ON THE DEGREE OF STRONG APPROXIMATION OF INTEGRABLE FUNCTIONS

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Abstract: We show the strong approximation version of some results of L. Leindler [3] connected with the theorems of P. Chandra [1, 2].

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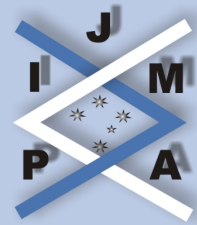
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1. Introduction

Let L^p $(1 < p < \infty)$ $[C]$ be the class of all 2π–periodic real–valued functions integrable in the Lebesgue sense with p–th power [continuous] over $Q = [-\pi, \pi]$ and let $X^p = L^p$ when $1 < p < \infty$ or $X^p = C$ when $p = \infty$. Let us define the norm of $f \in X^p$ as

$$\|f\|_{X^p} = \|f(\cdot)\|_{X^p} = \begin{cases} \left(\int_Q |f(x)|^p \, dx \right)^{\frac{1}{p}} & \text{when } 1 < p < \infty, \\ \sup_{x \in Q} |f(x)| & \text{when } p = \infty, \end{cases}$$

and consider its trigonometric Fourier series

$$Sf(x) = \frac{a_o(f)}{2} + \sum_{\nu=0}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

with the partial sums $S_k f$.

Let $A := (a_{n,k})$ $(k, n = 0, 1, 2, \dots)$ be a lower triangular infinite matrix of real numbers and let the A–transforms of $(S_k f)$ be given by

$$T_{n,A}f(x) := \left| \sum_{k=0}^{n} a_{n,k} S_k f(x) - f(x) \right| \qquad (n = 0, 1, 2, \dots)$$

and

$$H_{n,A}^q f(x) := \left\{ \sum_{k=0}^{n} a_{n,k} |S_k f(x) - f(x)|^q \right\}^{\frac{1}{q}} \qquad (q > 0, \ n = 0, 1, 2, \dots).$$

As a measure of approximation, by the above quantities we use the pointwise

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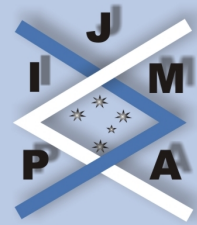
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characteristic

$$w_x f(\delta)_{L^p} := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p \, dt \right\}^{\frac{1}{p}},$$

where

$$\varphi_x(t) := f(x+t) + f(x-t) - 2f(x).$$

$w_x f(\delta)_{L^p}$ is constructed based on the definition of Lebesgue points (L^p−points), and the modulus of continuity for f in the space X^p defined by the formula

$$\omega f(\delta)_{X^p} := \sup_{0 \leq |h| \leq \delta} \|\varphi \cdot (h)\|_{X^p}.$$

We can observe that with $\widetilde{p} \geq p$, for $f \in X^{\widetilde{p}}$, by the Minkowski inequality

$$\|w \cdot f(\delta)_p\|_{X^{\widetilde{p}}} \leq \omega f(\delta)_{X^{\widetilde{p}}}.$$

The deviation $T_{n,A}f$ was estimated by P. Chandra [1, 2] in the norm of $f \in C$ and for monotonic sequences $a_n = (a_{n,k})$. These results were generalized by L. Leindler [3] who considered the sequences of bounded variation instead of monotonic ones. In this note we shall consider the strong means $H_{n,A}^q f$ and the integrable functions. We shall also give some results on norm approximation.

By K we shall designate either an absolute constant or a constant depending on some parameters, not necessarily the same of each occurrence.
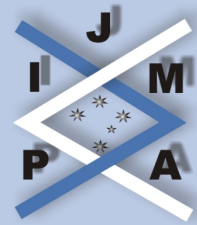
2. Statement of the Results

Let us consider a function w_x of modulus of continuity type on the interval $[0, +\infty)$, i.e., a nondecreasing continuous function having the following properties: $w_x(0) = 0$, $w_x(\delta_1 + \delta_2) \leq w_x(\delta_1) + w_x(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2$ and let

$$L^p(w_x) = \{f \in L^p : w_x f(\delta)_{L^p} \leq w_x(\delta)\}.$$

We can now formulate our main results.

To start with, we formulate the results on pointwise approximation.

Theorem 2.1. *Let $a_n = (a_{n,m})$ satisfy the following conditions:*

(2.1)
$$a_{n,m} \geq 0, \qquad \sum_{k=0}^{n} a_{n,k} = 1$$

and

(2.2)
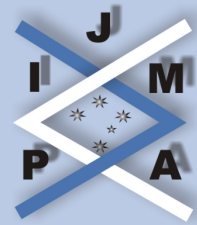$$\sum_{k=0}^{m-1} |a_{n,k} - a_{n,k+1}| \leq K a_{n,m},$$

where

$$m = 0, 1, \ldots, n \quad and \quad n = 0, 1, 2, \ldots \quad \left(\sum_{k=0}^{-1} = 0\right).$$

Suppose w_x is such that

(2.3)
$$\left\{ u^{\frac{p}{q}} \int_{u}^{\pi} \frac{(w_x(t))^p}{t^{1+\frac{p}{q}}} dt \right\}^{\frac{1}{p}} = O(u H_x(u)) \quad as \quad u \to 0+,$$

where $H_x(u) \geq 0$, $1 < p \leq q$ and

(2.4)
$$\int_{0}^{t} H_x(u)\, du = O(t H_x(t)) \quad as \quad t \to 0+.$$

If $f \in L^p(w_x)$, then

$$H_{n,A}^{q'} f(x) = O(a_{n,n} H_x(a_{n,n}))$$

with $q' \in (0, q]$ and q such that $1 < q(q-1) \le p \le q$.

Theorem 2.2. *Let (2.1), (2.2) and (2.3) hold. If $f \in L^p(w_x)$ then*

$$H_{n,A}^{q'} f(x) = O\left(w_x\left(\frac{\pi}{n+1}\right)\right) + O\left(a_{n,n} H_x\left(\frac{\pi}{n+1}\right)\right)$$

and if, in addition, (2.4) holds then

$$H_{n,A}^{q'} f(x) = O\left(a_{n,n} H_x\left(\frac{\pi}{n+1}\right)\right)$$

with $q' \in (0, q]$ and q such that $1 < q(q-1) \le p \le q$.

Theorem 2.3. *Let (2.1), (2.3), (2.4) and*

(2.5)
$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+1}| \le K a_{n,m},$$
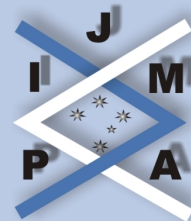
where

$$m = 0, 1, \ldots, n \ \text{ and } \ n = 0, 1, 2, \ldots$$

hold. If $f \in L^p(w_x)$ then

$$H_{n,A}^{q'} f(x) = O(a_{n,0} H_x(a_{n,0}))$$

with $q' \in (0, q]$ and q such that $1 < q(q-1) \le p \le q$.

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Theorem 2.4. *Let us assume that* (2.1), (2.3) *and* (2.5) *hold. If* $f \in L^p(w_x)$, *then*

$$H_{n,A}^{q'} f(x) = O\left(w_x(\frac{\pi}{n+1})\right) + O\left(a_{n,0} H_x\left(\frac{\pi}{n+1}\right)\right).$$

If, in addition, (2.4) *holds then*

$$H_{n,A}^{q'} f(x) = O\left(a_{n,0} H_x\left(\frac{\pi}{n+1}\right)\right)$$

with $q' \in (0, q]$ *and* q *such that* $1 < q(q-1) \leq p \leq q$.

Consequently, we formulate the results on norm approximation.

Theorem 2.5. *Let* $a_n = (a_{n,m})$ *satisfy the conditions* (2.1) *and* (2.2). *Suppose* $\omega f(\cdot)_{X^{\widetilde{p}}}$ *is such that*

$$(2.6) \qquad \left\{ u^{\frac{p}{q}} \int_u^\pi \frac{(\omega f(t)_{X^{\widetilde{p}}})^p}{t^{1+\frac{p}{q}}} dt \right\}^{\frac{1}{p}} = O\left(uH(u)\right) \quad as \quad u \to 0+$$
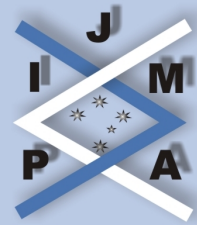
holds, with $1 < p \leq q$ *and* $\widetilde{p} \geq p$, *where* $H \ (\geq 0)$ *instead of* H_x *satisfies the condition* (2.4). *If* $f \in X^{\widetilde{p}}$ *then*

$$\left\| H_{n,A}^{q'} f(\cdot) \right\|_{X^{\widetilde{p}}} = O\left(a_{n,n} H(a_{n,n})\right)$$

with $q' \leq q$ *and* $p \leq \widetilde{p}$ *such that* $1 < q(q-1) \leq p \leq q$.
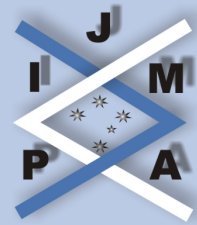
Theorem 2.6. *Let* (2.1), (2.2) *and* (2.6) *hold. If* $f \in X^{\widetilde{p}}$ *then*

$$\left\| H_{n,A}^{q'} f(\cdot) \right\|_{X^{\widetilde{p}}} = O\left(\omega f\left(\frac{\pi}{n+1}\right)_{X^{\widetilde{p}}}\right) + O\left(a_{n,n} H\left(\frac{\pi}{n+1}\right)\right).$$

If, in addition, $H \ (\geq 0)$ instead of H_x satisfies the condition (2.4) *then*

$$\left\| H_{n,A}^{q'} f\left(\cdot\right) \right\|_{X^{\widetilde{p}}} = O\left(a_{n,n} H\left(\frac{\pi}{n+1}\right)\right)$$

with $q' \leq q$ and $p \leq \widetilde{p}$ such that $1 < q\left(q-1\right) \leq p \leq q$.

Theorem 2.7. *Let* (2.1), (2.4) *with a function $H \ (\geq 0)$ instead of H_x,* (2.5) *and* (2.6) *hold. If $f \in X^{\widetilde{p}}$ then*

$$\left\| H_{n,A}^{q'} f\left(\cdot\right) \right\|_{X^{\widetilde{p}}} = O\left(a_{n,0} H\left(a_{n,0}\right)\right)$$

with $q' \leq q$ and $p \leq \widetilde{p}$ such that $1 < q\left(q-1\right) \leq p \leq q$.

Theorem 2.8. *Let* (2.1), (2.5) *and* (2.6) *hold. If $f \in X^{\widetilde{p}}$ then*

$$\left\| H_{n,A}^{q'} f\left(\cdot\right) \right\|_{X^{\widetilde{p}}} = O\left(\omega f\left(\frac{\pi}{n+1}\right)_{X^{\widetilde{p}}}\right) + O\left(a_{n,0} H\left(\frac{\pi}{n+1}\right)\right).$$

If, in addition, $H \ (\geq 0)$ instead of H_x satisfies the condition (2.4), *then*

$$\left\| H_{n,A}^{q'} f\left(\cdot\right) \right\|_{X^{\widetilde{p}}} = O\left(a_{n,0} H\left(\frac{\pi}{n+1}\right)\right)$$

with $q' \leq q$ and $p \leq \widetilde{p}$ such that $1 < q\left(q-1\right) \leq p \leq q$.

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3. Auxiliary Results

In tis section we denote by ω a function of modulus of continuity type.

Lemma 3.1. *If* (2.3) *with* $0 < p \leq q$ *and* (2.4) *with functions* ω *and* H (≥ 0) *instead of* w_x *and* H_x, *respectively, hold then*

$$(3.1) \qquad \int_0^u \frac{\omega(t)}{t} dt = O(uH(u)) \qquad (u \to 0+).$$

Proof. Integrating by parts in the above integral we obtain

$$\int_0^u \frac{\omega(t)}{t} dt = \int_0^u t \frac{d}{dt} \left(\int_t^\pi \frac{\omega(s)}{s^2} ds \right) dt$$

$$= \left[-t \int_t^\pi \frac{\omega(s)}{s^2} ds \right]_0^u + \int_0^u \left(\int_t^\pi \frac{\omega(s)}{s^2} ds \right) dt$$

$$\leq u \int_u^\pi \frac{\omega(s)}{s^2} ds + \int_0^u \left(\int_t^\pi \frac{\omega(s)}{s^2} ds \right) dt$$

$$= u \int_u^\pi \frac{\omega(s)}{s^{1+\frac{p}{q}+1-\frac{p}{q}}} ds + \int_0^u \left(\int_t^\pi \frac{\omega(s)}{s^{1+\frac{p}{q}+1-\frac{p}{q}}} ds \right) dt$$

$$\leq u^{\frac{p}{q}} \int_u^\pi \frac{\omega(s)}{s^{1+p/q}} ds + \int_0^u \frac{1}{t} \left(t^{\frac{p}{q}} \int_t^\pi \frac{(\omega(s))^p}{s^{1+p/q}} ds \right)^{\frac{1}{p}} dt,$$
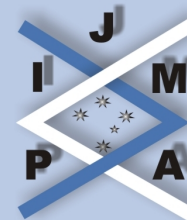
since $1 - \frac{p}{q} \geq 0$. Using our assumptions we have

$$\int_0^u \frac{\omega(t)}{t} dt = O(uH(u)) + \int_0^u \frac{1}{t} O(tH(t)) dt = O(uH(u))$$

and thus the proof is completed. $\qquad \square$

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Lemma 3.2 ([4, Theorem 5.20 II, Ch. XII]). *Suppose that $1 < q\,(q-1) \leq p \leq q$ and $\xi = 1/p + 1/q - 1$. If $\left| t^{-\xi} g\,(t) \right| \in L^p$ then*

$$(3.2) \qquad \left\{ \frac{|a_o(g)|^q}{2} + \sum_{k=0}^{\infty} (|a_k(g)|^q + |b_k(g)|^q) \right\}^{\frac{1}{q}} \leq K \left\{ \int_{-\pi}^{\pi} \left| t^{-\xi} g\,(t) \right|^p dt \right\}^{\frac{1}{p}}.$$
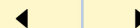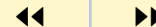
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4. Proofs of the Results

Since $H^q_{n,A}f$ is the monotonic function of q we shall consider, in all our proofs, the quantity $H^q_{n,A}f$ instead of $H^{q'}_{n,A}f$.

Proof of Theorem 2.1. Let

$$H^q_{n,A}f(x) = \left\{ \sum_{k=0}^{n} a_{n,k} \left| \frac{1}{\pi} \int_0^{\pi} \varphi_x(t) \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q \right\}^{\frac{1}{q}}$$

$$\leq \left\{ \sum_{k=0}^{n} a_{n,k} \left| \frac{1}{\pi} \int_0^{a_{n,n}} \varphi_x(t) \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q \right\}^{\frac{1}{q}}$$

$$+ \left\{ \sum_{k=0}^{n} a_{n,k} \left| \frac{1}{\pi} \int_{a_{n,n}}^{\pi} \varphi_x(t) \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q \right\}^{\frac{1}{q}}$$

$$= I(a_{n,n}) + J(a_{n,n})$$
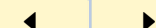
and, by (2.1), integrating by parts, we obtain,

$$I(a_{n,n}) \leq \int_0^{a_{n,n}} \frac{|\varphi_x(t)|}{2t} dt$$

$$= \int_0^{a_{n,n}} \frac{1}{2t} \frac{d}{dt} \left(\int_0^t |\varphi_x(s)| \, ds \right) dt$$

$$= \frac{1}{2a_{n,n}} \int_0^{a_{n,n}} |\varphi_x(t)| \, dt + \int_0^{a_{n,n}} \frac{w_x f(t)_1}{2t} dt$$

$$= \frac{1}{2} \left(w_x f(a_{n,n})_1 + \int_0^{a_{n,n}} \frac{w_x f(t)_1}{t} dt \right)$$
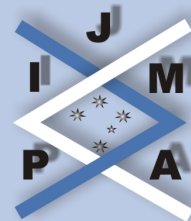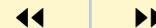
$$= K \left(a_{n,n} \int_{a_{n,n}}^{\pi} \frac{w_x f(t)_1}{t^2} dt + \int_0^{a_{n,n}} \frac{w_x f(t)_1}{t} dt \right)$$

$$\leq K \left(a_{n,n} \int_{a_{n,n}}^{\pi} \frac{(w_x f(t)_{L^1})^p}{t^2} dt \right)^{\frac{1}{p}} + K \left(\int_0^{a_{n,n}} \frac{w_x f(t)_{L^1}}{t} dt \right)$$

$$\leq K \left(a_{n,n}^{p/q} \int_{a_{n,n}}^{\pi} \frac{(w_x f(t)_{L^p})^p}{t^{1+p/q}} dt \right)^{\frac{1}{p}} + K \left(\int_0^{a_{n,n}} \frac{w_x f(t)_{L^p}}{t} dt \right).$$

Since $f \in L^p(w_x)$ and (2.4) holds, Lemma 3.1 and (2.3) give

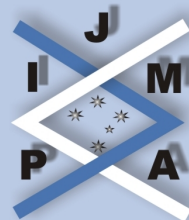$$I(a_{n,n}) = O(a_{n,n} H_x(a_{n,n})).$$

The Abel transformation shows that

$$(J(a_{n,n}))^q = \sum_{k=0}^{n-1} (a_{n,k} - a_{n,k+1}) \sum_{\nu=0}^{k} \left| \frac{1}{\pi} \int_{a_{n,n}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu + \frac{1}{2}\right) t}{2 \sin \frac{1}{2} t} dt \right|^q$$

$$+ a_{n,n} \sum_{\nu=0}^{n} \left| \frac{1}{\pi} \int_{a_{n,n}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu + \frac{1}{2}\right) t}{2 \sin \frac{1}{2} t} dt \right|^q,$$

whence, by (2.2),

$$(J(a_{n,n}))^q \leq (K+1) a_{n,n} \sum_{\nu=0}^{n} \left| \frac{1}{\pi} \int_{a_{n,n}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu + \frac{1}{2}\right) t}{2 \sin \frac{1}{2} t} dt \right|^q.$$
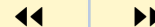
Using inequality (3.2), we obtain

$$J(a_{n,n}) \leq K (a_{n,n})^{\frac{1}{q}} \left\{ \int_{a_{n,n}}^{\pi} \frac{|\varphi_x(t)|^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}.$$

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Integrating by parts, we have

$$J\left(a_{n,n}\right) \leq K\left(a_{n,n}\right)^{\frac{1}{q}} \left\{ \left[\frac{1}{t^{p/q}} \left(w_x f\left(t\right)_{L^p}\right)^p \right]_{t=a_{n,n}}^{\pi} \right.$$

$$\left. + \left(1 + \frac{p}{q}\right) \int_{a_{n,n}}^{\pi} \frac{\left(w_x f\left(t\right)_{L^p}\right)^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}$$

$$\leq K\left(a_{n,n}\right)^{\frac{1}{q}} \left\{ \left(w_x f\left(\pi\right)_{L^p}\right) + \int_{a_{n,n}}^{\pi} \frac{\left(w_x f\left(t\right)_{L^p}\right)^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}.$$

Since $f \in L^p\left(w_x\right)$, by (2.3),

$$J\left(a_{n,n}\right) \leq K\left(a_{n,n}\right)^{\frac{1}{q}} \left\{ \left(w_x\left(\pi\right)\right) + \int_{a_{n,n}}^{\pi} \frac{\left(w_x\left(t\right)\right)^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}$$

$$\leq K \left\{ \left(a_{n,n}\right)^{\frac{p}{q}} \int_{a_{n,n}}^{\pi} \frac{\left(w_x\left(t\right)\right)^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}$$

$$= O\left(a_{n,n} H_x\left(a_{n,n}\right)\right).$$

Thus our result is proved. □

Proof of Theorem 2.2. Let, as before,

$$H_{n,A}^q f\left(x\right) \leq I\left(\frac{\pi}{n+1}\right) + J\left(\frac{\pi}{n+1}\right)$$

and

$$I\left(\frac{\pi}{n+1}\right) \leq \frac{n+1}{\pi} \int_0^{\frac{\pi}{n+1}} \left|\varphi_x\left(t\right)\right| dt = w_x\left(\frac{\pi}{n+1}\right)_{L^1}.$$

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In the estimate of $J\left(\frac{\pi}{n+1}\right)$ we again use the Abel transformation and (2.2). Thus

$$\left(J\left(\frac{\pi}{n+1}\right)\right)^q \le (K+1)\, a_{n,n} \sum_{\nu=0}^{n} \left| \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \varphi_x(t)\, \frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q$$

and, by inequality (3.2),

$$J\left(\frac{\pi}{n+1}\right) \le K\,(a_{n,n})^{\frac{1}{q}} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}.$$

Integrating (2.3) by parts, with the assumption $f \in L^p(w_x)$, we obtain

$$J\left(\frac{\pi}{n+1}\right) \le K\,((n+1)\,a_{n,n})^{\frac{1}{q}} \left\{ \left(\frac{\pi}{n+1}\right)^{\frac{p}{q}} \int_{\frac{\pi}{n+1}}^{\pi} \frac{(w_x(t))^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}$$

$$= O\left(((n+1)\,a_{n,n})^{\frac{1}{q}} \frac{\pi}{n+1} H_x\left(\frac{\pi}{n+1}\right) \right)$$

as in the previous proof, with $\frac{\pi}{n+1}$ instead of $a_{n,n}$.

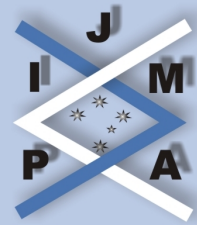Finally, arguing as in [3, p.110], we can see that, for $j = 0, 1, \ldots, n-1$,

$$|a_{n,j} - a_{n,n}| \le \left| \sum_{k=j}^{n-1} (a_{n,k} - a_{n,k+1}) \right| \le \sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+1}| \le K a_{n,n},$$

whence

$$a_{n,j} \le (K+1)\, a_{n,n}$$

and therefore

$$(K+1)\,(n+1)\, a_{n,n} \ge \sum_{j=0}^{n} a_{n,j} = 1.$$

This inequality implies that

$$J\left(\frac{\pi}{n+1}\right) = O\left(a_{n,n}H_x\left(\frac{\pi}{n+1}\right)\right)$$

and the proof of the first part of our statement is complete.

To prove of the second part of our assertion we have to estimate the term $I\left(\frac{\pi}{n+1}\right)$ once more.

Proceeding analogously to the proof of Theorem 2.1, with $a_{n,n}$ replaced by $\frac{\pi}{n+1}$, we obtain

$$I\left(\frac{\pi}{n+1}\right) = O\left(\frac{\pi}{n+1}H_x\left(\frac{\pi}{n+1}\right)\right).$$

By the inequality from the first part of our proof, the relation $(n+1)^{-1} = O\left(a_{n,n}\right)$ holds, whence the second statement follows. $\square$

Proof of Theorem 2.3. As usual, let

$$H_{n,A}^q f\left(x\right) \le I\left(a_{n,0}\right) + J\left(a_{n,0}\right).$$

Since $f \in L^p\left(w_x\right)$, by the same method as in the proof of Theorem 2.1, Lemma 3.1 and (2.3) yield

$$I\left(a_{n,0}\right) = O\left(a_{n,0}H_x\left(a_{n,0}\right)\right).$$

By the Abel transformation

$$\left(J\left(a_{n,0}\right)\right)^q \le \sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+1}| \sum_{\nu=0}^{k} \left|\frac{1}{\pi}\int_{a_{n,0}}^{\pi} \varphi_x\left(t\right)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^q$$
$$+ a_{n,n}\sum_{\nu=0}^{n}\left|\frac{1}{\pi}\int_{a_{n,0}}^{\pi}\varphi_x\left(t\right)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^q$$

$$\leq \left(\sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| + a_{n,n} \right) \sum_{\nu=0}^{\infty} \left| \frac{1}{\pi} \int_{a_{n,0}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu + \frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q.$$

Arguing as in [3, p.110], by (2.5), we have

$$|a_{n,n} - a_{n,0}| \leq \sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+1}| \leq \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| \leq K a_{n,0},$$

whence $a_{n,n} \leq (K+1)\, a_{n,0}$ and therefore

$$\sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| + a_{n,n} \leq (2K+1)\, a_{n,0}$$

and

$$(J(a_{n,0}))^q \leq (2K+1)\, a_{n,0} \sum_{\nu=0}^{\infty} \left| \frac{1}{\pi} \int_{a_{n,0}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu + \frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q.$$

Finally, by (3.2),

$$J(a_{n,0}) \leq K (a_{n,0})^{\frac{1}{q}} \left\{ \int_{a_{n,0}}^{\pi} \frac{|\varphi_x(t)|^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}$$
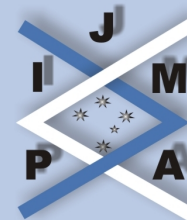
and, by (2.3),

$$J(a_{n,0}) = O\left(a_{n,0} H_x(a_{n,0})\right).$$

This completes of our proof. $\qquad\square$

Proof of Theorem 2.4. We start as usual with the simple transformation

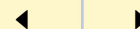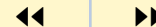$$H_{n,A}^q f(x) \leq I\left(\frac{\pi}{n+1}\right) + J\left(\frac{\pi}{n+1}\right).$$

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Similarly, as in the previous proofs, by (2.1) we have

$$I\left(\frac{\pi}{n+1}\right) \le w_x f\left(\frac{\pi}{n+1}\right)_{L^1}.$$

We estimate the term J in the following way

$$\left(J\left(\frac{\pi}{n+1}\right)\right)^q \le \sum_{k=0}^{n-1} |a_{n,k} - a_{n,k+1}| \sum_{\nu=0}^{k} \left| \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q$$

$$+ a_{n,n} \sum_{\nu=0}^{n} \left| \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q$$

$$\le \left(\sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+1}| + a_{n,n} \right) \sum_{\nu=0}^{\infty} \left| \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q.$$

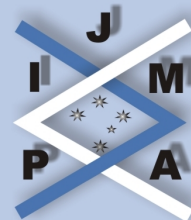From the assumption (2.5), arguing as before, we can see that

$$J\left(\frac{\pi}{n+1}\right) \le K \left(a_{n,0} \sum_{\nu=0}^{\infty} \left| \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q \right)^{\frac{1}{q}}$$

and, by (3.2),

$$J\left(\frac{\pi}{n+1}\right) \le K \left(a_{n,0}\right)^{\frac{1}{q}} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}.$$
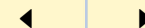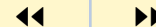
From (2.5), it follows that $a_{n,k} \le (K+1) a_{n,0}$ for any $k \le n$, and therefore

$$(K+1)(n+1) a_{n,0} \ge \sum_{k=0}^{n} a_{n,k} = 1,$$

whence

$$J\left(\frac{\pi}{n+1}\right) \le K\left((n+1)\,a_{n,0}\right)^{\frac{1}{q}} \left\{\left(\frac{\pi}{n+1}\right)^{\frac{p}{q}} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|^p}{t^{1+p/q}}\,dt\right\}^{\frac{1}{p}}.$$

$$\le K\,(n+1)\,a_{n,0} \left\{\left(\frac{\pi}{n+1}\right)^{\frac{p}{q}} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|^p}{t^{1+p/q}}\,dt\right\}^{\frac{1}{p}}.$$

Since $f \in L^p(w_x)$, integrating by parts we obtain

$$J\left(\frac{\pi}{n+1}\right) \le K\,(n+1)\,a_{n,0}\frac{\pi}{n+1}H_x\left(\frac{\pi}{n+1}\right)$$

$$\le K a_{n,0} H_x\left(\frac{\pi}{n+1}\right)$$

and the proof of the first part of our statement is complete.

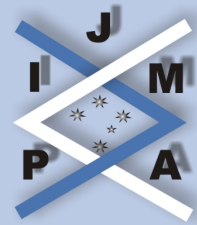To prove the second part, we have to estimate the term $I\left(\frac{\pi}{n+1}\right)$ once more.

Proceeding analogously to the proof of Theorem 2.1 we obtain

$$I\left(\frac{\pi}{n+1}\right) = O\left(\frac{\pi}{n+1}H_x\left(\frac{\pi}{n+1}\right)\right).$$

From the start of our proof we have $(n+1)^{-1} = O(a_{n,0})$, whence the second assertion also follows. $\square$

Proof of Theorem 2.5. We begin with the inequality

$$\left\|H_{n,A}^q f(\cdot)\right\|_{X^{\widetilde{p}}} \le \left\|I(a_{n,n})\right\|_{X^{\widetilde{p}}} + \left\|J(a_{n,n})\right\|_{X^{\widetilde{p}}}.$$

By (2.1), Lemma 3.1 gives

$$\begin{aligned}
\|I(a_{n,n})\|_{X^{\tilde{p}}} &\leq \int_0^{a_{n,n}} \frac{\|\varphi(t)\|_{X^{\tilde{p}}}}{2t} dt \\
&\leq \int_0^{a_{n,n}} \frac{\omega f(t)_{X^{\tilde{p}}}}{2t} dt \\
&= O(a_{n,n} H(a_{n,n})).
\end{aligned}$$

As in the proof of Theorem 2.1,

$$\begin{aligned}
\|J(a_{n,n})\|_{X^{\tilde{p}}} &\leq K(a_{n,n})^{\frac{1}{q}} \left\| \left\{ \int_{a_{n,n}}^{\pi} \frac{|\varphi(t)|^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \right\|_{X^{\tilde{p}}} \\
&\leq K(a_{n,n})^{\frac{1}{q}} \left\{ \int_{a_{n,n}}^{\pi} \frac{\|\varphi(t)\|_{X^{\tilde{p}}}^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \\
&\leq K(a_{n,n})^{\frac{1}{q}} \left\{ \int_{a_{n,n}}^{\pi} \frac{\omega f(t)_{X^{\tilde{p}}}}{t^{1+p/q}} dt \right\}^{\frac{1}{p}},
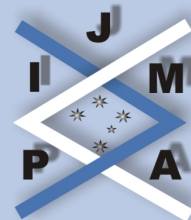\end{aligned}$$

whence, by (2.6),

$$\|J(a_{n,n})\|_{X^{\tilde{p}}} = O(a_{n,n} H(a_{n,n}))$$

holds and our result follows. $\qquad\square$

Proof of Theorem 2.6. It is clear that

$$\left\| H_{n,A}^q f(\cdot) \right\|_{X^{\tilde{p}}} \leq \left\| I\left(\frac{\pi}{n+1}\right) \right\|_{X^{\tilde{p}}} + \left\| J\left(\frac{\pi}{n+1}\right) \right\|_{X^{\tilde{p}}}$$

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and immediately

$$\left\| I \left(\frac{\pi}{n+1} \right) \right\|_{X^{\tilde{p}}} \leq \left\| w \cdot f \left(\frac{\pi}{n+1} \right)_1 \right\|_{X^{\tilde{p}}} \leq \omega f \left(\frac{\pi}{n+1} \right)_{X^{\tilde{p}}}$$

and

$$\left\| J \left(\frac{\pi}{n+1} \right) \right\|_{X^{\tilde{p}}} \leq K \left(a_{n,n} \right)^{\frac{1}{q}} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \frac{\| \varphi(t) \|_{X^{\tilde{p}}}^{p}}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}$$

$$\leq K \left((n+1) a_{n,n} \right)^{\frac{1}{q}} \left\{ \frac{\pi}{n+1} \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega f(t)_{X^{\tilde{p}}}}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}$$

$$\leq K \left((n+1) a_{n,n} \right)^{\frac{1}{q}} \frac{\pi}{n+1} H \left(\frac{\pi}{n+1} \right)$$

$$\leq K (n+1) a_{n,n} \frac{\pi}{n+1} H \left(\frac{\pi}{n+1} \right)$$

$$= O \left(a_{n,n} H \left(\frac{\pi}{n+1} \right) \right).$$
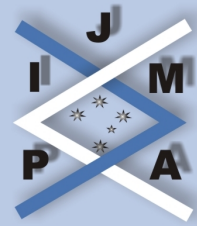
Thus our first statement holds. The second one follows on using a similar process to that in the proof of Theorem 2.1. We have to only use the estimates obtained in the proof of Theorem 2.5, with $\frac{\pi}{n+1}$ instead of $a_{n,n}$, and the relation

$$(n+1)^{-1} = O \left(a_{n,n} \right).$$

$\square$

Proof of Theorem 2.7. As in the proof of Theorem 2.5, we have

$$\left\| H_{n,A}^{q} f (\cdot) \right\|_{X^{\tilde{p}}} \leq \left\| I \left(a_{n,0} \right) \right\|_{X^{\tilde{p}}} + \left\| J \left(a_{n,0} \right) \right\|_{X^{\tilde{p}}}$$

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and
$$\left\| I\left(a_{n,0}\right)\right\|_{X^{\widetilde{p}}} = O\left(a_{n,0}H\left(a_{n,0}\right)\right) .$$

Also, from the proof of Theorem 2.3,

$$\left\| J\left(a_{n,0}\right)\right\|_{X^{\widetilde{p}}} \le K\left(a_{n,0}\right)^{\frac{1}{q}} \left\| \left\{ \int_{a_{n,0}}^{\pi} \frac{|\varphi_{\cdot}(t)|^p}{t^{1+p/q}}dt \right\}^{\frac{1}{p}} \right\|_{X^{\widetilde{p}}}$$

$$\le K\left(a_{n,0}\right)^{\frac{1}{q}} \left\{ \int_{a_{n,0}}^{\pi} \frac{\omega f(t)_{X^{\widetilde{p}}}}{t^{1+p/q}}dt \right\}^{\frac{1}{p}}$$

and, by (2.6),

$$\left\| J\left(a_{n,0}\right)\right\|_{X^{\widetilde{p}}} = O\left(a_{n,0}H\left(a_{n,0}\right)\right) .$$

Thus our result is proved. $\qquad\square$

Proof of Theorem 2.8. We recall, as in the previous proof, that

$$\left\| H_{n,A}^q f\left(\cdot\right)\right\|_{X^{\widetilde{p}}} \le \left\| I\left(\frac{\pi}{n+1}\right)\right\|_{X^{\widetilde{p}}} + \left\| J\left(\frac{\pi}{n+1}\right)\right\|_{X^{\widetilde{p}}}$$

and

$$\left\| I\left(\frac{\pi}{n+1}\right)\right\|_{X^{\widetilde{p}}} \le \omega f\left(\frac{\pi}{n+1}\right)_{X^{\widetilde{p}}} .$$

We apply a similar method as that used in the proof of Theorem 2.4 to obtain an estimate for the quantity $\left\| J\left(\frac{\pi}{n+1}\right)\right\|_{X^{\widetilde{p}}}$,

$$\left\| J\left(\frac{\pi}{n+1}\right)\right\|_{X^{\widetilde{p}}} \le K\left(n+1\right)a_{n,0} \left\| \left\{ \left(\frac{\pi}{n+1}\right)^{\frac{p}{q}} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x\left(t\right)|^p}{t^{1+p/q}}dt \right\}^{\frac{1}{p}} \right\|_{X^{\widetilde{p}}}$$

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$$\leq K\left(n+1\right)a_{n,0}\left\{\left(\frac{\pi}{n+1}\right)^{\frac{p}{q}}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\left\|\varphi_{\cdot}\left(t\right)\right\|_{X^{\tilde{p}}}^{p}}{t^{1+p/q}}dt\right\}^{\frac{1}{p}}$$

$$\leq K\left(n+1\right)a_{n,0}\left\{\left(\frac{\pi}{n+1}\right)^{\frac{p}{q}}\int_{\frac{\pi}{n+1}}^{\pi}\frac{\omega f(t)_{X^{\tilde{p}}}}{t^{1+p/q}}dt\right\}^{\frac{1}{p}}$$
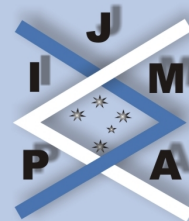
and, by (2.6),

$$\left\|J\left(\frac{\pi}{n+1}\right)\right\|_{X^{\tilde{p}}}\leq K\left(n+1\right)a_{n,0}\frac{\pi}{n+1}H\left(\frac{\pi}{n+1}\right)$$

$$\leq Ka_{n,0}H\left(\frac{\pi}{n+1}\right).$$

Thus the proof of the first part of our statement is complete.

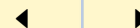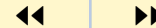To prove of the second part, we follow the line of the proof of Theorem 2.6. $\square$

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