A SIMPLE PROOF OF SCHIPP'S THEOREM

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Abstract: In this paper we give a simple proof of Schipp's theorem by using a basic martingale inequality.

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1. Introduction

The property Δ of operators was introduced by F. Schipp in [1] and he proved that, if $(T_n, n \in \mathbb{P})$ are a series of operators with property Δ and some boundedness, then the operator $T = \sum_{n=1}^{\infty} T_n$ is of type (p,p) $(p \geq 2)$. We resume this result as Theorem 1.1. F. Schipp applied Theorem 1.1 to prove the significant result that the Fourier-Vilenkin expansions of the function $f \in L^p$ converge to f in L^p-norms $(1 < p < \infty)$.

Throughout this paper $\mathbb{P}$ and $\mathbb{N}$ denote the set of positive integers and the set of nonnegative integers, respectively. We always use C, C_1 and C_2 to denote constants which may be different in different contexts.

Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space and $\{\mathcal{F}_n, n \in \mathbb{N}\}$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$ with $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. Denote by $\mathbb{E}$ and $\mathbb{E}_n$ expectation operator and conditional expectation operators relative to $\mathcal{F}_n$ for $n \in \mathbb{N}$, respectively. We briefly write L^p instead of the complex $L^p(\Omega, \mathcal{F}, \mu)$ while the norm (or quasinorm) of this space is defined by $\|f\|_p = (\mathbb{E}[|f|^p])^{\frac{1}{p}}$. A martingale $f = (f_n, n \in \mathbb{N})$ is an adapted, integrable sequence with $\mathbb{E}_n f_m = f_n$ for all $n \leq m$. For a martingale $f = (f_n)_{n\geq 0}$ we say that $f = (f_n)_{n\geq 0}$ is L_p $(1 \leq p < \infty)$-bounded if $\|f\|_p = \sup_n \|f_n\|_p < \infty$. If $1 < p < \infty$ and $f \in L^p$ then $\tilde{f} = (\mathbb{E}_n f)_{n\geq 0}$ is a L^p-bounded martingale, and $\|f\|_p = \left\|\tilde{f}\right\|_p$ (see [2]). We denote the maximal function and the martingale differences of a martingale $f = (f_n, n \in \mathbb{N})$ by $f^* = \sup_{n\in\mathbb{N}} |f_n|$ and $df_n = f_n - f_{n-1}$ $(n \in \mathbb{P})$, $df_0 = f_0$, respectively. We recall that for a L_p-bounded martingale $f = (f_n)_{n\geq 0}$ $(p > 1)$:

$$\|f\|_p \leq \|f^*\|_p \leq C \|f\|_p . \tag{1.1}$$

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We will use the following martingale inequality (see Weisz [2]):

$$(1.2)\quad \|f^*\|_p \le C_1 \left\| \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1}\left[|df_n|^2\right] \right)^{\frac{1}{2}} \right\|_p + C_1 \left\| \sup_{n\in\mathbb{N}} |df_n| \right\|_p \le C_2 \|f^*\|_p \quad (2 \le p < \infty).$$

Now let $\Delta_0 = \mathbb{E}$, $\Delta_n = \mathbb{E}_n - \mathbb{E}_{n-1}(n \in \mathbb{P})$. It is easy to see that

$$(1.3)\qquad \mathbb{E}_n \circ \mathbb{E}_m = \mathbb{E}_{\min(n,m)}, \qquad \Delta_n \circ \Delta_m = \delta_{mn}\Delta_n(n, m \in \mathbb{P}),$$

where δ_{mn} is the Kronecker symbol and $\circ$ denotes the composition of functions. Let $\{T_n, n \in \mathbb{P}\}$, $T_n : L^p \to L^q$ $(1 \le p, q < \infty)$, be a sequence of operators. We say that the operators $\{T_n, n \in \mathbb{P}\}$ are uniformly of type $(\mathcal{F}_{n-1}, p, q)$ if there exists a constant $C > 0$ such that for all $f \in L^p$

$$(\mathbb{E}_{n-1}[|T_n f|^q])^{\frac{1}{q}} \le C(\mathbb{E}_{n-1}[|f|^p])^{\frac{1}{p}}.$$

A sequence of operators $\{T_n, n \in \mathbb{P}\}$ is said to satisfy the Δ-condition, if

$$(1.4)\qquad T_n \circ \Delta_n = \Delta_n \circ T_n = T_n \ (n \in \mathbb{P}).$$

From the equations in (1.3) it is easy to see that the $\Delta-$condition is equivalent to the following conditions:

$$(1.5)\qquad T_n \circ \mathbb{E}_n = \mathbb{E}_n \circ T_n = T_n, \qquad T_n \circ \mathbb{E}_{n-1} = \mathbb{E}_{n-1} \circ T_n = 0(n \in \mathbb{P}).$$

For $f \in L^p$, set $Tf = \sum_{n=1}^{\infty} T_n f$ and $T^* f = \sup |\sum_{n=1}^{m} T_n f|$. It is obvious that the operator series $\sum_{n=1}^{\infty} T_n f$ is convergent at each point of $L = \bigcup_n L^p(\mathcal{F}_n)$ if $\{T_n, n \in \mathbb{P}\}$ satisfy the Δ-condition, since for $f \in L^p(\mathcal{F}_N)$, $T_n f = T_n \circ \Delta_n \circ \mathbb{E}_N f = 0$. We resume Schipp's theorem as follows:

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Theorem 1.1 ([1]). *Let* $(T_n, n \in \mathbb{P})$ *be a sequence of operators with the property* Δ*, and let* $p \geq 2$*. If for* $r = 2, p$ *and* $n \in \mathbb{P}$*, the operators* $(T_n, n \in \mathbb{P})$ *are uniformly of type* $(\mathcal{F}_{n-1}, r, r)$*, then the operator* T *is of type* (r, r)*, i.e., there exists a constant* $C > 0$ *such that for all* $f \in L^r$*:*

$$\|Tf\|_r \leq C \|f\|_r .$$

Proof of Schipp's Theorem
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2. Proof of Theorem 1.1

Proof. Let $f \in L^r$ $(r \geq 2)$. Then by (1.5), it is easy to see that the stochastic sequence $(\sum_{k=1}^{n} T_k f, \mathcal{F}_n)$ is a martingale. By (1.1) we only need to prove that

$$\|T^* f\|_r \leq C \|f^*\|_r .$$

Since the operators T_n are uniformly of type $(\mathcal{F}_{n-1}, 2, 2)$ and $(\mathcal{F}_{n-1}, r, r)$, it follows from (1.2) and (1.4) that

$$\begin{aligned}
\|T^* f\|_r &\leq C \left\| \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1} \left[|T_n f|^2 \right] \right)^{\frac{1}{2}} \right\|_r + C \left\| \sup_{n\in\mathbb{N}} |T_n f| \right\|_r \\
&= C \left\| \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1} \left[|T_n \circ \Delta_n f|^2 \right] \right)^{\frac{1}{2}} \right\|_r + C \left\| \sup_{n\in\mathbb{N}} |T_n \circ \Delta_n f| \right\|_r \\
&\leq C \left\| \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1} \left[|\Delta_n f|^2 \right] \right)^{\frac{1}{2}} \right\|_r + C \left\| \sup_{n\in\mathbb{N}} |\Delta_n f| \right\|_r \\
&\leq C \|f^*\|_r .
\end{aligned}$$

□

Remark 1. The theorem is proved for $r = 2$ and $r > 2$ in a unified way, which differs from the original proof.

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