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# ENERGY DECAY OF SOLUTIONS OF A WAVE EQUATION OF $p-$LAPLACIAN TYPE WITH A WEAKLY NONLINEAR DISSIPATION

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Abstract

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## Abstract

In this paper we study decay properties of the solutions to the wave equation of $p-$Laplacian type with a weak nonlinear dissipative.

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## Contents

## 1. Introduction

We consider the initial boundary problem for the nonlinear wave equation of $p-$Laplacian type with a weak nonlinear dissipation of the type

$$(P) \qquad \begin{cases} (|u'|^{l-2}u')' - \Delta_p u + \sigma(t)g(u') = 0 \text{ in } \Omega \times [0,+\infty[, \\ u = 0 \text{ on } \partial\Omega \times [0,+\infty[, \\ u(x,0) = u_0(x), \;\; u'(x,0) = u_1(x) \text{ in } \Omega. \end{cases}$$

where $\Delta_p u = div(|\nabla_x u|^{p-2}\nabla_x u)$, $p$, $l \geq 2$, $g : \mathbb{R} \to \mathbb{R}$ is a continuous non-decreasing function and $\sigma$ is a positive function.

When $p = 2, l = 2$ and $\sigma \equiv 1$, for the case $g(x) = \delta x$ $(\delta > 0)$, Ikehata and Suzuki [5] investigated the dynamics, showing that for sufficiently small initial data $(u_0, u_1)$, the trajectory $(u(t), u'(t))$ tends to $(0,0)$ in $H_0^1(\Omega) \times L^2(\Omega)$ as $t \to +\infty$. When $g(x) = \delta|x|^{m-1}x$ $(m \geq 1)$, Nakao [8] investigated the decay property of the problem $(P)$. In [8] the author has proved the existence of global solutions to the problem $(P)$.

For the problem $(P)$ with $\sigma \equiv 1, l = 2$, when $g(x) = \delta|x|^{m-1}x$ $(m \geq 1)$, Yao [1] proved that the energy decay rate is $E(t) \leq (1+t)^{-\frac{p}{(mp-m-1)}}$ for $t \geq 0$ by using a general method introduced by Nakao [8]. Unfortunately, this method does not seem to be applicable in the case of more general functions $\sigma$ and is more complicated.

Our purpose in this paper is to give energy decay estimates of the solutions to the problem $(P)$ for a weak nonlinear dissipation. We extend the results obtained by Yao and prove in some cases an exponential decay when $p > 2$ and the dissipative term is not necessarily superlinear near the origin.

We use a new method recently introduced by Martinez [7] (see also [2]) to study the decay rate of solutions to the wave equation $u'' - \Delta_x u + g(u') = 0$ in $\Omega \times \mathbb{R}^+$, where $\Omega$ is a bounded domain of $\mathbb{R}^n$. This method is based on a new integral inequality that generalizes a result of Haraux [4].

Throughout this paper the functions considered are all real valued. We omit the space variable $x$ of $u(t,x)$, $u_t(t,x)$ and simply denote $u(t,x)$, $u_t(t,x)$ by $u(t)$, $u'(t)$, respectively, when no confusion arises. Let $l$ be a number with $2 \leq l \leq \infty$. We denote by $\|\cdot\|_l$ the $L^l$ norm over $\Omega$. In particular, the $L^2$ norm is denoted by $\|\cdot\|_2$. $(\cdot)$ denotes the usual $L^2$ inner product. We use familiar function spaces $W_0^{1,p}$.

## 2. Preliminaries and Main Results

First assume that the solution exists in the class

$$u \in C(\mathbb{R}_+, W_0^{1,p}(\Omega)) \cap C^1(\mathbb{R}_+, L^l(\Omega)). \tag{2.1}$$

$\lambda(x), \sigma(t)$ and $g$ satisfy the following hypotheses:
**(H1)** $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a non increasing function of class $C^1$ on $\mathbb{R}_+$ satisfying

$$\int_0^{+\infty} \sigma(\tau) d\tau = +\infty. \tag{2.2}$$

**(H2)** Consider $g : \mathbb{R} \to \mathbb{R}$ a non increasing $C^0$ function such that

$$g(v)v > 0 \text{ for all } v \neq 0.$$

and suppose that there exist $c_i > 0$; $i = 1, 2, 3, 4$ such that

$$c_1|v|^m \leq |g(v)| \leq c_2|v|^{\frac{1}{m}} \text{ if } |v| \leq 1, \tag{2.3}$$

$$c_3|v|^s \leq |g(v)| \leq c_4|v|^r \text{ for all } |v| \geq 1, \tag{2.4}$$

where $m \geq 1$, $l - 1 \leq s \leq r \leq \frac{n(p-1)+p}{n-p}$.
We define the energy associated to the solution given by (2.1) by the following formula

$$E(t) = \frac{l-1}{l}\|u'\|_l^l + \frac{1}{p}\|\nabla_x u\|_p^p.$$

We first state two well known lemmas, and then state and prove a lemma that will be needed later.

**Lemma 2.1 (Sobolev-Poincaré inequality).** *Let $q$ be a number with $2 \le q < +\infty$ $(n = 1, 2, \dots, p)$ or $2 \le q \le \frac{np}{(n-p)}$ $(n \ge p+1)$, then there is a constant $c_* = c(\Omega, q)$ such that*

$$\|u\|_q \le c_* \|\nabla u\|_p \quad \textit{for} \quad u \in W_0^{1,p}(\Omega).$$

**Lemma 2.2 ([6]).** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $q \ge 0$ and $A > 0$ such that*

$$\int_S^{+\infty} E^{q+1}(t)\,dt \le AE(S), \quad 0 \le S < +\infty,$$

*then we have*

$$E(t) \le cE(0)(1+t)^{\frac{-1}{q}} \;\; \forall t \ge 0, \quad \textit{if} \;\; q > 0$$

*and*

$$E(t) \le cE(0)e^{-\omega t} \;\; \forall t \ge 0, \quad \textit{if} \;\; q = 0,$$

*where $c$ and $\omega$ are positive constants independent of the initial energy $E(0)$.*

**Lemma 2.3 ([7]).** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing $C^2$ function such that*

$$\phi(0) = 0 \quad \textit{and} \quad \phi(t) \to +\infty \;\; \textit{as} \;\; t \to +\infty.$$

*Assume that there exist $q \ge 0$ and $A > 0$ such that*

$$\int_S^{+\infty} E(t)^{q+1}(t)\phi'(t)\,dt \le AE(S), \quad 0 \le S < +\infty,$$

*then we have*

$$E(t) \leq cE(0)(1+\phi(t))^{\frac{-1}{q}} \ \ \forall t \geq 0, \quad \text{if} \ \ q>0$$

*and*

$$E(t) \leq cE(0)e^{-\omega\phi(t)} \ \ \forall t \geq 0, \quad \text{if} \ \ q=0,$$

*where $c$ and $\omega$ are positive constants independent of the initial energy $E(0)$.*

*Proof of Lemma 2.3.* Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$. $f$ is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned} \int_{\phi(S)}^{\phi(T)} f(x)^{q+1}\,dx &= \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{q+1}\,dx \\ &= \int_S^T E(t)^{q+1}\phi'(t)\,dt \\ &\leq AE(S) = Af(\phi(S)) \quad 0 \leq S < T < +\infty. \end{aligned}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\int_s^{+\infty} f(x)^{q+1}\,dx \leq Af(s) \quad 0 \leq s < +\infty.$$

By Lemma 2.2, we can deduce the desired results. □

Our main result is the following

**Theorem 2.4.** *Let $(u_0, u_1) \in W_0^{1,p} \times L^l(\Omega)$ and suppose that (**H1**) and (**H2**) hold. Then the solution $u(x,t)$ of the problem $(P)$ satisfies*

*(1) If* $l \geq m+1$, *we have*

$$E(t) \leq C(E(0)) \exp\left(1-\omega \int_0^t \sigma(\tau)\, d\tau\right) \quad \forall t>0.$$

*(2) If* $l < m+1$, *we have*

$$E(t) \leq \left(\frac{C(E(0))}{\int_0^t \sigma(\tau)\, d\tau}\right)^{\frac{p}{(mp-m-1)}} \quad \forall t>0.$$

## Examples

1) If $\sigma(t) = \frac{1}{t^\theta}$ $(0 \leq \theta \leq 1)$, by applying Theorem 2.4 we obtain

$$E(t) \leq C(E(0)) e^{1-\omega t^{1-\theta}} \quad \text{if } \theta \in [0,1[,\ l \geq m+1,$$

$$E(t) \leq C(E(0)) t^{-\frac{(1-\theta)p}{mp-m-1}} \quad \text{if } 0 \leq \theta < 1,\ l < m+1$$

and

$$E(t) \leq C(E(0)) (\ln t)^{-\frac{p}{(mp-m-1)}} \quad \text{if } \theta = 1,\ l < m+1.$$

2) If $\sigma(t) = \frac{1}{t^\theta \ln t \ln_2 t \ldots \ln_k t}$, where $k$ is a positive integer and

$$\begin{cases} \ln_1(t) = \ln(t) \\ \ln_{k+1}(t) = \ln(\ln_k(t)), \end{cases}$$

by applying Theorem 2.4, we obtain

$$E(t) \leq C(E(0))(\ln_{k+1} t)^{-\frac{p}{(mp-m-1)}} \qquad \text{if } \theta = 1,\ l < m+1,$$

$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{mp-m-1}} (\ln t \ln_2 t \dots \ln_k t)^{\frac{p}{mp-m-1}} \ \text{if } 0 \leq \theta < 1,\ l < m+1.$$

3) If $\sigma(t) = \dfrac{1}{t^\theta (\ln t)^\gamma}$, by applying Theorem 2.4, we obtain

$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{mp-m-1}} (\ln t)^{\frac{\gamma p}{mp-m-1}} \qquad \text{if } 0 \leq \theta < 1,\ l < m+1,$$

$$E(t) \leq C(E(0))(\ln t)^{-\frac{(1-\gamma)p}{mp-m-1}} \qquad \text{if } \theta = 1, 0 \leq \gamma < 1,\ l < m+1,$$

$$E(t) \leq C(E(0))(\ln_2 t)^{-\frac{p}{mp-m-1}} \ \text{if } \theta = 1, \gamma = 1,\ l < m+1.$$

**Proof of Theorem 2.4.**

First we have the following energy identity to the problem $(P)$

**Lemma 2.5 (Energy identity).** *Let $u(t,x)$ be a local solution to the problem $(P)$ on $[0,\infty)$ as in Theorem 2.4. Then we have*

$$E(t) + \int_\Omega \int_0^t \sigma(s) u'(s) g(u'(s))\, ds\, dx = E(0)$$

*for all $t \in [0,\infty)$.*

*Proof of the energy decay.* From now on, we denote by $c$ various positive constants which may be different at different occurrences. We multiply the first equation of $(P)$ by $E^q\phi' u$, where $\phi$ is a function satisfying all the hypotheses of Lemma 2.3 to obtain

$$\begin{aligned}
0 &= \int_S^T E^q\phi' \int_\Omega u((|u'|^{l-2}u')_t - \Delta_p u + \sigma(t)g(u'))\,dx\,dt \\
&= \left[E^q\phi' \int_\Omega uu'|u'|^{l-2}\,dx\right]_S^T - \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'|u'|^{l-2}\,dxdt \\
&\quad - \frac{3l-2}{l}\int_S^T E^q\phi' \int_\Omega |u'|^2\,dxdt + 2\int_S^T E^q\phi' \int_\Omega \left(\frac{l-1}{l}u'^2 + \frac{1}{p}|\nabla u|^p\right)\,dxdt \\
&\quad + \int_S^T E^q\phi' \int_\Omega \sigma(t)ug(u')\,dxdt + \left(1-\frac{2}{p}\right)\int_S^T E^q\phi' \|\nabla u\|_p^p\,dxdt.
\end{aligned}$$

We deduce that

$$\begin{aligned}
(2.5)\quad 2\int_S^T E^{q+1}\phi'\,dt \le &- \left[E^q\phi' \int_\Omega uu'|u'|^{l-2}\,dx\right]_S^T \\
&+ \int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'|u'|^{l-2}\,dxdt \\
&+ \frac{3l-2}{l}\int_S^T E^q\phi' \int_\Omega |u'|^l\,dxdt - \int_S^T E^q\phi' \int_\Omega \sigma(t)ug(u')\,dxdt.
\end{aligned}$$

Since $E$ is nonincreasing, $\phi'$ is a bounded nonnegative function on $\mathbb{R}_+$ (and we denote by $\mu$ its maximum), using the Hölder inequality, we have

$$\left|E(t)^q\phi'\int_\Omega uu'|u'|^{l-2}\,dx\right| \le c\mu E(S)^{q+\frac{l-1}{l}+\frac{1}{p}} \qquad \forall t\ge S.$$

$$\begin{aligned}\int_S^T (qE'E^{q-1}\phi' + E^q\phi'')\int_\Omega uu'|u'|^{l-2}\,dxdt \\ &\le c\mu\int_S^T -E'(t)E(t)^{q-\frac{1}{l}+\frac{1}{p}}\,dt + c\int_S^T E(t)^{q+\frac{l-1}{l}+\frac{1}{p}}(-\phi''(t))\,dt \\ &\le c\mu E(S)^{q+\frac{l-1}{l}+\frac{1}{p}}.\end{aligned}$$

Using these estimates we conclude from the above inequality that

$$\begin{aligned}2\int_S^T E(t)^{1+q}\phi'(t)\,dt \\ &\le cE(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + \frac{3l-2}{l}\int_S^T E^q\phi'\int_\Omega |u'|^l\,dxdt \\ &\quad - \int_S^T E^q\phi'\int_\Omega \sigma(t)ug(u')\,dxdt \\ &\le cE(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + \frac{3l-2}{l}\int_S^T E^q\phi'\int_\Omega |u'|^l\,dxdt \\ &\quad - \int_S^T E^q\phi'\int_{|u'|\le 1} \sigma(t)ug(u')\,dxdt \\ &\quad - \int_S^T E^q\phi'\int_{|u'|>1} \sigma(t)ug(u')\,dxdt.\end{aligned} \tag{2.6}$$

Define

$$\phi(t) = \int_0^t \sigma(s)\, ds.$$

It is clear that $\phi$ is a non decreasing function of class $C^2$ on $\mathbb{R}_+$. The hypothesis (2.2) ensures that

$$\phi(t) \to +\infty \text{ as } t \to +\infty. \tag{2.7}$$

Now, we estimate the terms of the right-hand side of (2.6) in order to apply the results of Lemma 2.3:

Using the Hölder inequality, we get for $l < m+1$

$$\begin{aligned}
&\int_S^T E^q \phi' \int_\Omega |u'|^l \, dx dt \\
&\le C \int_S^T E^q \phi' \int_\Omega \frac{1}{\sigma(t)} u' \rho(t, u')\, dx\, dt \\
&\qquad + C' \int_S^T E^q \phi' \int_\Omega \left( \frac{1}{\sigma(t)} u' \rho(t, u') \right)^{\frac{l}{(m+1)}} dx\, dt \\
&\le C \int_S^T E^q \frac{\phi'}{\sigma(t)} (-E')\, dt + C'(\Omega) \int_S^T E^q \frac{\phi'}{\sigma^{\frac{l}{m+1}}(t)} (-E')^{\frac{l}{m+1}}\, dt \\
&\le C E^{q+1}(S) + C'(\Omega) \int_S^T E^q \phi'^{\frac{m+1-l}{m+1}} \left( \frac{\phi'}{\sigma(t)} \right)^{\frac{l}{m+1}} (-E')^{\frac{l}{m+1}}\, dt.
\end{aligned}$$

Now, fix an arbitrarily small $\varepsilon > 0$ (to be chosen later). By applying Young's inequality, we obtain

$$\int_S^T E^q \phi' \int_\Omega |u'|^l \, dxdt \leq CE^{q+1}(S) + C'(\Omega) \frac{m+-l}{m+1} \varepsilon^{\frac{(m+1)}{(m+1-l)}} \int_S^T E^{q \frac{m+1}{m+1-l}} \phi' \, dt + C'(\Omega) \frac{l}{m+1} \frac{1}{\varepsilon^{\frac{(m+1)}{l}}} E(S). \tag{2.8}$$

If $l \geq m+1$, we easily obtain from (2.3) and (2.4)

$$\int_S^T E^q \phi' \int_\Omega |u'|^l \, dxdt \leq CE^{q+1}(S). \tag{2.9}$$

Next, we estimate the third term of the right-hand of (2.6). We get for $l < m+1$

$$\begin{aligned} \int_S^T E^q \phi' \int_{|u'| \leq 1} \sigma(t) u g(u') \, dxdt &\leq \varepsilon_1 \int_S^T E^q \phi' \int_{|u'| \leq 1} \|u\|_p^p \, dt + C(\varepsilon_1) \int_S^T E^q \phi' \int_{|u'| \leq 1} (\sigma g(u'))^{\frac{p}{p-1}} \, dx \\ &\leq c\varepsilon_1 \int_S^T E^{q+1} \phi' \, dt + C(\varepsilon_1) \int_S^T E^q \phi' \int_{|u'| \leq 1} (\sigma g(u'))^{\frac{p}{p-1}} \, dx. \end{aligned} \tag{2.10}$$

We now estimate the last term of the above inequality to get

$$\begin{aligned}(2.11)\qquad \int_S^T E^q\phi' \int_{|u'|\le 1} (\sigma g(u'))^{\frac{p}{p-1}}\,dx\,dt &\le \int_S^T E^q\phi' \int_{|u'|\le 1} (u' g(u'))^{\frac{p}{(m+1)(p-1)}}\,dx\,dt \\ &\le \int_S^T E^q\phi' \frac{1}{\sigma^{\frac{p}{(m+1)(p-1)}}} \int_{|u'|\le 1} (\sigma u' g(u'))^{\frac{p}{(m+1)(p-1)}}\,dx\,dt \\ &\le C(\Omega)\int_S^T E^q\phi' \frac{1}{\sigma^{\frac{p}{(m+1)(p-1)}}} (-E')^{\frac{p}{(m+1)(p-1)}}\,dt.\end{aligned}$$

Set $\varepsilon_2 > 0$; due to Young's inequality, we obtain

$$\begin{aligned}(2.12)\qquad &\int_S^T E^q\phi' \int_{|u'|\le 1} (\sigma g(u'))^{\frac{p}{p-1}}dxdt \\ &\le C(\Omega)\frac{(m+1)(p-1)-p}{(m+1)(p-1)}\varepsilon_2^{\frac{(m+1)(p-1)}{(m+1)(p-1)-p}} \int_S^T E^{q\frac{(m+1)(p-1)}{(m+1)(p-1)-p}}\phi' dt \\ &\qquad + \frac{C(\Omega)\,p}{(m+1)(p-1)}\frac{1}{\varepsilon_2^{\frac{(m+1)(p-1)}{p}}}E(S),\end{aligned}$$

we chose $q$ such that

$$q\frac{(m+1)(p-1)}{(m+1)(p-1)-p} = q+1.$$

thus we find $q = \frac{mp-m-1}{p}$ and thus $q\frac{m+1}{m+1-l} = q+1+\alpha$ with $\alpha = \frac{(m+1)(p\,l-p-l)}{p(m+1-l)}$.

Using the Hölder inequality, the Sobolev imbedding and the condition (2.4), we obtain

$$\begin{aligned}\int_S^T E^q\phi' \int_{|u'|\geq 1} \sigma(t)ug(u')\,dxdt &\leq \int_S^T E^q\phi'\sigma(t)\left(\int_\Omega |u|^{r+1}\,dx\right)^{\frac{1}{(r+1)}}\left(\int_{|u'|>1}|g(u')|^{\frac{r+1}{r}}\,dx\right)^{\frac{r}{r+1}}\,dt\\ &\leq c\int_S^T E^{q+\frac{1}{p}}\phi'\sigma^{\frac{1}{(r+1)}}(t)\left(\int_{|u'|>1}\sigma u' g(u')\,dx\right)^{\frac{r}{r+1}}\,dt\\ &\leq c\int_S^T E^{q+\frac{1}{p}}\phi'\sigma^{\frac{1}{(r+1)}}(t)(-E')^{\frac{r}{r+1}}\,dt.\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}\int_S^T E^q\phi' \int_{|u'|\geq 1} \sigma(t)ug(u')\,dxdt &\leq \varepsilon_3\int_S^T (E^{q+\frac{1}{p}}\phi'\sigma^{\frac{1}{(r+1)}}(t))^{r+1}\,dt + c(\varepsilon_3)\int_S^T(-E')\,dt\\ &\leq \varepsilon_3\mu^{r+1}E^{\frac{(p-1)(mr-1)}{p}}(0)\int_S^T E^{q+1}\phi'\,dt + c(\varepsilon_3)E(S).\end{aligned}\tag{2.13}$$

If $l \geq m+1$, the last inequality is also valid in the domain $\{|u'|<1\}$ and with $m$ instead of $r$.

Choosing $\varepsilon, \varepsilon_1, \varepsilon_2$ and $\varepsilon_3$ small enough, we deduce from (2.6), (2.8), (2.10), (2.12) and (2.13) for $l < m+1$

$$\int_S^T E(t)^{1+q}\phi'(t)\,dt \le CE(S)^{q+1} + C'E(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + C''E(S)$$
$$+ C'''E(0)^{\frac{(p\;l-p-l)(m+1)}{p\;l}}E(S) + C''''E(0)^{\frac{(m\;r-1)(p-1)}{p\;r}}E(S),$$

where $C, C', C'', C''', C'''$ are different positive constants independent of $E(0)$.

Choosing $\varepsilon_3$ small enough, we deduce from (2.6), (2.9) and (2.13) for $l \ge m+1$

$$\int_S^T E(t)^{1+q}\phi'(t)\,dt \le CE(S)^{q+1} + C'E(S)^{q+\frac{l-1}{l}+\frac{1}{p}} + C''E(0)^{\frac{(m^2-1)(p-1)}{p\;m}}E(S),$$

where $C, C', C''$ are different positive constants independent of $E(0)$, we may thus complete the proof by applying Lemma 2.3. □

**Remark 1.** *We obtain the same results for the following problem*

$$\begin{cases} (|u'|^{l-2}u')' - e^{-\Phi(x)}div(e^{\Phi(x)}|\nabla_x u|^{p-2}\nabla_x u) + \sigma(t)g(u') = 0 \textit{ in } \Omega\times[0,+\infty[, \\ u = 0 \textit{ on } \partial\Omega\times[0,+\infty[, \\ u(x,0) = u_0(x),\;\; u'(x,0) = u_1(x) \textit{ in } \Omega, \end{cases}$$

*where $\Phi$ is a positive function such that $\Phi \in L^\infty(\Omega)$, in this case $(u_0,u_1) \in W^{1,p}_{0,\Phi}\times L^l_\Phi$, where*

$$W^{1,p}_{0,\Phi}(\Omega) = \left\{u \in W^{1,p}_0(\Omega),\;\; \int_\Omega e^{\Phi(x)}|\nabla_x u|^p\,dx < \infty\right\},$$
$$L^l_\Phi(\Omega) = \left\{u \in L^l(\Omega),\;\; \int_\Omega e^{\Phi(x)}|u|^l\,dx < \infty\right\}.$$

*Thus the energy associated to the solution is given by the following formula*

$$E(t) = \frac{l-1}{l}\|e^{\Phi(x)/l}u'\|_l^l + \frac{1}{p}\|e^{\Phi(x)/p}\nabla_x u\|_p^p.$$

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