

Journal of Inequalities in Pure and Applied Mathematics

ON SOME NEW RETARD INTEGRAL INEQUALITIES IN n INDEPENDENT VARIABLES AND THEIR APPLICATIONS

XUEQIN ZHAO AND FANWEI MENG

Department of Mathematics
Qufu Normal University
Qufu 273165
People's Republic of China.

EMail: xqzhao1972@126.com

©2000 Victoria University
ISSN (electronic): 1443-5756
318-05



volume 7, issue 3, article 111,
2006.

*Received 24 October, 2005;
accepted 24 May, 2006.*

Communicated by: S.S. Dragomir

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

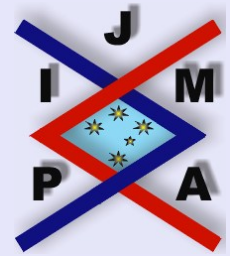
In this paper, we established some retard integral inequalities in n independent variables and by means of examples we show the usefulness of our results.

2000 Mathematics Subject Classification: 26D15, 26D10.

Key words: Retard integral inequalities; integral equation; Partial differential equation.

Contents

1	Introduction	3
2	Preliminaries and Lemmas	4
3	Main Results	5
4	Some Applications	22
	References	



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

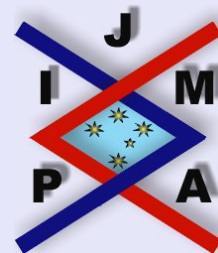
Close

Quit

Page 2 of 27

1. Introduction

The study of integral inequalities involving functions of one or more independent variables is an important tool in the study of existence, uniqueness, bounds, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations. During the past few years, many new inequalities have been discovered (see [1, 3, 4, 7, 8]). In the qualitative analysis of some classes of partial differential equations, the bounds provided by the earlier inequalities are inadequate and it is necessary to seek some new inequalities in order to achieve a diversity of desired goals. Our aim in this paper is to establish some new inequalities in n independent variables, meanwhile, some applications of our results are also given.



**On Some New Retard Integral
Inequalities in n Independent
Variables and Their
Applications**

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

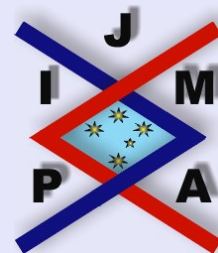
Page 3 of 27

2. Preliminaries and Lemmas

In this paper, we suppose $\mathbb{R}_+ = [0, \infty)$, is subset of real numbers \mathbb{R} , $\tilde{0} = (0, \dots, 0)$, $\tilde{\alpha}(t) = (\alpha_1(t_1), \dots, \alpha_n(t_n)) \in \mathbb{R}_+^n$, $t = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, $s = (s_1, \dots, s_n) \in \mathbb{R}_+^n$, $\tilde{r} = (r_1, \dots, r_n)$, $\tilde{r}_0 = (r_{10}, \dots, r_{n0})$, $\tilde{z} = (z_1, \dots, z_n)$, $\tilde{z}_0 = (z_{10}, \dots, z_{n0})$, $T = (T_1, \dots, T_n) \in \mathbb{R}_+^n$.

If $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$, we suppose

1. $s \leq t \Leftrightarrow s_i \leq t_i \quad (i = 1, 2, \dots, n)$;
2. $\int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds = \int_0^{\alpha_1(t_1)} \dots \int_0^{\alpha_n(t_n)} f(s_1, \dots, s_n) ds_n \dots ds_1$;
3. $D_i = \frac{d}{dt_i}$, $i = 1, 2, \dots, n$.



**On Some New Retard Integral
Inequalities in n Independent
Variables and Their
Applications**

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 4 of 27

3. Main Results

In this part, we obtain our main results as follows:

Theorem 3.1. *Let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\psi(u) > 0$ on $(0, \infty)$, and let c be a nonnegative constant. Let $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on \mathbb{R}_+ ($i = 1, \dots, n$). If $u, f \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ and*

$$(3.1) \quad u(t) \leq c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)\psi(u(s))ds,$$

for $\tilde{0} \leq t < T$, then

$$(3.2) \quad u(t) \leq G^{-1} \left[G(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds \right],$$

where

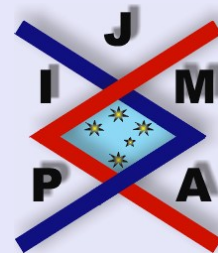
$$G(\tilde{z}) = \int_{\tilde{z}_0}^{\tilde{z}} \frac{ds}{\psi(s)}, \quad \tilde{z} \geq \tilde{z}_0 > 0.$$

G^{-1} is the inverse of G , $T \in \mathbb{R}_+^n$ is chosen so that

$$(3.3) \quad G(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds \in \text{Dom}(G^{-1}), \quad \tilde{0} \leq t < T.$$

Define the nondecreasing positive function $z(t)$ and make

$$(3.4) \quad z(t) = c + \varepsilon + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)\psi(u(s))ds, \quad \tilde{0} \leq t < T,$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 5 of 27

where ε is an arbitrary small positive number. We know that

$$(3.5) \quad u(t) \leq z(t), \quad D_1 D_2 \cdots D_n z(t) = f(\tilde{\alpha}) \psi(u(\tilde{\alpha})) \alpha'_1 \alpha'_2 \cdots \alpha'_n.$$

Using (3.5), we have

$$(3.6) \quad \frac{D_1 D_2 \cdots D_n z(t)}{\psi(z(t))} \leq f(\tilde{\alpha}) \alpha'_1 \alpha'_2 \cdots \alpha'_n.$$

For

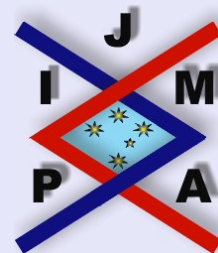
$$(3.7) \quad D_n \left(\frac{D_1 D_2 \cdots D_{n-1} z(t)}{\psi(z(t))} \right) = \frac{D_1 D_2 \cdots D_n z(t) \psi(z(t)) - D_1 D_2 \cdots D_{n-1} z(t) \psi' D_n z(t)}{\psi^2(z(t))},$$

using $D_1 D_2 \cdots D_{n-1} z(t) \geq 0$, $\psi' \geq 0$, $D_n z(t) \geq 0$ in (3.7), we get

$$(3.8) \quad D_n \left(\frac{D_1 D_2 \cdots D_{n-1} z(t)}{\psi(z(t))} \right) \leq \frac{D_1 D_2 \cdots D_n z(t)}{\psi(z(t))} \leq f(\tilde{\alpha}) \alpha'_1 \alpha'_2 \cdots \alpha'_n.$$

Fixing t_1, \dots, t_{n-1} , setting $t_n = s_n$, integrating from t_n to ∞ , yields

$$(3.9) \quad \frac{D_1 D_2 \cdots D_{n-1} z(t)}{\psi(z(t))} \leq \int_0^{\alpha_n(t_n)} f(\alpha_1(t_1), \dots, \alpha_{n-1}(t_{n-1}), s_n) \alpha'_1 \alpha'_2 \cdots \alpha'_{n-1} ds_n.$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 6 of 27

Using the same method, we deduce that

$$(3.10) \quad \frac{D_1 z(t)}{\psi(z(t))} \leq \int_0^{\alpha_2(t_2)} \cdots \int_0^{\alpha_n(t_n)} f(t_1, s_2, \dots, s_n) \alpha_1' ds_n \dots ds_2,$$

and integration on $[t_1, \infty)$ yields

$$(3.11) \quad G(z(t)) \leq G(c + \varepsilon) + \int_0^{\alpha_1(t_1)} \cdots \int_0^{\alpha_n(t_n)} f(s_1, s_2, \dots, s_n) ds_n \dots ds_1, \quad t \in \mathbb{R}_+^n.$$

From the definition of G and letting $\varepsilon \rightarrow 0$, we can obtain inequality (3.2).

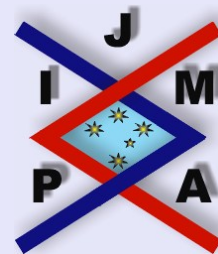
Remark 1. If we let $G(z) \rightarrow \infty$, $z \rightarrow \infty$, then condition (3.3) can be omitted.

Corollary 3.2. If we let $\psi = s^r$, $0 < r \leq 1$ in Theorem 3.1, then for $t \in \mathbb{R}_+^n$, we have

$$(3.12) \quad u(t) \leq \begin{cases} \left[c^{1-r} + (1-r) \int_0^{\tilde{\alpha}(t)} f(s) ds \right]^{\frac{1}{1-r}}, & 0 < r < 1; \\ c \exp \left(\int_0^{\tilde{\alpha}(t)} f(s) ds \right), & r = 1. \end{cases}$$

Remark 2. If we let $n = 1$, $r = 1$, $\tilde{\alpha}(t) = t$, in Corollary 3.2, we obtain the Mate-Nevali inequality.

Theorem 3.3. Let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$. Let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function and let c be a nonnegative



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 7 of 27

constant. Let $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on \mathbb{R}_+ ($i = 1, \dots, n$). If $u, f \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ and

$$(3.13) \quad \varphi(u(t)) \leq c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)\psi(u(s))ds,$$

for $\tilde{0} \leq t < T$, then

$$(3.14) \quad u(t) \leq \varphi^{-1} \left\{ G^{-1} \left[G(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds \right] \right\},$$

where $G(\tilde{z}) = \int_{\tilde{z}_0}^{\tilde{z}} \frac{ds}{\psi[\varphi^{-1}(s)]}$, $\tilde{z} \geq \tilde{z}_0 > 0$, φ^{-1}, G^{-1} are respectively the inverse of φ and G , $T \in \mathbb{R}_+^n$ is chosen so that

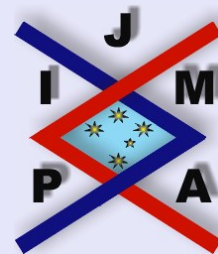
$$(3.15) \quad G(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds \in \text{Dom}(G^{-1}), \quad \tilde{0} \leq t < T.$$

Proof. From the definition of the φ , we know (3.13) can be restated as

$$(3.16) \quad \varphi(u(t)) \leq c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)\psi[\varphi^{-1}(\varphi(u(s)))]ds, \quad t \in \mathbb{R}_+^n.$$

Now an application of Theorem 3.1 gives

$$(3.17) \quad \varphi(u(t)) \leq G^{-1} \left[G(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds \right], \quad \tilde{0} \leq t < T.$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 8 of 27

So,

$$(3.18) \quad u(t) \leq \varphi^{-1} \left\{ G^{-1} [G(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds] \right\}, \quad \tilde{0} \leq t < T.$$

□

Corollary 3.4. *If we let $\varphi = s^p$, $\psi = s^q$, p, q are constants, and $p \geq q > 0$ in Theorem 3.3, for $\tilde{0} \leq t < T$, then*

$$(3.19) \quad u(t) \leq \begin{cases} \left[c^{1-\frac{q}{p}} + \left(1 - \frac{q}{p}\right) \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds \right]^{\frac{1}{p-q}}, & \text{when } p > q; \\ c^{\frac{1}{p}} \exp \left(\frac{1}{p} \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds \right), & \text{when } p = q. \end{cases}$$

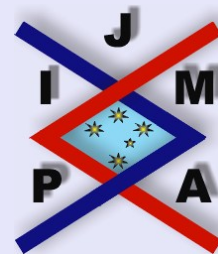
Theorem 3.5. *Let u, f and g be nonnegative continuous functions defined on \mathbb{R}_+^n , let c be a nonnegative constant. Moreover, let $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_i(u) > 0$ ($i = 1, 2$) on $(0, \infty)$. Let $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on \mathbb{R}_+ ($i = 1, \dots, n$). If*

$$(3.20) \quad u(t) \leq c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) w_1(u(s)) ds + \int_{\tilde{0}}^t g(s) w_2(u(s)) ds,$$

for $\tilde{0} \leq t < T$, then

(i) *for the case $w_2(u) \leq w_1(u)$,*

$$(3.21) \quad u(t) \leq G_1^{-1} \left\{ G_1(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds + \int_{\tilde{0}}^t g(s) ds \right\},$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents

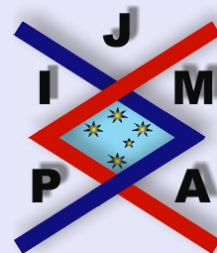


Go Back

Close

Quit

Page 9 of 27



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 10 of 27

(ii) for the case $w_1(u) \leq w_2(u)$,

$$(3.22) \quad u(t) \leq G_2^{-1} \left\{ G_2(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \right\},$$

where

$$(3.23) \quad G_i(\tilde{z}) = \int_{\tilde{z}_0}^{\tilde{z}} \frac{ds}{w_i(s)}, \quad \tilde{z} \geq \tilde{z}_0 > 0, \quad (i = 1, 2)$$

and G_i^{-1} ($i = 1, 2$) is the inverse of G_i , $T \in \mathbb{R}_+^n$ is chosen so that

$$(3.24) \quad G_i(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \in \text{Dom}(G_i^{-1}), \quad (i = 1, 2), \quad \tilde{0} \leq t < T.$$

Proof. Define the nonincreasing positive function $z(t)$ and make

$$(3.25) \quad z(t) = c + \varepsilon + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)w_1(u(s))ds + \int_{\tilde{0}}^t g(s)w_2(u(s))ds,$$

where ε is an arbitrary small positive number. From inequality (3.20), we know

$$(3.26) \quad u(t) \leq z(t)$$

and

$$(3.27) \quad D_1 D_2 \cdots D_n z(t) = [f(\tilde{\alpha})w_1(u(\tilde{\alpha}))\alpha'_1 \alpha'_2 \cdots \alpha'_n + g(t)w_2(u(t))].$$

The rest of the proof can be completed by following the proof of Theorem 3.1 with suitable modifications. \square

Theorem 3.6. *Let u, f and g be nonnegative continuous functions defined on \mathbb{R}_+^n , and let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$ and let c be a nonnegative constant. Moreover, let $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_i(u) > 0$ ($i = 1, 2$) on $(0, \infty)$, $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on \mathbb{R}_+ ($i = 1, \dots, n$). If*

$$(3.28) \quad \varphi(u(t)) \leq c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)w_1(u(s))ds + \int_{\tilde{0}}^t g(s)w_2(u(s))ds,$$

for $\tilde{0} \leq t < T$, then

(i) for the case $w_2(u) \leq w_1(u)$,

$$(3.29) \quad u(t) \leq \varphi^{-1} \left\{ G_1^{-1} \left[G_1(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \right] \right\};$$

(ii) for the case $w_1(u) \leq w_2(u)$,

$$(3.30) \quad u(t) \leq \varphi^{-1} \left\{ G_2^{-1} \left[G_2(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \right] \right\},$$

where

$$G_i(\tilde{z}) = \int_{\tilde{z}_0}^{\tilde{z}} \frac{ds}{w_i(\varphi^{-1}(s))}, \quad \tilde{z} > \tilde{z}_0, \quad (i = 1, 2),$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents

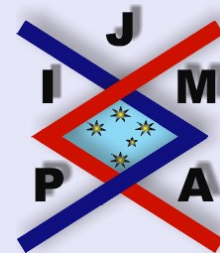


Go Back

Close

Quit

Page 11 of 27



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents

⏪

⏩

◀

▶

Go Back

Close

Quit

Page 12 of 27

and φ^{-1}, G_i^{-1} ($i = 1, 2$) are respectively the inverse of $G_i, \varphi, T \in \mathbb{R}_+^n$ is chosen so that

$$(3.31) \quad G_i(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \in \text{Dom}(G_i^{-1}), \quad (i = 1, 2), \quad \tilde{0} \leq t < T.$$

Proof. From the definition of φ , we know (3.28) can be restated as

$$(3.32) \quad \varphi(u(t)) \leq c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)w_1[\varphi^{-1}(\varphi(u(s)))]ds + \int_{\tilde{0}}^t g(s)w_2[\varphi^{-1}(\varphi(u(s)))]ds, \quad t \in \mathbb{R}_+^n.$$

Now an application of Theorem 3.5 gives

$$\varphi(u(t)) \leq G_i^{-1} \left\{ G_i(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \right\}, \quad \tilde{0} \leq t < T,$$

where T satisfies (3.31). We can obtain the desired inequalities (3.29) and (3.30). \square

Theorem 3.7. Let u, f and g be nonnegative continuous functions defined on \mathbb{R}_+^n and let c be a nonnegative constant. Moreover, let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty, \psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 13 of 27

function with $\psi(u) > 0$ on $(0, \infty)$ and $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on \mathbb{R}_+ ($i = 1, \dots, n$). If

$$(3.33) \quad \varphi(u(t)) \leq c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} [f(s)u(s)\psi(u(s)) + g(s)u(s)]ds,$$

for $\tilde{0} \leq t < T$, then

$$(3.34) \quad u(t) \leq \varphi^{-1} \left\{ \Omega^{-1} \left[G^{-1} \left(G[\Omega(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} g(s)ds] + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds \right) \right] \right\},$$

where

$$\Omega(\tilde{r}) = \int_{\tilde{r}_0}^{\tilde{r}} \frac{ds}{\varphi^{-1}(s)}, \quad \tilde{r} \geq \tilde{r}_0 > 0,$$

$$G(\tilde{z}) = \int_{\tilde{z}_0}^{\tilde{z}} \frac{ds}{\psi\{\varphi^{-1}[\Omega^{-1}(s)]\}}, \quad \tilde{z} \geq \tilde{z}_0 > 0,$$

$\Omega^{-1}, \varphi^{-1}, G^{-1}$ are respectively the inverse of Ω, φ, G . And $T \in \mathbb{R}_+$ is chosen so that

$$G \left[\Omega(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} g(s)ds \right] + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds \in \text{Dom}(G^{-1}), \quad \tilde{0} \leq t < T,$$

and

$$G^{-1} \left\{ G \left[\Omega(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} g(s)ds \right] + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds \right\} \in \text{Dom}(\Omega^{-1}), \quad \tilde{0} \leq t < T.$$

Proof. Let us first assume that $c > 0$. Defining the nondecreasing positive function $z(t)$ by the right-hand side of (3.33)

$$z(t) = c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} [f(s)u(s)\psi(u(s)) + g(s)u(s)]ds,$$

we know

$$(3.35) \quad u(t) \leq \varphi^{-1}[z(t)]$$

and

$$(3.36) \quad D_1 D_2 \cdots D_n z(t) = [f(\tilde{\alpha})u(\tilde{\alpha})\psi(u(\tilde{\alpha})) + g(\tilde{\alpha})u(\tilde{\alpha})] \alpha'_1 \alpha'_2 \cdots \alpha'_n.$$

Using (3.35), we have

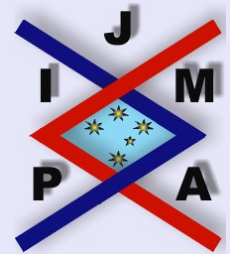
$$(3.37) \quad \frac{D_1 D_2 \cdots D_n z(t)}{\varphi^{-1}(z(t))} \leq [f(\tilde{\alpha})\psi(\varphi^{-1}(z(\alpha))) + g(\tilde{\alpha})] \alpha'_1 \alpha'_2 \cdots \alpha'_n.$$

For

$$(3.38) \quad D_n \left(\frac{D_1 D_2 \cdots D_{n-1} z(t)}{\varphi^{-1}(z(t))} \right) = \frac{D_1 D_2 \cdots D_n z(t) \varphi^{-1}(z(t)) - D_1 D_2 \cdots D_{n-1} z(t) (\varphi^{-1}(z(t)))' D_n z(t)}{(\varphi^{-1}(z(t)))^2},$$

using $D_1 D_2 \cdots D_{n-1} z(t) \geq 0$, $D_n z(t) \geq 0$, $(\varphi^{-1}(z(t)))' \geq 0$ in (3.38), we get

$$(3.39) \quad D_n \left(\frac{D_1 D_2 \cdots D_{n-1} z(t)}{\varphi^{-1}(z(t))} \right) \leq \frac{D_1 D_2 \cdots D_n z(t)}{\varphi^{-1}(z(t))} \leq [f(\tilde{\alpha})\psi(\varphi^{-1}z(\tilde{\alpha})) + g(\tilde{\alpha})] \alpha'_1 \alpha'_2 \cdots \alpha'_n.$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page	
Contents	
⏪	⏩
◀	▶
Go Back	
Close	
Quit	
Page 14 of 27	

Fixing t_1, \dots, t_{n-1} , setting $t_n = s_n$, integrating from 0 to t_n with respect to s_n yields

$$(3.40) \quad \frac{D_1 D_2 \cdots D_{n-1} z(t)}{\varphi^{-1}(z(t))} \\ \leq \int_0^{\alpha_n(t_n)} [f(\alpha_1(t_1), \dots, \alpha_{n-1}(t_{n-1}), s_n) \psi(\varphi^{-1}(z(\alpha_1(t_1), \dots, \alpha_{n-1}(t_{n-1}), s_n))) \\ + g(\alpha_1(t_1), \dots, \alpha_{n-1}(t_{n-1}), s_n)] \alpha'_1 \alpha'_2 \cdots \alpha'_{n-1} ds_n.$$

Using the same method, we deduce that

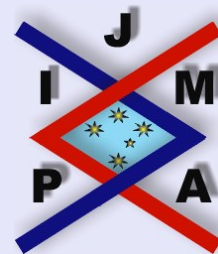
$$(3.41) \quad \frac{D_1 z(t)}{\varphi^{-1}(z(t))} \\ \leq \int_0^{\alpha_2(t_2)} \cdots \int_0^{\alpha_n(t_n)} [f(\alpha_1(t_1), s_2, \dots, s_n) \psi(\varphi(z(\alpha_1(t_1), s_2, \dots, s_n))) \\ + g(\alpha_1(t_1), s_2, \dots, s_n)] \alpha'_1 ds_n \dots ds_2.$$

Setting $t_1 = s_1$, and integrating it from 0 to t_1 with respect to s_1 yields

$$(3.42) \quad \Omega(z(t)) \leq \Omega(c) + \int_0^{\tilde{\alpha}(t)} f(s) \psi(\varphi^{-1}(z(s))) ds + \int_0^{\tilde{\alpha}(t)} g(s) ds,$$

Let $T_1 \leq T$ be arbitrary, we denote $p(T_1) = \Omega(c) + \int_0^{\tilde{\alpha}(T_1)} g(s) ds$, from (3.42), we deduce that

$$\Omega(z(t)) \leq p(T_1) + \int_0^{\tilde{\alpha}(t)} f(s) \psi[\varphi^{-1}(z(s))] ds, \quad \tilde{0} \leq t \leq T_1 \leq T.$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 15 of 27

Now an application of Theorem 3.3 gives

$$z(t) \leq \Omega^{-1} \left\{ G^{-1} \left[G(p(T_1)) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds \right] \right\}, \quad \tilde{0} \leq t \leq T_1 \leq T,$$

so

$$u(t) \leq \varphi^{-1} \left\{ \Omega^{-1} \left[G^{-1} \left(G(p(T_1)) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds \right) \right] \right\}, \quad \tilde{0} \leq t \leq T_1 \leq T.$$

Taking $t = T_1$ in the above inequality, since T_1 is arbitrary, we can prove the desired inequality (3.34). \square

If $c = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \rightarrow 0$.

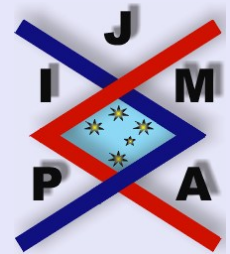
Setting $f(t) = 0$, $n = 1$, we can obtain a retarded Ou-Iang inequality.

Let u, f and g be nonnegative continuous functions defined on \mathbb{R}_+^n and let c be a nonnegative constant. Moreover, let $\psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function with $\psi(u) > 0$ on $(0, \infty)$ and $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on \mathbb{R}_+ ($i = 1, \dots, n$). If

$$u^2(t) \leq c^2 + \int_{\tilde{0}}^{\tilde{\alpha}(t)} [f(s)u(s)\psi(u(s)) + g(s)u(s)] ds,$$

for $\tilde{0} \leq t < T$, then

$$u(t) \leq \Omega^{-1} \left[\Omega \left(c + \frac{1}{2} \int_{\tilde{0}}^{\tilde{\alpha}(t)} g(s) ds \right) + \frac{1}{2} \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds \right],$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 16 of 27

where

$$\Omega(\tilde{z}) = \int_{\tilde{z}_0}^{\tilde{z}} \frac{ds}{\psi(s)} \quad \tilde{z} > \tilde{z}_0,$$

Ω^{-1} is the inverse of Ω , and $T \in \mathbb{R}_+^n$ is chosen so that

$$\Omega \left(c + \frac{1}{2} \int_{\tilde{0}}^{\tilde{\alpha}(t)} g(s) ds \right) + \frac{1}{2} \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds \in \text{Dom}(\Omega^{-1}), \quad \tilde{0} \leq t < T.$$

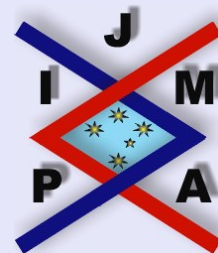
Corollary 3.8. *Let u, f and g be nonnegative continuous functions defined on \mathbb{R}_+^n and let c be a nonnegative constant. Moreover, let p, q be positive constants with $p \geq q, p \neq 1$. Let $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ on \mathbb{R}_+ ($i = 1, \dots, n$). If*

$$u^p(t) \leq c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} [f(s)u^q(s) + g(s)u(s)] ds, \quad t \geq 0$$

for $\tilde{0} \leq t < T$ then

$$u(t) \leq \begin{cases} \left(c^{(1-\frac{1}{p})} + \frac{p-1}{p} \int_{\tilde{0}}^{\tilde{\alpha}(t)} g(s) ds \right)^{\frac{p}{p-1}} \exp \left[\frac{1}{p} \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds \right], & \text{when } p = q; \\ \left[\left(c^{(1-\frac{1}{p})} + \frac{p-1}{p} \int_{\tilde{0}}^{\tilde{\alpha}(t)} g(s) ds \right)^{\frac{p-q}{p-1}} + \frac{p-q}{p} \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds \right]^{\frac{1}{p-q}}, & \text{when } p > q. \end{cases}$$

Theorem 3.9. *Let u, f and g be nonnegative continuous functions defined on \mathbb{R}_+^n , and let $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function with $\varphi(\infty) = \infty$ and*



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 17 of 27

let c be a nonnegative constant. Moreover, let $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_i(u) > 0$ ($i = 1, 2$) on $(0, \infty)$, and $\alpha_i \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\alpha_i(t_i) \leq t_i$ ($i = 1, \dots, n$). If

$$(3.43) \quad \varphi(u(t)) \leq c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)u(s)w_1(u(s))ds + \int_{\tilde{0}}^t g(s)u(s)w_2(u(s))ds,$$

for $\tilde{0} \leq t < T$, then

(i) for the case $w_2(u) \leq w_1(u)$,

$$(3.44) \quad u(t) \leq \varphi^{-1} \left\{ \Omega^{-1} \left[G_1^{-1} \left(G_1(\Omega(c)) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \right) \right] \right\},$$

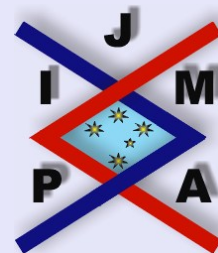
(ii) for the case $w_1(u) \leq w_2(u)$,

$$(3.45) \quad u(t) \leq \varphi^{-1} \left\{ \Omega^{-1} \left[G_2^{-1} \left(G_2(\Omega(c)) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \right) \right] \right\},$$

where

$$\Omega(\tilde{r}) = \int_{\tilde{r}_0}^{\tilde{r}} \frac{ds}{\varphi^{-1}(s)}, \quad \tilde{r} \geq \tilde{r}_0 > 0,$$

$$G_i(\tilde{z}) = \int_{\tilde{z}_0}^{\tilde{z}} \frac{ds}{w_i\{\varphi^{-1}[\Omega^{-1}(s)]\}}, \quad \tilde{z} \geq \tilde{z}_0 > 0 \quad (i = 1, 2)$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents

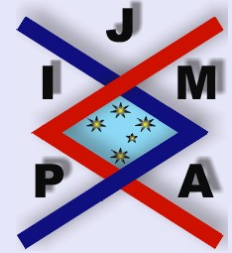


Go Back

Close

Quit

Page 18 of 27



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents

⏪ ⏩

◀ ▶

Go Back

Close

Quit

Page 19 of 27

$\Omega^{-1}, \varphi^{-1}, G^{-1}$ are respectively the inverse of Ω, φ, G , and $T \in \mathbb{R}_+$ is chosen so that

$$G_i \left(\Omega(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \right) \in \text{Dom}(G_i^{-1}), \quad \tilde{0} \leq t \leq T,$$

and

$$G_i^{-1} \left[G_i \left(\Omega(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)ds + \int_{\tilde{0}}^t g(s)ds \right) \right] \in \text{Dom}(\Omega^{-1}), \quad \tilde{0} \leq t \leq T.$$

Proof. Let $c > 0$ and define the nonincreasing positive function $z(t)$ and make

$$(3.46) \quad z(t) = c + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s)u(s)w_1(u(s))ds + \int_{\tilde{0}}^t g(s)u(s)w_2(u(s))ds.$$

From inequality (3.43), we know

$$(3.47) \quad u(t) \leq \varphi^{-1}[z(t)],$$

and

$$(3.48) \quad D_1 D_2 \cdots D_n z(t) = [f(\tilde{\alpha})u(\tilde{\alpha})w_1(u(\tilde{\alpha}))\alpha'_1 \alpha'_2 \cdots \alpha'_n + g(t)u(t)w_2(u(t))].$$

Using (3.47), we have

$$(3.49) \quad \frac{D_1 D_2 \cdots D_n z(t)}{\varphi^{-1}(z(t))} \leq f(\tilde{\alpha})w_1(u(\tilde{\alpha}))\alpha'_1 \alpha'_2 \cdots \alpha'_n + g(t)w_2(u(t)).$$

For

$$(3.50) \quad D_n \left(\frac{D_1 D_2 \cdots D_{n-1} z(t)}{\varphi^{-1}(z(t))} \right) = \frac{D_1 D_2 \cdots D_n z(t) \varphi^{-1}(z(t)) - D_1 D_2 \cdots D_{n-1} z(t) (\varphi^{-1}(z(t)))' D_n z(t)}{(\varphi^{-1}(z(t)))^2},$$

using $D_1 D_2 \cdots D_{n-1} z(t) \geq 0$, $(\varphi^{-1}(z(t)))' \geq 0$, $D_n z(t) \geq 0$ in (3.50), we get

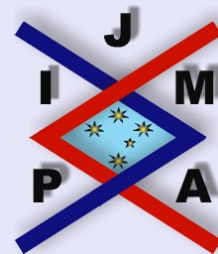
$$D_n \left(\frac{D_1 D_2 \cdots D_{n-1} z(t)}{\varphi^{-1}(z(t))} \right) \leq \frac{D_1 D_2 \cdots D_n z(t)}{\varphi^{-1}(z(t))} \leq f(\tilde{\alpha}) \alpha'_1 \alpha'_2 \cdots \alpha'_n w_1(\varphi^{-1}(\tilde{\alpha}(t))) + g(t) w_2(\varphi^{-1}(t)).$$

Fixing t_1, \dots, t_{n-1} , setting $t_n = s_n$, integrating from t_n to ∞ , yields

$$\begin{aligned} \frac{D_1 D_2 \cdots D_{n-1} z(t)}{\varphi^{-1}(z(t))} &\leq \int_0^{\alpha_n(t_n)} f(\alpha_1(t_1), \dots, \alpha_{n-1}(t_{n-1}), s_n) \\ &\quad \times w_1(\varphi^{-1}(\alpha_1(t_1), \dots, \alpha_{n-1}(t_{n-1}), s_n)) \alpha'_1 \alpha'_2 \cdots \alpha'_{n-1} ds_n \\ &\quad + \int_0^{t_n} g(t_1, \dots, t_{n-1}, s_n) w_2(\varphi^{-1}(t_1, \dots, t_{n-1}, s_n)) ds_n. \end{aligned}$$

Deductively

$$(3.51) \quad \frac{D_1 z(t)}{\varphi^{-1}(z(t))} \leq \int_0^{\alpha_2(t_2)} \cdots \int_0^{\alpha_n(t_n)} f(\alpha_1(t_1), s_2, \dots, s_n) \times w_1(\varphi^{-1}(\alpha_1(t_1), s_2, \dots, s_n)) \alpha'_1 ds_n \dots ds_2 + \int_0^{t_2} \cdots \int_0^{t_n} g(t_1, s_2, \dots, s_n) w_2(\varphi^{-1}(t_1, s_2, \dots, s_n)) ds_n \dots ds_2.$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 20 of 27

Fixing t_2, \dots, t_n , setting $t_1 = s_1$, integrating from 0 to t_1 with respect to s_1 yields

$$(3.52) \quad \Omega(z(t)) \leq \Omega(c) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s_1, \dots, s_n) w_1(\varphi^{-1}(z(s))) \\ + \int_{\tilde{0}}^t g(s) w_2(\varphi^{-1}(z(s))) ds, \quad t \in \mathbb{R}_+^n.$$

From Theorem 3.6, we obtain

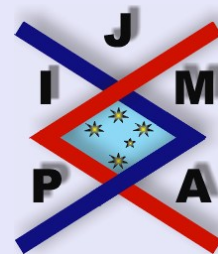
$$z(t) \leq \Omega^{-1} \left[G_1^{-1} \left(G_1(\Omega(c)) + \int_{\tilde{0}}^{\tilde{\alpha}(t)} f(s) ds + \int_{\tilde{0}}^t g(s) ds \right) \right],$$

using (3.47), we get the inequality (3.44).

If $c = 0$ we carry out the above procedure with $\varepsilon > 0$ instead of c and subsequently let $\varepsilon \rightarrow 0$.

(ii) when $w_1(u) \leq w_2(u)$.

The proof can be completed with suitable changes. □



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 21 of 27

4. Some Applications

Example 4.1. Consider the integral equation:

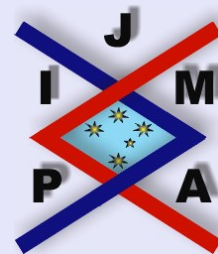
$$(4.1) \quad u^p(t_1, \dots, t_n) \\ = f(t_1, \dots, t_n) + \int_0^{\tilde{\alpha}(t)} K(s_1, \dots, s_n) g(s_1, \dots, s_n, u(s_1, \dots, s_n)) ds_1 \dots ds_n,$$

where $f, K : \mathbb{R}_+^n \rightarrow \mathbb{R}$, $g : \mathbb{R}_+^n \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions and $p > 0$ and $p \neq 1$ is constant, $\tilde{\alpha}_i(t) \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is nondecreasing with $\alpha_i(t) \leq t_i$ on \mathbb{R}_+ ($i = 1, \dots, n$). In [8] B.G. Pachpatte studied the problem when $\alpha(t) = t$, $n = 1$. Here we assume that every solution under discussion exists on an interval \mathbb{R}_+^n . We suppose that the functions f, K, g in (4.1) satisfy the following conditions

$$(4.2) \quad |f(t_1, \dots, t_n)| \leq c_1, \quad |K(t_1, \dots, t_n)| \leq c_2, \\ |g(t_1, \dots, t_n, u)| \leq r(t_1, \dots, t_n)|u|^q + h(t_1, \dots, t_n)|u|,$$

where c_1, c_2 , are nonnegative constants, and $p \geq q > 0$, and $r : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$, $h : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ are continuous functions. From (4.1) and using (4.2), we get

$$(4.3) \quad |u^p(t_1, \dots, t_n)| \\ \leq c_1 + \int_0^{\tilde{\alpha}(t)} [c_2 r(s_1, \dots, s_n)|u|^q + c_2 h(s_1, \dots, s_n)|u|] ds_1 \dots ds_n.$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 22 of 27

Now an application of Corollary 3.8 yields

$$|u(t)| \leq \begin{cases} \left(c_1^{(1-\frac{1}{p})} + \frac{c_2(p-1)}{p} \int_0^{\tilde{\alpha}(t)} h(s) ds_1 \dots ds_n \right)^{\frac{p-1}{p}} \\ \quad \times \exp \left[\frac{c_2}{p} \int_0^{\tilde{\alpha}(t)} r(s) ds_1 \dots ds_n \right] \\ \quad \text{when } p = q, \\ \\ \left[\left(c_1^{(1-\frac{1}{p})} + \frac{c_2(p-1)}{p} \int_0^{\tilde{\alpha}(t)} h(s) ds_1 \dots ds_n \right)^{\frac{p-q}{p-1}} \right. \\ \quad \left. + \frac{c_2(p-q)}{p} \int_0^{\tilde{\alpha}(t)} r(s) ds_1 \dots ds_n \right]^{\frac{1}{p-q}} \\ \quad \text{when } p > q. \end{cases}$$

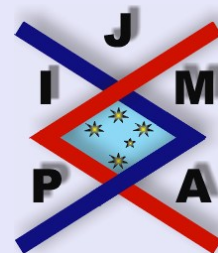
If the integrals of $r(s), h(s)$ are bounded, then we can have the bound of the solution $u(t)$ of (4.1). Similarly, we can obtain many other kinds of estimates.

Example 4.2. Consider the partial delay differential equation:

$$(4.4) \quad \frac{\partial^2 u^p(x, y)}{\partial x_1 \partial x_2} = f(x, y, u(x, y), u(x - h_1(x), y - h_2(y))),$$

$$(4.5) \quad u^p(x, 0) = a_1(x) \quad u^p(0, y) = a_2(y)$$

$$(4.6) \quad a_1(0) = a_2(0) = 0, \quad |a_1(x) + a_2(y)| \leq c,$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 23 of 27

where $f \in C(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2, \mathbb{R})$, $a_1 \in C^1(\mathbb{R}_+, \mathbb{R})$, $a_2 \in C^1(\mathbb{R}_+, \mathbb{R})$, c and p are nonnegative constants. $h_1 \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, $h_2 \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, such that

$$\begin{aligned} x - h_1(x) &\geq 0, & y - h_2(y) &\geq 0, \\ h_1'(x) &< 1, & h_2'(y) &< 1. \end{aligned}$$

Suppose that

$$(4.7) \quad |f(x, y, u, v)| \leq a(x, y)|v|^q + b(x, y)|v|,$$

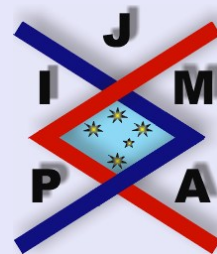
where $a, b \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ and let

$$(4.8) \quad M_1 = \max_{x \in \mathbb{R}_+} \frac{1}{1 - h_1'(x)}, \quad M_2 = \max_{y \in \mathbb{R}_+} \frac{1}{1 - h_2'(y)}.$$

If $u(x, y)$ is any solution of (4.4) – (4.7), then

(i) if $p = q$, we have

$$(4.9) \quad |u(x, y)| \leq \left(c^{(1-\frac{1}{p})} + \frac{M_1 M_2 (p-1)}{p} \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \tilde{b}(\sigma, \tau) d\sigma d\tau \right)^{\frac{p}{p-1}} \times \exp \left[\frac{M_1 M_2}{p} \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \tilde{a}(\sigma, \tau) d\sigma d\tau \right]$$

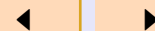


On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 24 of 27

(ii) if $p > q$, we have

$$(4.10) \quad |u(x, y)| \leq \left[\left(c^{(1-\frac{1}{p})} + \frac{M_1 M_2 (p-1)}{p} \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \tilde{b}(\sigma, \tau) d\sigma d\tau \right)^{\frac{p-q}{p-1}} + \frac{M_1 M_2 (p-q)}{p} \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \tilde{a}(\sigma, \tau) d\sigma d\tau \right]^{\frac{1}{p-q}}.$$

In which $\phi_1(x) = x - h_1(x)$, $x \in \mathbb{R}_+^n$, $\phi_2(y) = y - h_2(y)$, $y \in \mathbb{R}_+^n$ and

$$\tilde{b}(\sigma, \tau) = b(\sigma + h_1(s), \tau + h_2(t)), \tilde{a}(\sigma, \tau) = a(\sigma + h_1(s), \tau + h_2(t)),$$

for $\sigma, s, \tau, t \in \mathbb{R}_+^n$.

In fact, if $u(x, y)$ is a solution of (4.4) – (4.7), then it satisfies the equivalent integral equation:

$$(4.11) \quad [u(x, y)]^p = a_1(x) + a_2(y) + \int_0^x \int_0^y f(s, t, u(s, t), u(s - h_1(s), t - h_2(t))) dt ds.$$

for $x, y \in (\mathbb{R}_+^n \times \mathbb{R}_+^n, \mathbb{R})$.

Using (4.5), (4.7) in (4.11) and making the change of variables, we have

$$(4.12) \quad |u(x, y)|^p \leq c + M_1 M_2 \int_0^{\phi_1(x)} \int_0^{\phi_1(y)} \tilde{a}(\sigma, \tau) |u(\sigma, \tau)|^q + \tilde{b}(\sigma, \tau) |u(\sigma, \tau)| d\sigma d\tau.$$



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



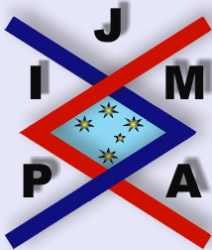
Go Back

Close

Quit

Page 25 of 27

Now a suitable application of the inequality in Corollary 3.8 to (4.12) yields (4.9) and (4.10).



On Some New Retard Integral Inequalities in n Independent Variables and Their Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents

◀◀ ▶▶

◀ ▶

Go Back

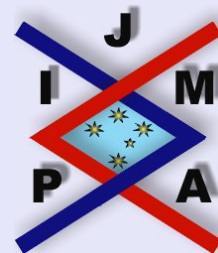
Close

Quit

Page 26 of 27

References

- [1] B.G. PACHPATTE, On some new inequalities related to certain inequalities in the theory of differential equations, *J. Math. Anal. Appl.*, **189** (1995), 128–144.
- [2] B.G. PACHPATTE, Explicit bounds on certain integral inequalities, *J. Math. Anal. Appl.*, **267** (2002), 48–61.
- [3] B.G. PACHPATTE, On some new inequalities related to a certain inequality arising in the theory of differential equations, *J. Math. Anal. Appl.*, **251** (2000), 736–751.
- [4] B.G. PACHPATTE, On a certain inequality arising in the theory of differential equations, *J. Math. Anal. Appl.*, **182** (1994), 143–157.
- [5] I. BIHARI, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta. Math. Acad. Sci. Hungar.*, **7** (1956), 71–94.
- [6] M. MEDVED, Nonlinear singular integral inequalities for functions in two and n independent variables, *J. Inequalities and Appl.*, **5** (2000), 287–308.
- [7] O. LIPOVAN, A retarded Gronwall-like inequality and its applications, *J. Math. Anal. Appl.*, **252** (2000), 389–401.
- [8] O. LIPOVAN, A retarded integral inequality and its applications, *J. Math. Anal. Appl.*, **285** (2003), 436–443.



On Some New Retard Integral
Inequalities in n Independent
Variables and Their
Applications

Xueqin Zhao and Fanwei Meng

Title Page

Contents



Go Back

Close

Quit

Page 27 of 27