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APPROXIMATION, NUMERICAL DIFFERENTIATION AND INTEGRATION BASED ON TAYLOR POLYNOMIAL

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ABSTRACT. We represent new estimates of errors of quadrature formula, formula of numerical differentiation and approximation using Taylor polynomial. To measure the errors we apply representation of the remainder in Taylor formula by least concave majorant of the modulus of continuity of the $n-$th derivative of an $n-$times differentiable function. Our quantitative estimates are special applications of a more general inequality for P_n-simple functionals.

Key words and phrases: Degree of approximation, Taylor polynomial, P_n-simple functionals, least concave majorant of $\omega(f,\cdot)$.

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1. INTRODUCTION

The present note is motivated by our recent estimates of the remainder in the Taylor formula for n-times differentiable functions (see [5, 10]). Different types of representations of the Taylor remainder are known in the literature (see [3, p. 230] or [8, p. 489]) where "little o" Landau notation is used. This abbreviation always appears at the end, since hardly any further quantitative results can be based on a little-o- statement. To illustrate this remark we recall Theorem 1.6.6 from Davis' book [1], where the remainder term is attributed to Young:

Theorem 1.1. *Let $f(x)$ be n-times differentiable at $x = x_0$. Then*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \cdots + \frac{1}{(n-1)!} f^{(n-1)}(x_0)(x-x_0)^{n-1} + \frac{(x-x_0)^n}{n!} \cdot \left[f^{(n)}(x_0) + \varepsilon(x) \right],$$

where $\lim_{x\to x_0} \varepsilon(x) = 0$.

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For a function $f \in C^n[a,b]$, the space of n-times continuously differentiable functions, the remainder in Taylor's formula is given by $(x_0, x \in [a,b],\ n \in \mathbb{N} = \{1, 2, \dots, \})$

$$R_n(f; x_0, x) := f(x) - P_n(f, x_0), \tag{1.1}$$

where

$$P_n(f, x_0) = \sum_{j=0}^{n} \frac{(x-x_0)^j}{j!} \cdot f^{(j)}(x_0) \tag{1.2}$$

is the Taylor polynomial - the special case of the Hermite interpolation polynomial. In [6] the Peano remainder is estimated in a different form. For a continuous function f defined on the compact interval $[a,b]$, i.e. $f \in C[a,b]$, the first order modulus of continuity is given for $\varepsilon \geq 0$ by

$$\omega(f, \varepsilon) := \{\sup |f(x) - f(y)| : x, y \in [a,b],\ |x-y| \leq \varepsilon\}.$$

Further we denote by $\tilde{\omega}(f, \cdot)$ the least concave majorant of $\omega(f, \cdot)$ as

$$\tilde{\omega}(f, \varepsilon) = \sup_{0 \leq x \leq \varepsilon \leq y \leq 1,\, x \neq y} \frac{(\varepsilon - x)\omega(f, y) + (y - \varepsilon)\omega(f, x)}{y - x},$$

for $0 \leq \varepsilon \leq 1$. It is known that

$$\omega(f, \varepsilon) \leq \tilde{\omega}(f, \varepsilon) \leq 2\omega(f, \varepsilon). \tag{1.3}$$

The first quantitative estimate for $R_n(f; x_0, x)$ using $\tilde{\omega}(f^{(n)}, \cdot)$ was proved in [6, Theorem 3.2] which we cite as

Theorem 1.2. *For $n \in \mathbb{N}_0$, let $f \in C^n[a,b]$, and $x, x_0 \in [a,b]$. Then for the remainder in Taylor formula we have*

$$|R_n(f; x_0, x)| \leq \frac{|x-x_0|^n}{n!} \cdot \tilde{\omega}\left(f^{(n)}; \frac{|x-x_0|}{n+1}\right). \tag{1.4}$$

Another appropriate tool to estimate $R_n(f; x_0, x)$ is the so-called local moduli of continuity, defined as

$$\omega(f, x_0; \varepsilon) = \{\sup |f(x_0 + h) - f(x_0)| :\ |h| \leq \varepsilon\}.$$

The properties of local and averaged moduli of continuity and their numerous applications in a broad class of problems in numerical analysis can be found in the monograph [9] and in the paper [7]. In [10] to obtain the quantitative variant of Voronovskaja's theorem for the Bernstein operator the following estimate was established:

Theorem 1.3. *For $n \in \mathbb{N}_0$, let $f \in C^n[a,b]$, and $x, x_0 \in [a,b]$. Then for the remainder in Taylor formula we have*

$$|R_n(f; x_0, x)| \leq \frac{|x-x_0|^n}{n!} \cdot \omega\left(f^{(n)}, x_0; |x-x_0|\right). \tag{1.5}$$

It is clear that

$$\omega(f, x_0; \varepsilon) \leq \omega(f, \varepsilon).$$

In Section 2 we study the approximation properties of the Taylor polynomial $P_n(f, x_0)$. In Section 3 we give new estimates of the error in the quadrature formula based on the Taylor polynomial. The formula for numerical differentiation will be studied in Section 4.

2. DEGREE OF APPROXIMATION BY TAYLOR POLYNOMIAL

In Theorem 6.1 of [7], the following was proved.

Theorem 2.1. *Let f have a bounded n-th derivative in $[0,1]$ and $P_n(f)$ be a Hermite interpolation polynomial for f w.r.t. the net $X = \{x_0, x_1, \dots, x_m\}$. Then*

$$\|f - P_n(f)\|_{C[0,1]} \le C \cdot \frac{1}{(n+1)^n} \cdot \omega\left(f^{(n)}, \frac{1}{n+1}\right), \tag{2.1}$$

where $C = O(n^n)$, $n \to \infty$.

If we take the $\sup$ norm in both sides of (1.4) with

$$q = \max\{|x_0|, |1 - x_0|\}$$

we arrive at

Theorem 2.2. *For $f \in C^n[0,1]$, $x_0 \in [0,1]$ we have*

$$\|f - P_n(f, x_0)\|_{C[0,1]} \le \frac{q}{n!} \cdot \tilde{\omega}\left(f^{(n)}, \frac{q}{n+1}\right). \tag{2.2}$$

If we compare the estimates in Theorems 2.1 and 2.2 it is clear that (2.2) is much better than (2.1) according to the term in front of the modulus of continuity of $f^{(n)}(x)$. One of the reasons is that in Theorem 2.2 we suppose that $f^{(n)}$ is continuous and in Theorem 2.1 only the boundedness of $f^{(n)}$ is supposed. However, Theorem 2.2 cannot be obtained as a corollary from Theorem 2.1.

In pointwise form the estimate (1.5) could be better than (1.4). For example let us consider the following function

$$f(x) = \begin{cases} \left(\frac{1}{4} - x\right)^3, & 0 \le x \le \frac{1}{4}, \\ 0, & \frac{1}{4} \le x \le \frac{3}{4}, \\ \left(x - \frac{3}{4}\right)^3, & \frac{3}{4} \le x \le 1. \end{cases}$$

Let $n = 2$ and $x_0 = 0.5$. It is clear that $f \in C^2[0,1]$ and $f''(x) = 6\left(\frac{1}{4} - x\right)$, for $x \in \left[0, \frac{1}{4}\right]$ and $f''(x) = 6\left(x - \frac{3}{4}\right)$ for $x \in \left[\frac{3}{4}, 1\right]$. We calculate $P_2\left(f, \frac{1}{2}\right) = 0$ and $R_2\left(f, \frac{1}{2}, x\right) = f(x)$ for $x \in [0,1]$. If $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$ we get

$$\left|x - \frac{1}{2}\right| \in \left[0, \frac{1}{4}\right]$$

and

$$R_2 = 0 = \omega\left(f'', \frac{1}{2}; \left|x - \frac{1}{2}\right|\right).$$

Hence the estimate (1.5) with local moduli is exact. On the other side

$$\tilde{\omega}\left(f'', \frac{\left|x - \frac{1}{2}\right|}{3}\right) \approx \omega\left(f'', \frac{\left|x - \frac{1}{2}\right|}{3}\right) = 2\left|x - \frac{1}{2}\right| \ge 0.$$

The advantage of (1.4) compared with (1.5) is the term $\frac{1}{n+1}$ in the argument of the modulus. Therefore we may conclude that for the "small" values of n (1.5) is preferable and for big values of n (1.4) is more appropriate.

3. QUADRATURE FORMULA

Let f have a bounded n-th derivative and

$$L(f)=\sum_{j=0}^{n}A_j\cdot f^{(j)}(x_0) \tag{3.1}$$

be a quadrature formula exact in H_n-the set of all algebraic polynomials of degree n. We denote the error of L by

$$R(f)=\int_0^1 f(x)dx-L(f). \tag{3.2}$$

There are numerous quadrature formulas which include the derivatives of the integrated function f. Based on the Hermite interpolation polynomial and as partial case of the Taylor polynomial we cite the following result:

Theorem 3.1 ([7, Theorem 9.1, p. 296])**.**

$$|R(f)|\le C\cdot\frac{1}{(n+1)^n}\cdot\omega\left(f^{(n)},\frac{1}{n+1}\right), \tag{3.3}$$

where $C=O(n^n)$, $n\to\infty$ *and* $f^{(n)}$ *is bounded on* $[0,1]$.

Recently I. Gavrea has proved estimates for P_n-simple functionals in terms of the least concave majorant of modulus of continuity $\tilde{\omega}(f^{(n)},\cdot)$ (see [4, 5]). The linear functional A is a P_n-simple functional if the following requirements hold:

(i) $A(e_{n+1})\neq 0$, where $e_i:[0,1]\to\mathbb{R}$, $e_i(x)=x^i, i\in\mathbb{N}$.

(ii) For any function $f\in C[0,1]$ there exist distinct points $t_i=t_i(f)\in[0,1]$, $i=1,2,\dots,n+2$ such that

$$A(f)=A(e_{n+1})[t_1,t_2,\dots,t_{n+2};f],$$

where $[t_1,t_2,\dots,t_{n+2};f]$ is the divided difference of f. If A is a P_n-simple functional then we notice that

$$A(e_i)=0,\quad i=0,1,\dots,n.$$

The main result in [5] is the following

Theorem 3.2. *Let A be a P_n-simple functional, $A:C[0,1]\to\mathbb{R}$. If $f\in C^{(n)}[0,1]$ then*

$$|A(f)|\le\frac{\|B\|}{2}\cdot\tilde{\omega}\left(f^{(n)},\frac{2|B(e_1)|}{\|B\|}\right), \tag{3.4}$$

where

$$\|B\|=\frac{1}{(n-1)!}\cdot\int_0^1|A((\cdot-y)_+^{n-1})|dy, \tag{3.5}$$

and

$$B(e_1)=\frac{1}{(n-1)!}\cdot\int_0^1 A((\cdot-y)_+^{n-1})ydy. \tag{3.6}$$

We recall that $(\cdot-y)_+^{n-1}$ is $(x-y)^{n-1}$ for $y\le x\le 1$ and 0 for $0\le x\le y$.

Next we construct a quadrature formula of type (3.1) based on the Taylor polynomial for $f(x)$ and represent the error of the quadrature formula $R(f)$ from (3.2) as P_n-simple functional $A(f)$. Let

$$L = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{(j+1)!}, \tag{3.7}$$

where L is obtained by integrating $P_n(f, x_0)$ for $x_0 = 0$. Consequently we denote the error of this quadrature formula by

$$A(f) := R(f) = \int_0^1 f(x)dx - L. \tag{3.8}$$

The functional $A(f)$ from (3.8) is P_n-simple and is 0 for all f-algebraic polynomials of degree n. To apply Theorem 3.1 we estimate $\|B\|$ and $B(e_1)$.

Lemma 3.3. *Under the conditions of Theorem 3.1, $A(f)$ from (3.8) and L from (3.7) we have*

$$\|B\| = \frac{1}{(n+1)!}. \tag{3.9}$$

Proof. For $f(x) = (x-y)_+^{n-1}$ we have $f^{(k)}(0) = 0$ for $0 \le k \le n$, $y \in [0,1]$. Hence

$$A(f) = \int_0^1 f(x)dx = \int_y^1 (x-y)^{n-1}dx = \frac{(1-y)^n}{n}.$$

Lastly

$$\|B\| = \frac{1}{(n-1)!} \cdot \int_0^1 \frac{(1-y)^n}{n} dy = \frac{1}{(n+1)!}.$$

□

Lemma 3.4. *Under conditions of Theorem 3.1 we have*

$$|B(e_1)| = \frac{1}{(n+2)!}. \tag{3.10}$$

Proof. We calculate

$$\begin{aligned} |B(e_1)| &= \frac{1}{(n-1)!} \int_0^1 \frac{(1-y)^n}{n} \cdot y dy \\ &= -\frac{1}{(n+1)!} \int_0^1 y d((1-y)^{n+1}). \end{aligned}$$

After integrating by parts we get

$$|B(e_1)| = \frac{1}{(n+2)!}.$$

□

If we apply Lemmas 3.3 and 3.4 in Theorem 3.2 we complete the proof of the following

Theorem 3.5. *Let A be the functional from (3.8) and $f \in C^{(n)}[0,1]$. Then*

$$|A(f)| \le \frac{1}{2(n+1)!} \cdot \tilde{\omega}\left(f^{(n)}, \frac{2}{n+2}\right).$$

Again, if we compare Theorem 3.5 with Theorem 3.1, we see that the condition of continuity of $f^{(n)}$ leads to a quadrature formula with an essentially smaller error.

4. Error of Numerical Differentiation

In this section we estimate the error committed when replacing the first derivative $f'(x)$ with the derivative of the Taylor polynomial $P_n(f, x_0)$ at the point $x_0 = 1$. Differentiating (1.2) we have

$$P'_n(f,1) = \sum_{j=0}^{n-1} \frac{(x-1)^j}{j!} \cdot f^{(j+1)}(1). \tag{4.1}$$

Consequently we denote the error of numerical differentiation by

$$A(f) := f'(x) - P'_n(f,1). \tag{4.2}$$

It is clear that $A(f) = 0$ for $f \in H_n$. To apply Theorem 3.2 for $A(f)$ from (4.2) we need to calculate the values of $\|B\|$ and $B(e_1)$.

Lemma 4.1. *Let* $f(x) = (x-y)_+^{n-1}$, *for* $x, y \in [0,1]$. *Then*

$$\|B\| = \frac{(1-x)^{n-1}}{(n-1)!}. \tag{4.3}$$

Proof. Obviously $f'(x) = (n-1)(x-y)_+^{n-2}$, $n > 2$. Let us calculate $P'_n(f,1)$ from (4.1). We get

$$\begin{aligned} P'_n(f,1) &= \sum_{j=0}^{n-1} \frac{(x-1)^j}{j!} \cdot f^{(j+1)}(1) \\ &= (n-1)\left[(x-1) + (1-y)\right]^{n-2} \\ &= (n-1)(x-y)^{n-2}. \end{aligned}$$

Following (4.2) we have

$$A(f) = (n-1)\left[(x-y)_+^{n-2} - (x-y)^{n-2}\right].$$

Therefore

$$\begin{aligned} \|B\| &= \frac{1}{(n-1)!}\int_0^1 |A((\cdot - y)_+^{n-1})| dy \\ &= \frac{1}{(n-2)!}\int_x^1 (y-x)^{n-2} dy \\ &= \frac{1}{(n-1)!} \cdot (1-x)^{n-1}. \end{aligned}$$

□

Lemma 4.2. *Let* $f(x) = (x-y)_+^{n-1}$. *Then*

$$B(e_1) = \frac{(x-1)^{n-1}}{(n-1)!} \cdot \left(1 + \frac{x-1}{n}\right). \tag{4.4}$$

Proof. We evaluate $B(e_1)$ from (3.6) as follows

$$B(e_1) = \frac{1}{(n-2)!}\int_x^1 \left(-(x-y)^{n-2}\right) y dy = \frac{1}{(n-1)!}\int_x^1 y d(x-y)^{n-1}.$$

Integration by parts of the last integral yields

$$\frac{(x-1)^{n-1}}{(n-1)!} + \frac{(x-1)^n}{n!} = \frac{(x-1)^{n-1}}{(n-1)!} \cdot \left(1 + \frac{x-1}{n}\right).$$

□

Further we apply (4.3) and (4.4) in Theorem 3.2.

Theorem 4.3. *For $f \in C^{(n)}[0,1]$ and the differentiation formula $A(f)$ from (4.2) we have*

$$|A(f)| \leq \frac{(1-x)^{(n-1)}}{2(n-1)!} \cdot \tilde{\omega}\left(f^{(n)}, 2\left(1+\frac{x-1}{n}\right)\right). \tag{4.5}$$

It is easy to observe that for $x = 1, A(f) = 0$ and the right side of (4.5) is also 0. The estimate (4.5) is pointwise, i.e. the formula of differentiation (4.2) gives the possibility of approximating $f'(x)$ at each $x \in [0,1]$ by the derivative of its Taylor polynomial at $x_0 = 1$. It would be interesting to establish similar formulas for numerical differentiation using the Taylor expansion formula at $x_0 < 1$, including higher order derivatives of f.

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