

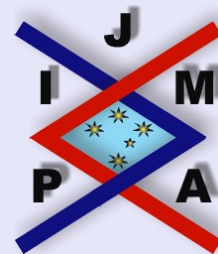
Further, we have that $f(z) \in \mathcal{H}^q(U)$ for $0 < q \leq 2$, where \mathcal{H}^q denotes the Hardy space (cf. Duren [1]).

Proof. Since,

$$\int_0^{2\pi} |g(z)|^\mu d\theta = \int_0^{2\pi} |z^p|^\mu |1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)}|^\mu d\theta,$$

applying Hölder's inequality for $0 < \mu < 2$, we obtain that

$$\begin{aligned} &\int_0^{2\pi} |g(z)|^\mu d\theta \\ &\leq \left\{ \int_0^{2\pi} (|z^p|^\mu)^{\frac{2}{2-\mu}} d\theta \right\}^{\frac{2-\mu}{2}} \left\{ \int_0^{2\pi} (|1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)}|^\mu)^{\frac{2}{\mu}} d\theta \right\}^{\frac{\mu}{2}} \\ &= \left\{ r^{\frac{2p\mu}{2-\mu}} \int_0^{2\pi} d\theta \right\}^{\frac{2-\mu}{2}} \left\{ \int_0^{2\pi} |1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)}|^2 d\theta \right\}^{\frac{\mu}{2}} \end{aligned}$$



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents

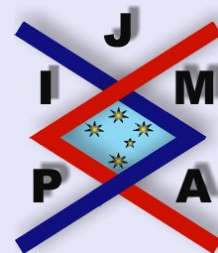


Go Back

Close

Quit

Page 9 of 22



Title Page

Contents



Go Back

Close

Quit

Page 10 of 22

$$\begin{aligned}
 &= \left\{ 2\pi r^{\frac{2p\mu}{2-\mu}} \right\}^{\frac{2-\mu}{2}} \left\{ 2\pi \left(1 + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)} \right) \right\}^{\frac{\mu}{2}} \\
 &= 2\pi r^{p\mu} \left(1 + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)} \right)^{\frac{\mu}{2}} \\
 &< 2\pi \left(1 + |b_j|^2 + |b_{2j-p}|^2 \right)^{\frac{\mu}{2}}.
 \end{aligned}$$

Further, it is easy to see that for $\mu = 2$,

$$\begin{aligned}
 \int_0^{2\pi} |f(z)|^2 d\theta &\leq 2\pi r^{p\mu} \left\{ 1 + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)} \right\} \\
 &< 2\pi \left\{ 1 + |b_j|^2 + |b_{2j-p}|^2 \right\}.
 \end{aligned}$$

From the above, we also have that, for $0 < \mu \leq 2$,

$$\sup_{z \in U} \frac{1}{2\pi} \int_0^{2\pi} |f(z)|^\mu d\theta \leq 2\pi r^{p\mu} \left\{ 1 + |b_j|^2 + |b_{2j-p}|^2 \right\}^{\frac{\mu}{2}} < \infty$$

which observe that $f(z) \in \mathcal{H}^2(\mathbb{U})$. Noting that $\mathcal{H}^q \subset \mathcal{H}^r$ ($0 < r < q < \infty$), we complete the proof of the corollary. \square

Example 2.1. Let $f(z) \in \mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.4) and

$$g(z) = z^p + \frac{n}{n+p-\alpha} \varepsilon z^j + \delta z^{2j-p} \quad (|\varepsilon| = |\delta| = 1)$$

with $0 \leq \alpha < p$. Note that $b_j = \frac{n\varepsilon}{n+p-\alpha}$ and $b_{2j-p} = \delta$.

By virtue of (1.4), we observe that

$$\sum_{k=p+n}^{\infty} |a_k| \leq \frac{p-\alpha}{p+n-\alpha} = 1 - \frac{n}{p+n-\alpha} = |b_{2j-p}| - |b_j|.$$

Therefore, if there exists the function $w(z)$ satisfying the condition in Theorem 2.1, then $f(z)$ and $g(z)$ satisfy the conditions in Theorem 2.1. Thus we have for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq 2\pi r^{p\mu} \left\{ 1 + \left(\frac{n}{p+n-\alpha} \right)^2 r^{2(j-p)} + r^{4(j-p)} \right\}^{\frac{\mu}{2}}$$

$$< 2\pi \left\{ 2 + \left(\frac{n}{p+n-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}.$$

Using the same technique as in the proof of Theorem 2.1, we also derive

Theorem 2.3. Let $f(z) \in \mathcal{A}_{p,n}$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies

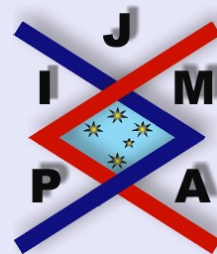
$$(2.5) \quad \sum_{k=p+n}^{\infty} k|a_k| \leq (2j-p)|b_{2j-p}| - j|b_j| \quad (|b_j| < |b_{2j-p}|)$$

and there exists an analytic function $w(z)$ such that

$$(2j-p)b_{2j-p}(w(z))^{2(j-p)} + jb_j(w(z))^{j-p} - \sum_{k=p+n}^{\infty} ka_k z^{k-p} = 0,$$

then for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$)

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq \int_0^{2\pi} |g'(z)|^\mu d\theta.$$



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 11 of 22

Further, with the help of Hölder's inequality, we have

Corollary 2.4. Let $f(z) \in \mathcal{A}_{p,n}$ and $g(z)$ be given by (2.1). If $f(z)$ satisfies conditions of Theorem 2.3, then for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$)

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq 2\pi r^{(p-1)\mu} \{p^2 + j^2|b_j|^2 r^{2(j-p)} + (2j-p)^2|b_{2j-p}|^2 r^{4(j-p)}\}^{\frac{\mu}{2}} < 2\pi \{p^2 + j^2|b_j|^2 + (2j-p)^2|b_{2j-p}|^2\}^{\frac{\mu}{2}}.$$

Example 2.2. Let $f(z) \in \mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.5) and

$$g(z) = z^p + \frac{n}{j(n+p-\alpha)} \varepsilon z^j + \frac{\delta}{2j-p} z^{2j-p} \quad (|\varepsilon| = |\delta| = 1)$$

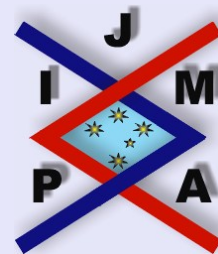
with $0 \leq \alpha < p$. Then

$$b_j = \frac{n\varepsilon}{j(n+p-\alpha)} \quad \text{and} \quad b_{2j-p} = \frac{\delta}{2j-p}.$$

Since

$$\sum_{k=p+n}^{\infty} k|a_k| \leq \frac{p-\alpha}{p+n-\alpha} = 1 - \frac{n}{p+n-\alpha} = (2j-p)|b_{2j-p}| - j|b_j|,$$

if there exists the function $w(z)$ satisfying the condition in Theorem 2.3, then $f(z)$ and $g(z)$ satisfy the conditions in Theorem 2.3. Thus by Corollary 2.4, we



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

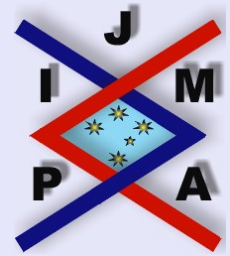
Quit

Page 12 of 22

have for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq 2\pi r^{(p-1)\mu} \left\{ p^2 + \left(\frac{n}{p+n-\alpha} \right)^2 r^{2(j-p)} + r^{4(j-p)} \right\}^{\frac{\mu}{2}}$$

$$< 2\pi \left\{ p^2 + 1 + \left(\frac{n}{p+n-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}.$$



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 13 of 22

3. Integral Means Inequalities for $f(z)$ and $h(z)$

In this section, we introduce an analytic and multivalent function $h(z)$ defined by

$$(3.1) \quad h(z) = z^p + b_j z^j + b_{2j-p} z^{2j-p} + b_{3j-2p} z^{3j-2p} \quad (j \geq n + p).$$

For the above function $h(z)$, we show

Theorem 3.1. *Let $f(z) \in \mathcal{A}_{p,n}$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies*

$$(3.2) \quad \sum_{k=p+n}^{\infty} |a_k| \leq |b_{3j-2p}| - |b_{2j-p}| - |b_j| \quad (|b_j| + |b_{2j-p}| < |b_{3j-2p}|)$$

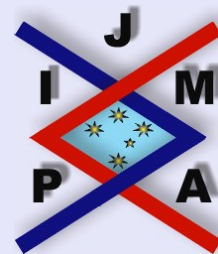
and there exists an analytic function $w(z)$ such that

$$(3.3) \quad b_{3j-2p} (w(z))^{3(j-p)} + b_{2j-p} (w(z))^{2(j-p)} + b_j (w(z))^{j-p} - \sum_{k=p+n}^{\infty} a_k z^{k-p} = 0,$$

then for $\mu > 0$ and $z = r e^{i\theta}$ ($0 < r < 1$)

$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq \int_0^{2\pi} |h(z)|^\mu d\theta.$$

Proof. In the same way as in the proof of Theorem 2.1, we have to show that there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$) such that $f(z) = h(w(z))$. Note that this function $w(z)$ is defined by (3.3).



Since, for $z = 0$,

$$(w(0))^{j-p} \left\{ b_{3j-2p} (w(0))^{2(j-p)} + b_{2j-p} (w(0))^{j-p} + b_j \right\} = 0,$$

we consider $w(z)$ satisfies $w(0) = 0$.

On the other hand, we have that

$$|b_{3j-2p}| |w(z)|^{3(j-p)} - |b_{2j-p}| |w(z)|^{2(j-p)} - |b_j| |w(z)|^{j-p} - \sum_{k=n+p}^{\infty} |a_k| < 0.$$

Putting $t = |w(z)|^{j-p}$ ($t \geq 0$), we define the function $H(t)$ by

$$H(t) = |b_{3j-2p}| t^3 - |b_{2j-p}| t^2 - |b_j| t - \sum_{k=n+p}^{\infty} |a_k|.$$

It follows that $H(0) \leq 0$ and

$$H'(t) = 3|b_{3j-2p}| t^2 - 2|b_{2j-p}| t - |b_j|.$$

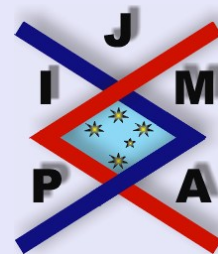
Since the discriminant of $H'(t) = 0$ is greater than 0, if $H'(1) \geq 0$, then $t < 1$ for $H(t) < 0$. Therefore, we need the following inequality:

$$H(1) = |b_{3j-2p}| - |b_{2j-p}| - |b_j| - \sum_{k=p+n}^{\infty} |a_k| \geq 0$$

or

$$\sum_{k=p+n}^{\infty} |a_k| \leq |b_{3j-2p}| - |b_{2j-p}| - |b_j|.$$

This completes the proof of Theorem 3.1. □



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 15 of 22

Corollary 3.2. Let $f(z) \in \mathcal{A}_{p,n}$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies conditions of Theorem 3.1, then for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$)

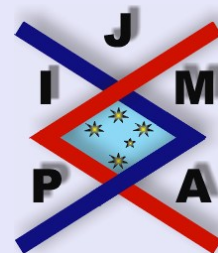
$$\int_0^{2\pi} |f(z)|^\mu d\theta \leq 2\pi r^{p\mu} \left\{ 1 + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)} + |b_{3j-2p}|^2 r^{6(j-p)} \right\}^{\frac{\mu}{2}} \\ < 2\pi \left\{ 1 + |b_j|^2 + |b_{2j-p}|^2 + |b_{3j-2p}|^2 \right\}^{\frac{\mu}{2}}.$$

Proof. Since

$$\int_0^{2\pi} |h(z)|^\mu d\theta = \int_0^{2\pi} |z^p|^\mu |1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)} + b_{3j-2p} z^{3(j-p)}|^\mu d\theta,$$

applying Hölder's inequality for $0 < \mu < 2$, we obtain that

$$\int_0^{2\pi} |h(z)|^\mu d\theta \\ \leq \left\{ \int_0^{2\pi} (|z^p|^\mu)^{\frac{2}{2-\mu}} d\theta \right\}^{\frac{2-\mu}{2}} \\ \times \left\{ \int_0^{2\pi} (|1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)} + b_{3j-2p} z^{3(j-p)}|^\mu)^{\frac{2}{\mu}} d\theta \right\}^{\frac{\mu}{2}} \\ = \left\{ r^{\frac{2p\mu}{2-\mu}} \int_0^{2\pi} d\theta \right\}^{\frac{2-\mu}{2}} \left\{ \int_0^{2\pi} |1 + b_j z^{j-p} + b_{2j-p} z^{2(j-p)} + b_{3j-2p} z^{3(j-p)}|^2 d\theta \right\}^{\frac{\mu}{2}} \\ = \left\{ 2\pi r^{\frac{2p\mu}{2-\mu}} \right\}^{\frac{2-\mu}{2}} \left\{ 2\pi (1 + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)} + |b_{3j-2p}|^2 r^{6(j-p)}) \right\}^{\frac{\mu}{2}} \\ = 2\pi r^{p\mu} \left(1 + |b_j|^2 r^{2(j-p)} + |b_{2j-p}|^2 r^{4(j-p)} + |b_{3j-2p}|^2 r^{6(j-p)} \right)^{\frac{\mu}{2}} \\ < 2\pi \left(1 + |b_j|^2 + |b_{2j-p}|^2 + |b_{3j-2p}|^2 \right)^{\frac{\mu}{2}}.$$



Title Page

Contents



Go Back

Close

Quit

Page 16 of 22

Further, we have that $f(z) \in \mathcal{H}^q(\mathbb{U})$ for $0 < q < 2$. □

We consider the example for Theorem 3.1.

Example 3.1. Let $f(z) \in \mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.4) and

$$h(z) = z^p + \frac{nt}{p+n-\alpha} \varepsilon z^j + \frac{n(1-t)}{p+n-\alpha} \delta z^{2j-p} + \sigma z^{3j-2p}$$

$$(|\varepsilon| = |\delta| = \sigma = 1; 0 \leq t \leq 1)$$

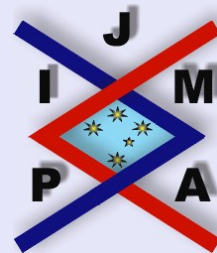
with $0 \leq \alpha < p$. Then

$$b_j = \frac{nt}{p+n-\alpha} \varepsilon, b_{2j-p} = \frac{n(1-t)}{p+n-\alpha} \delta \quad \text{and} \quad b_{3j-2p} = \sigma.$$

In view of (1.4), we see that

$$\begin{aligned} \sum_{k=p+n}^{\infty} |a_k| &\leq \frac{p-\alpha}{p+n-\alpha} \\ &= 1 - \frac{n(1-t)}{p+n-\alpha} - \frac{nt}{p+n-\alpha} \\ &= |b_{3j-2p}| - |b_{2j-p}| - |b_j|. \end{aligned}$$

Therefore, if there exists the function $w(z)$ satisfying the condition in Theorem 3.1, then $f(z)$ and $g(z)$ satisfy the conditions in Theorem 3.1. Thus applying



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 17 of 22

Corollary 3.2, we have for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f(z)|^\mu d\theta$$

$$\leq 2\pi r^{p\mu} \left\{ 1 + \left(\frac{nt}{p+n-\alpha} \right)^2 r^{2(j-p)} + \left(\frac{n(1-t)}{p+n-\alpha} \right)^2 r^{4(j-p)} + r^{6(j-p)} \right\}^{\frac{\mu}{2}}$$

$$< 2\pi \left\{ 2 + (2t^2 - 2t + 1) \left(\frac{n}{p+n-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}.$$

Next, we derive

Theorem 3.3. Let $f(z) \in \mathcal{A}_{p,n}$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies

$$(3.4) \quad \sum_{k=p+n}^{\infty} k|a_k| \leq (3j-2p)|b_{3j-2p}| - (2j-p)|b_{2j-p}| - j|b_j|$$

$$(j|b_j| + (2j-p)|b_{2j-p}| < (3j-2p)|b_{3j-2p}|)$$

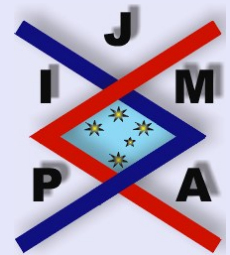
and there exists an analytic function $w(z)$ such that

$$(3j-2p)b_{3j-2p}(w(z))^{3(j-p)} + (2j-p)b_{2j-p}(w(z))^{2(j-p)}$$

$$+ jb_j(w(z))^{j-p} - \sum_{k=p+n}^{\infty} ka_k z^{k-p} = 0,$$

then for $\mu > 0$ and $z = re^{i\theta}$ ($0 < r < 1$)

$$\int_0^{2\pi} |f'(z)|^\mu d\theta \leq \int_0^{2\pi} |h'(z)|^\mu d\theta.$$



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker and Shigeyoshi Owa

Title Page
Contents
◀◀
▶▶
◀
▶
Go Back
Close
Quit
Page 18 of 22

Corollary 3.4. Let $f(z) \in \mathcal{A}_{p,n}$ and $h(z)$ be given by (3.1). If $f(z)$ satisfies conditions in Theorem 3.3, then for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$)

$$\begin{aligned} & \int_0^{2\pi} |f'(z)|^\mu d\theta \\ & \leq 2\pi r^{(p-1)\mu} \left\{ p^2 + j^2 |b_j|^2 r^{2(j-p)} + (2j-p)^2 |b_{2j-p}|^2 r^{4(j-p)} \right. \\ & \quad \left. + (3j-2p)^2 |b_{3j-2p}|^2 r^{6(j-p)} \right\}^{\frac{\mu}{2}} \\ & < 2\pi \left\{ p^2 + j^2 |b_j|^2 + (2j-p)^2 |b_{2j-p}|^2 + (3j-2p)^2 |b_{3j-2p}|^2 \right\}^{\frac{\mu}{2}}. \end{aligned}$$

Finally, we show

Example 3.2. Let $f(z) \in \mathcal{A}_{p,n}$ satisfy the coefficient inequality (1.5) and

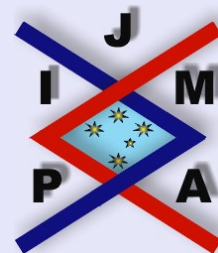
$$h(z) = z^p + \frac{nt}{j(p+n-\alpha)} \varepsilon z^j + \frac{n(1-t)}{(2j-p)(p+n-\alpha)} \delta z^{2j-p} + \frac{\sigma}{3j-2p} z^{3j-2p}$$

($|\varepsilon| = |\delta| = |\sigma| = 1; 0 \leq t \leq 1$)

with $0 \leq \alpha < p$. Then

$$b_j = \frac{nt}{p+n-\alpha} \varepsilon, b_{2j-p} = \frac{n(1-t)}{(2j-p)(p+n-\alpha)} \delta \quad \text{and}$$

$$b_{3j-2p} = \frac{\sigma}{3j-2p}.$$



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 19 of 22

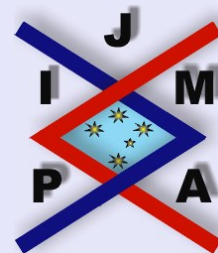
Since

$$\begin{aligned} \sum_{k=p+n}^{\infty} k|a_k| &\leq \frac{p-\alpha}{p+n-\alpha} \\ &= 1 - \frac{n}{p+n-\alpha} \\ &= (3j-2p)|b_{3j-2p}| - (2j-p)|b_{2j-p}| - j|b_j|, \end{aligned}$$

if there exists the function $w(z)$ satisfying the condition in Theorem 3.3, then $f(z)$ and $g(z)$ satisfy the conditions in Theorem 3.3. Thus by Corollary 3.4, we have for $0 < \mu \leq 2$ and $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} &\int_0^{2\pi} |f'(z)|^\mu d\theta \\ &\leq 2\pi r^{(p-1)\mu} \left\{ p^2 + \left(\frac{nt}{p+n-\alpha} \right)^2 r^{2(j-p)} + \left(\frac{n(1-t)}{p+n-\alpha} \right)^2 r^{4(j-p)} + r^{6(j-p)} \right\}^{\frac{\mu}{2}} \\ &< 2\pi \left\{ p^2 + 1 + (2t^2 - 2t + 1) \left(\frac{n}{p+n-\alpha} \right)^2 \right\}^{\frac{\mu}{2}}. \end{aligned}$$

Remark 1. We have not been able to prove that the analytic function $w(z)$ satisfying each condition of the theorems in this paper exists. However, if we consider some special function $f(z)$ in our theorems, then we know that there is the analytic function $w(z)$ satisfying each condition of our theorems. Thus, if we prove that such a function $w(z)$ exists for any function $f(z) \in \mathcal{A}_{p,n}$, then we do not need to give the condition for $w(z)$ in our theorems.



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



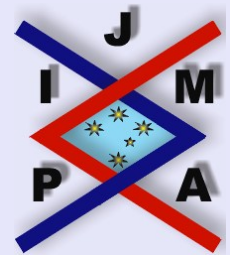
Go Back

Close

Quit

Page 20 of 22

Remark 2. *In the above theorems and examples, if we take $p = 1$, we obtain the results by Owa and Sekine [5]. Therefore, the results of our paper are a generalization of the results in [5].*



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

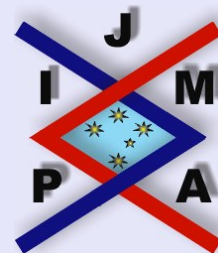
Close

Quit

Page 21 of 22

References

- [1] P.L. DUREN, *Theory of \mathcal{H}^p Space*, Academic Press, New York, 1970.
- [2] P.L. DUREN, *Univalent Functions*, Springer-Verlag, New York, 1983.
- [3] J.E. LITTLEWOOD, On inequalities in the theory of functions, *Proc. London Math. Soc.*, (2) **23** (1925), 481–519.
- [4] S. OWA, On certain classes of p -valent functions with negative coefficients, *Simon Stevin*, **59** (1985), 385–402.
- [5] S. OWA AND T. SEKINE, Integral means for analytic functions, *J. Math. Anal. Appl.*, **304** (2005), 772–782.



Integral Means of Multivalent Functions

H. Özlem Güney, S. Sümer Eker
and Shigeyoshi Owa

Title Page

Contents



Go Back

Close

Quit

Page 22 of 22