

# A SIMPLE PROOF OF SCHIPP'S THEOREM

YANBO REN AND JUNYAN REN

Department of Mathematics and Physics  
Henan University of Science and Technology  
Luoyang 471003, China.  
EMail: ryb7945@sina.com.cn renjy03@lzu.edu.cn

- Received:* 13 September, 2007
- Accepted:* 06 November, 2007
- Communicated by:* S.S. Dragomir
- 2000 AMS Sub. Class.:* 60G42.
- Key words:* Martingale inequality, Property  $\Delta$ .
- Abstract:* In this paper we give a simple proof of Schipp's theorem by using a basic martingale inequality.
- Acknowledgements:* The authors thank referees for their valuable comments.



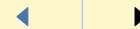
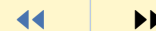
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Proof of Schipp's Theorem  
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vol. 8, iss. 4, art. 117, 2007

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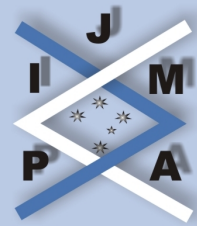
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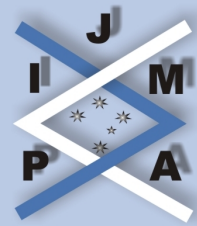
## 1. Introduction

The property  $\Delta$  of operators was introduced by F. Schipp in [1] and he proved that, if  $(T_n, n \in \mathbb{P})$  are a series of operators with property  $\Delta$  and some boundedness, then the operator  $T = \sum_{n=1}^{\infty} T_n$  is of type  $(p, p)$  ( $p \geq 2$ ). We resume this result as Theorem 1.1. F. Schipp applied Theorem 1.1 to prove the significant result that the Fourier-Vilenkin expansions of the function  $f \in L^p$  converge to  $f$  in  $L^p$ -norms ( $1 < p < \infty$ ).

Throughout this paper  $\mathbb{P}$  and  $\mathbb{N}$  denote the set of positive integers and the set of nonnegative integers, respectively. We always use  $C, C_1$  and  $C_2$  to denote constants which may be different in different contexts.

Let  $(\Omega, \mathcal{F}, \mu)$  be a complete probability space and  $\{\mathcal{F}_n, n \in \mathbb{N}\}$  an increasing sequence of sub- $\sigma$ -algebras of  $\mathcal{F}$  with  $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$ . Denote by  $\mathbb{E}$  and  $\mathbb{E}_n$  expectation operator and conditional expectation operators relative to  $\mathcal{F}_n$  for  $n \in \mathbb{N}$ , respectively. We briefly write  $L^p$  instead of the complex  $L^p(\Omega, \mathcal{F}, \mu)$  while the norm (or quasinorm) of this space is defined by  $\|f\|_p = (\mathbb{E}[|f|^p])^{\frac{1}{p}}$ . A martingale  $f = (f_n, n \in \mathbb{N})$  is an adapted, integrable sequence with  $\mathbb{E}_n f_m = f_n$  for all  $n \leq m$ . For a martingale  $f = (f_n)_{n \geq 0}$  we say that  $f = (f_n)_{n \geq 0}$  is  $L_p$  ( $1 \leq p < \infty$ )-bounded if  $\|f\|_p = \sup_n \|f_n\|_p < \infty$ . If  $1 < p < \infty$  and  $f \in L^p$  then  $\tilde{f} = (\mathbb{E}_n f)_{n \geq 0}$  is a  $L^p$ -bounded martingale, and  $\|f\|_p = \|\tilde{f}\|_p$  (see [2]). We denote the maximal function and the martingale differences of a martingale  $f = (f_n, n \in \mathbb{N})$  by  $f^* = \sup_{n \in \mathbb{N}} |f_n|$  and  $df_n = f_n - f_{n-1}$  ( $n \in \mathbb{P}$ ),  $df_0 = f_0$ , respectively. We recall that for a  $L_p$ -bounded martingale  $f = (f_n)_{n \geq 0}$  ( $p > 1$ ):

$$(1.1) \quad \|f\|_p \leq \|f^*\|_p \leq C \|f\|_p.$$



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We will use the following martingale inequality (see Weisz [2]):

$$(1.2) \quad \|f^*\|_p \leq C_1 \left\| \left( \sum_{n=0}^{\infty} \mathbb{E}_{n-1} [|df_n|^2] \right)^{\frac{1}{2}} \right\|_p + C_1 \left\| \sup_{n \in \mathbb{N}} |df_n| \right\|_p \leq C_2 \|f^*\|_p \quad (2 \leq p < \infty).$$

Now let  $\Delta_0 = \mathbb{E}$ ,  $\Delta_n = \mathbb{E}_n - \mathbb{E}_{n-1}$  ( $n \in \mathbb{P}$ ). It is easy to see that

$$(1.3) \quad \mathbb{E}_n \circ \mathbb{E}_m = \mathbb{E}_{\min(n,m)}, \quad \Delta_n \circ \Delta_m = \delta_{mn} \Delta_n \quad (n, m \in \mathbb{P}),$$

where  $\delta_{mn}$  is the Kronecker symbol and  $\circ$  denotes the composition of functions. Let  $\{T_n, n \in \mathbb{P}\}$ ,  $T_n : L^p \rightarrow L^q$  ( $1 \leq p, q < \infty$ ), be a sequence of operators. We say that the operators  $\{T_n, n \in \mathbb{P}\}$  are uniformly of type  $(\mathcal{F}_{n-1}, p, q)$  if there exists a constant  $C > 0$  such that for all  $f \in L^p$

$$(\mathbb{E}_{n-1}[|T_n f|^q])^{\frac{1}{q}} \leq C (\mathbb{E}_{n-1}[|f|^p])^{\frac{1}{p}}.$$

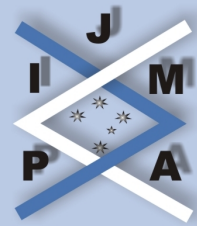
A sequence of operators  $\{T_n, n \in \mathbb{P}\}$  is said to satisfy the  $\Delta$ -condition, if

$$(1.4) \quad T_n \circ \Delta_n = \Delta_n \circ T_n = T_n \quad (n \in \mathbb{P}).$$

From the equations in (1.3) it is easy to see that the  $\Delta$ -condition is equivalent to the following conditions:

$$(1.5) \quad T_n \circ \mathbb{E}_n = \mathbb{E}_n \circ T_n = T_n, \quad T_n \circ \mathbb{E}_{n-1} = \mathbb{E}_{n-1} \circ T_n = 0 \quad (n \in \mathbb{P}).$$

For  $f \in L^p$ , set  $Tf = \sum_{n=1}^{\infty} T_n f$  and  $T^*f = \sup |\sum_{n=1}^m T_n f|$ . It is obvious that the operator series  $\sum_{n=1}^{\infty} T_n f$  is convergent at each point of  $L = \bigcup_n L^p(\mathcal{F}_n)$  if  $\{T_n, n \in \mathbb{P}\}$  satisfy the  $\Delta$ -condition, since for  $f \in L^p(\mathcal{F}_N)$ ,  $T_n f = T_n \circ \Delta_n \circ \mathbb{E}_N f = 0$ . We resume Schipp's theorem as follows:



**Theorem 1.1 ([1]).** Let  $(T_n, n \in \mathbb{P})$  be a sequence of operators with the property  $\Delta$ , and let  $p \geq 2$ . If for  $r = 2, p$  and  $n \in \mathbb{P}$ , the operators  $(T_n, n \in \mathbb{P})$  are uniformly of type  $(\mathcal{F}_{n-1}, r, r)$ , then the operator  $T$  is of type  $(r, r)$ , i.e., there exists a constant  $C > 0$  such that for all  $f \in L^r$ :

$$\|Tf\|_r \leq C \|f\|_r.$$

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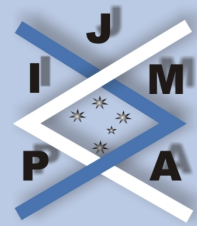
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## 2. Proof of Theorem 1.1

*Proof.* Let  $f \in L^r$  ( $r \geq 2$ ). Then by (1.5), it is easy to see that the stochastic sequence  $(\sum_{k=1}^n T_k f, \mathcal{F}_n)$  is a martingale. By (1.1) we only need to prove that

$$\|T^* f\|_r \leq C \|f^*\|_r.$$

Since the operators  $T_n$  are uniformly of type  $(\mathcal{F}_{n-1}, 2, 2)$  and  $(\mathcal{F}_{n-1}, r, r)$ , it follows from (1.2) and (1.4) that

$$\begin{aligned} \|T^* f\|_r &\leq C \left\| \left( \sum_{n=0}^{\infty} \mathbb{E}_{n-1} [|T_n f|^2] \right)^{\frac{1}{2}} \right\|_r + C \left\| \sup_{n \in \mathbb{N}} |T_n f| \right\|_r \\ &= C \left\| \left( \sum_{n=0}^{\infty} \mathbb{E}_{n-1} [|T_n \circ \Delta_n f|^2] \right)^{\frac{1}{2}} \right\|_r + C \left\| \sup_{n \in \mathbb{N}} |T_n \circ \Delta_n f| \right\|_r \\ &\leq C \left\| \left( \sum_{n=0}^{\infty} \mathbb{E}_{n-1} [|\Delta_n f|^2] \right)^{\frac{1}{2}} \right\|_r + C \left\| \sup_{n \in \mathbb{N}} |\Delta_n f| \right\|_r \\ &\leq C \|f^*\|_r. \end{aligned}$$

□

*Remark 1.* The theorem is proved for  $r = 2$  and  $r > 2$  in a unified way, which differs from the original proof.

## References

- [1] F. SCHIPP, On  $L^p$ -norm convergence of series with respect to product systems, *Analysis Math.*, **2**(1976), 49–64.
- [2] F. WEISZ, *Martingale Hardy spaces and their applications in Fourier analysis*, Lecture Notes in Math., Vol. 1568, Berlin: Springer, 1994.



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